KYUNGPOOK Math. J. 46(2006), 399-408

The Action of the Handlebody Group on the First Homology Group of the Surface

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Dedicated to Professor Takao Matumoto on his sixtieth birthday

Abstract. The action of mapping class groups of a handlebody on the first homology group is investigated. A homological analogy of mapping class groups of a handlebody is defined and its system of generators is given.

1. Introduction

Let Σ_g be a closed orientable surface of genus g, bounding a 3-dimensional handlebody H_g. Griffiths [1] showed that a homeomorphism of Σ_g extends to a homeomorphism of H_g exactly when its induced automorphism on $\pi_1(\Sigma_g)$ preserves the normal closure of the set of boundary circles of a standard collection of meridian disks for H_g. In this case, its induced automorphism on $H_1(\Sigma_g)$ is a symplectic block matrix of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, with respect to a standard basis of $H_1(\Sigma_g)$ whose first g elements are the boundary circles of the meridian disks. Our first main result is that every such matrix is induced by a homeomorphism that extends to H_g. We also consider the mapping classes of Σ_g whose induced automorphisms preserve the subgroup of $H_1(\Sigma_g)$ generated by these boundary circles. These might be called *homologically extendible* homeomorphisms. Our second main result gives an efficient generating set for the group of isotopy classes of homologically extendible homeomorphisms (which we call *homological handlebody group*), when $g \geq 3$.

2. Preliminaries and results

A 3-dimensional handlebody H_g is an orientable 3-manifold constructed from a 3-ball by attaching g 1-handles. The boundary of H_g is an orientable closed surface

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Received March 10, 2005.
2000 Mathematics Subject Classification: 57N10, 57N05, 20F38.
Key words and phrases: mapping class group, 3-dimensional handlebody, symplectic group.
This research was partially supported by Grant-in-Aid for Encouragement of Young Scientists (No. 16740038), Ministry of Education, Science, Sports and Culture, Japan.

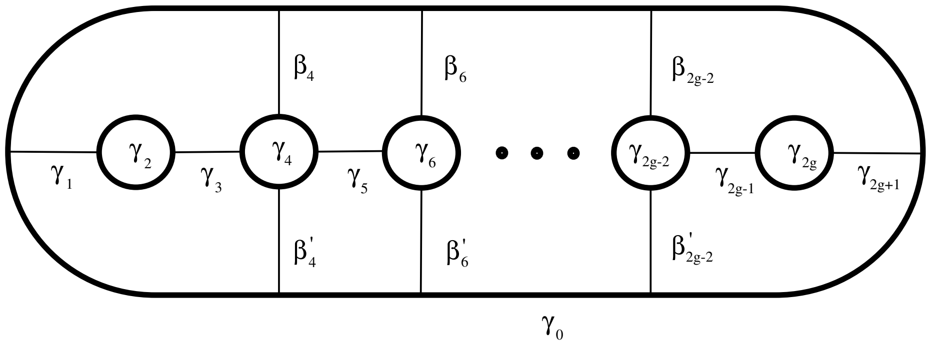

Figure 1:

Σ_g of genus g. Let $\mathcal{M}_g$ be the mapping class group of Σ_g and $\mathcal{H}_g$ be the mapping class group of H_g, for short, we call this group the *handlebody group*. For elements a, b and c of a group, we write $\bar{c} = c^{-1}$, and $a * b = ab\bar{a}$.

Let P_g be a planar surface constructed from a 2-disk by removing g copies of disjoint 2-disks. As indicated in Figure 1, the boundary components of P_g are denoted by $\gamma_0, \gamma_2, \cdots, \gamma_{2g}$, and some properly embedded arcs of P_g by $\gamma_1, \gamma_3, \cdots, \gamma_{2g+1}$, $\beta_4, \cdots, \beta_{2g-2}$ and $\beta'_4, \cdots, \beta'_{2g-2}$. The 3-manifold $P_g \times [-1, 1]$ is homeomorphic to H_g. On $\partial(P_g \times [-1, 1]) = \Sigma_g$, we define $c_{2i-1} = \partial(\gamma_{2i-1} \times [-1, 1])$ $(1 \leq i \leq g+1)$, $b_{2j} = \partial(\beta_{2j} \times [-1, 1])$, $b'_{2j} = \partial(\beta'_{2j} \times [-1, 1])$ $(2 \leq j \leq g-1)$, and $c_{2k} = \gamma_{2k} \times \{0\}$ $(1 \leq k \leq g)$. In Figures 2 and 3, these circles are illustrated and oriented.

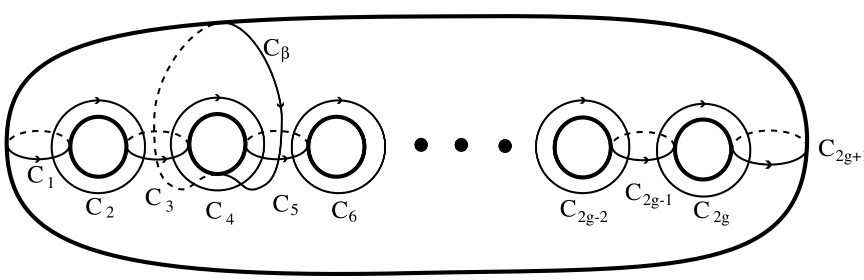

Figure 2:

For a simple closed curve a on Σ_g, we define the Dehn twist T_a about a as indicated in Figure 4. For short, we write C_i for T_{c_i}, and B_{2i} for $T_{b_{2i}}$. As elements of $H_1(\Sigma_g, \mathbb{Z})$, we take

$$\begin{aligned}
x_1 &= -c_1, \quad y_1 = -c_2 \\
x_i &= b_{2i}, \quad y_i = -c_{2i}, \text{ where } 2 \leq i \leq g-1, \\
x_g &= c_{2g+1}, \quad y_g = -c_{2g}.
\end{aligned}$$

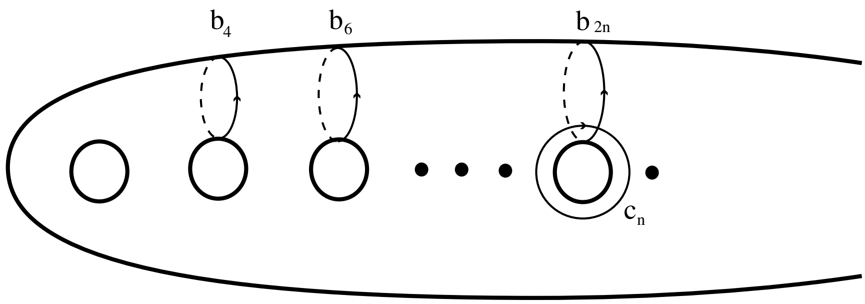

Figure 3:

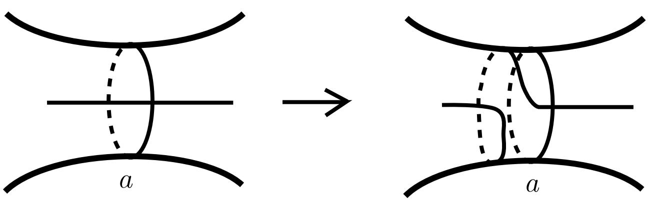

Figure 4:

Then, $\{x_1, y_1, \cdots, x_g, y_g\}$ is a basis of $H_1(\Sigma_g, \mathbb{Z})$, and satisfy $(x_i, y_j) = \delta_{i,j}$, $(x_i, x_j) = (y_i, y_j) = 0$ for the intersection form $(,)$. Let E_g be a identity $g \times g$ matrix, and

$$J = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

We define $\mathrm{S}p(2g) = \{M \in GL(2g, \mathbb{Z}) \mid M'JM = J\}$, where M' means the transpose of M.

We can characterize the handlebody group $\mathcal{H}_g$ by the actions of its elements on the fundamental group $\pi_1(\Sigma_g, p)$, where p is a point on Σ_g. Let l_1 be an arc on Σ_g which begins from p and ends on c_1, l_i $(2 \leq i \leq g-1)$ be an arc on Σ_g which begins from p and ends on b_{2i}, and l_g be an arc on Σ_g which begins from p and ends on c_{2g}. We denote by $\mathcal{N}$ the normal closure of $\{l_1 c_1 \overline{l_1}, l_2 b_4 \overline{l_2}, \cdots, l_{g-1} b_{2g-2} \overline{l_{g-1}}, l_g c_{2g+1} \overline{l_g}\}$, then $\mathcal{H}_g = \{\phi \in \mathcal{M}_g \mid \phi(\mathcal{N}) = \mathcal{N}\}$.

We define a homological analogue of $\mathcal{H}_g$. Let N be the $\mathbb{Z}$-submodule of $H_1(\Sigma_g, \mathbb{Z})$ generated by $\{x_1, \cdots, x_g\}$, and $\mathcal{HH}_g$ be a subgroup of $\mathcal{M}_g$ defined by $\mathcal{HH}_g = \{\phi \in \mathcal{M}_g \mid \phi_*(N) = N\}$. We call $\mathcal{HH}_g$ the *homological handlebody group* of genus g.

For each element ϕ of $\mathcal{M}_g$, we define a $2g \times 2g$ matrix M_ϕ by

$$(\phi(x_1), \phi(x_2), \cdots, \phi(x_g), \phi(y_1), \phi(y_2), \cdots, \phi(y_g)) = (x_1, x_2, \cdots, x_g, y_1, y_2, \cdots, y_g) M_\phi.$$

Then, M_ϕ is an element of $\mathrm{S}p(2g)$, and the map μ from $\mathcal{M}_g$ to $\mathrm{S}p(2g)$ defined by mapping ϕ to M_ϕ is a surjection. On the other hand, $\mu|_{\mathcal{H}_g}$ is not a surjection. We define a subgroup $ur\mathrm{S}p(2g)$ of $\mathrm{S}p(2g)$ by

$$ur\mathrm{S}p(2g) = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{S}p(2g) \right\},$$

where A, B, and D are $g \times g$ matrices, 0 is a $g \times g$ zero matrix, and ur stands for "upper right". We show the following theorem

Theorem 2.1. $\mu(\mathcal{H}_g) = ur\mathrm{S}p(2g)$.

Since $ur\mathrm{S}p(2g)$ is an infinite index subgroup, we have

Corollary 2.2. *$\mathcal{H}_g$ is an infinite index subgroup of $\mathcal{M}_g$.*

Remark 2.3. An embedding of Σ_g into the 3-sphere S^3 is called *trivial* if $S^3 \setminus \Sigma_g$ is the disjoint union of the interior of 3-dimensional handlebodies of genera g. For this embedding, we define

$$\mathcal{E}(S^3, \Sigma_g) = \left\{ \phi \in \mathcal{M}_g \;\middle|\; \begin{array}{l} \text{there is an element } \Phi \in \mathrm{Diff}^+(S^3) \\ \text{such that } \Phi|_{\Sigma_g} \text{ represents } \phi \end{array} \right\}.$$

Since $\mathcal{E}(S^3, \Sigma_g) \subset \mathcal{H}_g$, $\mathcal{E}(S^3, \Sigma_g)$ is an infinite index subgroup of $\mathcal{M}_g$. This is much different from the case for surfaces trivially embedded in S^4. In this case, $\mathcal{E}(S^4, \Sigma_g)$ is a finite index subgroup of $\mathcal{M}_g$ [2].

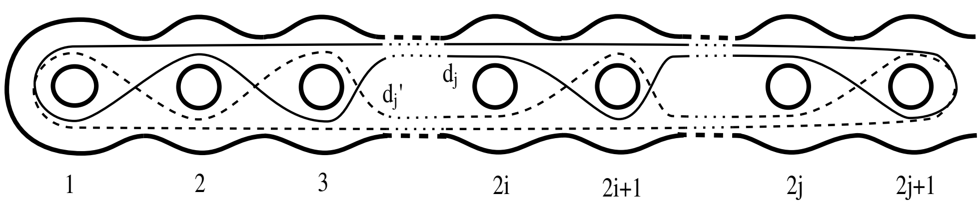

Figure 5:

By definition, $\mathcal{HH}_g = \mu^{-1}(ur\mathrm{S}p(2g))$. Let $[a]$ be the largest integer n which satisfies $n \leq a$, and d_j, d'_j, e_k, e'_k are indicated in Figures 5 and 6. We show

Theorem 2.4. *If $g \geq 3$, $\mathcal{HH}_g$ is generated by C_1, $C_2C_1^2C_2$, $C_2C_1C_3C_2$, $C_{2i}C_{2i-1}B_{2i}C_{2i}$, $C_{2i}C_{2i+1}B_{2i}C_{2i}$ $(2 \leq i \leq g-1)$, $C_{2g}C_{2g-1}C_{2g+1}C_{2g}$, $T_{d_j}\overline{T_{d'_j}}$ $(1 \leq j \leq [\frac{g-1}{2}])$, and $T_{e_k}\overline{T_{e'_k}}$ $(1 \leq k \leq [\frac{g-2}{2}])$.*

The author does not know whether $\mathcal{HH}_2$ is finitely generated, and whether $\mathcal{HH}_g$ are finitely presented.

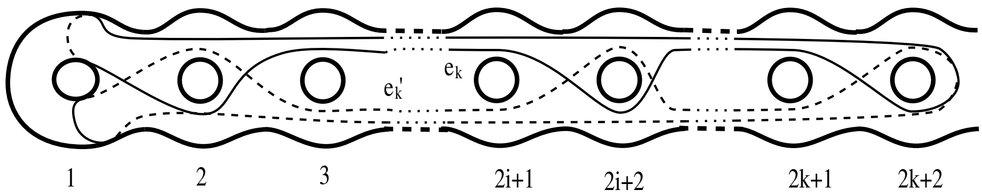

Figure 6:

3. Proof of Theorem 2.1

By definition $\mu(\mathcal{HH}_g) \subset ur\mathrm{S}p(2g)$. We show that $ur\mathrm{S}p(2g) \subset \mu(\mathcal{HH}_g)$. Let S_0 be the $g \times g$ symmetric matrix, and U_1, U_2, U_3 the $g \times g$ unimodular matrices given by

$$S_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Lemma 3.1. *The group $ur\mathrm{S}p(2g)$ is generated by*

$$\left\{ \begin{pmatrix} E_g & S_0 \\ 0 & E_g \end{pmatrix}, \begin{pmatrix} U_i & 0 \\ 0 & (U_i')^{-1} \end{pmatrix}, \text{ where } i = 1, 2, 3 \right\}.$$

Proof. If we set $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, then the condition $M'JM = J$ is equivalent to $A'D = E_g$, $B'D = D'B$. Let $S = B'D$, then S is symmetric, for $S' = D'B'' = D'B = B'D = S$. Since $A'D = E_g$, $A = U$ is unimodular, and $D = (U')^{-1}$. By the above equations, we see $US = US' = AD'B = E_gB = B$, therefore,

$$\begin{pmatrix} U & 0 \\ 0 & (U')^{-1} \end{pmatrix} \begin{pmatrix} E_g & S \\ 0 & E_g \end{pmatrix} = \begin{pmatrix} U & US \\ 0 & (U')^{-1} \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

This equation shows that $ur\mathrm{S}p(2g)$ is generated by the following types of elements:

(I) $\begin{pmatrix} E_g & S \\ 0 & E_g \end{pmatrix}$, where S is symmetric.

(II) $\begin{pmatrix} U & 0 \\ 0 & (U')^{-1} \end{pmatrix}$, where U is unimodular.

By applying the argument by Hua and Reiner [3, p.420 and p.421], we see that elements of (I) and (II) are generated by four elements listed in the statement of this lemma. □

Suzuki [6] introduced elements ρ (cyclic translation of handles), ω_1 (twisting a knob), ρ_{12} (interchanging two knobs), and θ_{12} (sliding) of $\mathcal{H}_g$. In [6], their actions on the fundamental group of Σ_g were listed. Using this list, we have

$$\mu(C_1) = \begin{pmatrix} E_g & S_0 \\ 0 & E_g \end{pmatrix}, \quad \mu(\rho) = \begin{pmatrix} U_1 & 0 \\ 0 & (U_1')^{-1} \end{pmatrix},$$
$$\mu(\rho_{12}\theta_{12}\rho_{12}^{-1}) = \begin{pmatrix} U_2 & 0 \\ 0 & (U_2')^{-1} \end{pmatrix}, \quad \mu(\omega_1) = \begin{pmatrix} U_3 & 0 \\ 0 & (U_3')^{-1} \end{pmatrix}.$$

Lemma 3.1 now implies that $ur\mathrm{S}p(2g) \subset \mu(\mathcal{H}_g)$.

4. Proof of Theorem 2.4

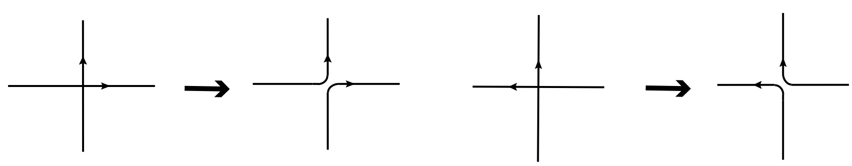

Figure 7:

Let $\mathcal{I}_g$ be the kernel of μ. This group $\mathcal{I}_g$ is called the *Torelli group*. Theorem 2.1 shows that $\mathcal{HH}_g$ is generated by $\mathcal{H}_g \cup \mathcal{I}_g$. Johnson [4] showed that, when g is larger than or equal to 3, $\mathcal{I}_g$ is finitely generated.

We review results in [4]. For the oriented simple closed curves shown in Figure 2, we refer to $(c_1, c_2, \cdots, c_{2g+1})$ and $(c_\beta, c_5, \cdots, c_{2g+1})$ as *chains*. For oriented simple closed curves d and e which intersect transversely in one point, we construct an oriented simple closed curve $d+e$ from $d \cup e$ as follows: choose a disk neighborhood of the intersection point and in it make a replacement as indicated in Figure 7. For a consecutive subset $\{c_i, c_{i+1}, \cdots, c_j\}$ of a chain, let $c_i + \cdots + c_j$ be the oriented simple closed curve constructed by repeated applications of the above operations.

Let $(i_1, \cdots, i_{r+1})$ be a subsequence of $(1, 2, \cdots, 2g+1)$ (Resp. $(\beta, 5, \cdots, 2g+1)$). We construct the union of circles $\mathcal{C} = c_{i_1} + \cdots + c_{i_2-1} \cup c_{i_2} + \cdots + c_{i_3-1} \cup \cdots \cup c_{i_r} + \cdots + c_{i_{r+1}-1}$. If r is odd, a regular neighborhood of $\mathcal{C}$ is an oriented compact surface with 2 boundary components. Let ϕ be the element of $\mathcal{M}_g$ defined as the composition of the positive Dehn twist along the boundary curve to the left of $\mathcal{C}$ and the negative Dehn twist along the boundary curve to the right of $\mathcal{C}$. Then, ϕ is an element of $\mathcal{I}_g$. We denote ϕ by $[i_1, \cdots, i_{r+1}]$, and call this *the odd subchain map* of $(c_1, c_2, \cdots, c_{2g+1})$ (Resp. $(c_\beta, c_5, \cdots, c_{2g+1})$). Johnson [4] showed the following theorem:

Theorem 4.1 ([4, Main Theorem]). *For $g \geq 3$, the odd subchain maps of the two chains $(c_1, c_2, \cdots, c_{2g+1})$ and $(c_\beta, c_5, \cdots, c_{2g+1})$ generate $\mathcal{I}_g$.*

From now on, we assume that $g \geq 3$. Since $c_\beta = B_4(c_4)$ and $B_4 \in \mathcal{H}_g$, it suffices for our purpose to consider the odd subchain maps of $(c_1, c_2, \cdots, c_{2g+1})$. By the word "an odd chain map", we mean an odd subchain map of $(c_1, c_2, \cdots, c_{2g+1})$. We use the following results by Johnson [4].

Lemma 4.2 ([4, Lemma 1]).

(a) *C_j commutes with $[i_1, i_2, \cdots]$ if and only if j and $j+1$ are either both contained in or are disjoint from the i's.*

(b) *If $i \neq j+1$, then $\overline{C_j} * [\cdots, j, i, \cdots] = [\cdots, j+1, i, \cdots]$.*

(c) *If $k \neq j$, then $C_j * [\cdots, k, j+1, \cdots] = [\cdots, k, j, \cdots]$.*

Lemma 4.3. *For any sequence of integers $5 \leq k_5 < k_6 < \cdots < k_{2n} \leq 2g+1$,*

$$\begin{aligned} & [1,2,3,4][1,2,k_5,k_6,\cdots,k_{2n}]B_4 * [3,4,k_5,\cdots,k_{2n}] \\ = \ & [k_5,k_6,\cdots,k_{2n}][1,2,3,4,k_5,k_6,\cdots,k_{2n}]. \end{aligned}$$

Proof. At first, we can show,

$$\begin{aligned} & [1,2,3,4][1,2,5,6,\cdots,2n]B_4 * [3,4,5,\cdots,2n] \\ = \ & [5,6,\cdots,2n][1,2,3,4,5,6,\cdots,2n]. \end{aligned}$$

Johnson showed this equation only in the case where $n = g$. But we can apply the proof of Lemma 10 of [4] for the case where $3 \leq n < g$ because we can regard each surfaces in Figure 18 of [4] as a surface of genus n which is a submanifold of Σ_g.

We take a proper conjugation of the above equation by $C_5, C_6, \cdots, C_{2g+1}$, then we have the equation which we need to show. □

Odd chain maps och_1, och_2 are *$\mathcal{H}_g$-equivalent* (denoted by $och_1 \sim_{\mathcal{H}_g} och_2$), if there is an element ϕ of $\mathcal{H}_g$ such that $\phi * och_1 = och_2$. For an odd chain $[n_1, n_2, \cdots, n_k]$, we introduce a sequence of integers $m_1, m_2, \cdots, m_g, m_{g+1}$ as follows

$$\begin{aligned} m_i &= \#\{n_1, n_2, \cdots, n_k\} \cap \{2i-1, 2i\}, \text{ when } 1 \leq i \leq g, \\ &= \#\{n_1, n_2, \cdots, n_k\} \cap \{2g+1\}, \text{ when } i = g+1. \end{aligned}$$

We define $\Phi([n_1, n_2, \cdots, n_k]) = [[m_1, \cdots, m_{g+1}]]$, and call $[[m_1, \cdots, m_{g+1}]]$ a *block sequence*. By applying Lemma 4.2 to C_1, C_3, C_5, $\cdots$, $C_{2g+1} \in \mathcal{H}_g$, we show the following

Lemma 4.4. *If odd chain maps och_1, och_2 satisfy $\Phi(och_1) = \Phi(och_2)$, then $och_1 \sim_{\mathcal{H}_g} och_2$.*

For any $[[m_1, m_2, \cdots, m_{g+1}]]$ so that $m_i = 0, 1, 2$, for $1 \leq i \leq g$ and $m_{g+1} = 0, 1$, there are odd chain maps och such that $\Phi(och) = [[m_1, m_2, \cdots, m_{g+1}]]$ and we call these och *representatives* of $[[m_1, m_2, \cdots, m_{g+1}]]$. Lemma 4.4 says that, for any $[[m_1, m_2, \cdots, m_{g+1}]]$ satisfying the above condition, any pair of representatives of it are $\mathcal{H}_g$-equivalent. We show

Lemma 4.5.

(a) *Odd chain maps och_1 and och_2 such that $\Phi(och_1) = [[\bullet\bullet\bullet, 0, 2, \circ\circ\circ]]$ and $\Phi(och_2) = [[\bullet\bullet\bullet, 2, 0, \circ\circ\circ]]$, where the parts indicated by $\bullet\bullet\bullet$ and $\circ\circ\circ$ are equal to each other, are $\mathcal{H}_g$-equivalent.*

(b) *Odd chain maps och_1 and och_2 such that $\Phi(och_1) = [[\bullet\bullet\bullet, 1, 2, \circ\circ\circ]]$ and $\Phi(och_2) = [[\bullet\bullet\bullet, 2, 1, \circ\circ\circ]]$, where the parts indicated by $\bullet\bullet\bullet$ and $\circ\circ\circ$ are equal to each other, are $\mathcal{H}_g$-equivalent.*

(c) *Odd chain maps och_1 and och_2 such that $\Phi(och_1) = [[\bullet\bullet\bullet, 0, 1, \circ\circ\circ]]$ and $\Phi(och_2) = [[\bullet\bullet\bullet, 1, 0, \circ\circ\circ]]$, where the parts indicated by $\bullet\bullet\bullet$ and $\circ\circ\circ$ are equal to each other, are $\mathcal{H}_g$-equivalent.*

Proof. For each $k = 1, \cdots, g$, we can show that $C_{2k}C_{2k-1}C_{2k+1}C_{2k}$ are elements of $\mathcal{H}_g$ by checking their actions on the fundamental group of Σ_g.

(a). As a representative of $[[\cdots, \overset{k}{0}, \overset{k+1}{2}, \cdots]]$, we take $[\cdots, i, 2k+1, 2k+2, \cdots]$, where $i \leq 2k-2$. By Lemma 4.2, $(C_{2k}C_{2k-1}C_{2k+1}C_{2k}) * [\cdots, i, 2k+1, 2k+2, \cdots] = [\cdots, i, 2k-1, 2k, \cdots]$. The odd chain map $[\cdots, i, 2k-1, 2k, \cdots]$ is a representative of $[[\cdots, \overset{k}{2}, \overset{k+1}{0}, \cdots]]$.

(b). As a representative of $[[\cdots, \overset{k}{1}, \overset{k+1}{2}, \cdots]]$, we take $[\cdots, i, 2k-1, 2k+1, 2k+2, \cdots]$, where $i \leq 2k-2$. By Lemma 4.2, $(C_{2k}C_{2k-1}C_{2k+1}C_{2k}) * [\cdots, i, 2k-1, 2k+1, 2k+2, \cdots] = [\cdots, i, 2k-1, 2k, 2k+1, \cdots]$. The odd chain map $[\cdots, i, 2k-1, 2k, 2k+1, \cdots]$ is a representative of $[[\cdots, \overset{k}{2}, \overset{k+1}{1}, \cdots]]$.

(c). As a representative of $[[\cdots, \overset{k}{0}, \overset{k+1}{1}, \cdots]]$, we take $[\cdots, i, 2k+2, \cdots]$, where $i \leq 2k-2$. By Lemma 4.2, $(C_{2k}C_{2k-1}C_{2k+1}C_{2k}) * [\cdots, i, 2k+2, \cdots] = [\cdots, i, 2k, \cdots]$. The odd chain map $[\cdots, i, 2k, \cdots]$ is a representative of $[[\cdots, \overset{k}{1}, \overset{k+1}{0}, \cdots]]$. □

By this lemma, we show that any odd chain map is $\mathcal{H}_g$-equivalent to a odd chain map och such that $\Phi(och) = [[2, \cdots, 2, 1, \cdots, 1, 0, \cdots, 0]]$.

Lemma 4.6. *Any odd subchain map representing a block sequence* $[[2,\cdots,2,1,\cdots,1,0,\cdots,0]]$ *is a product of* $B_4 \in \mathcal{H}_g$ *and odd chain maps representing* $[[2,2,0,\cdots,0]]$, $[[1,\cdots,1,0,\cdots,0]]$ *and* $[[2,1,\cdots,1,0,\cdots,0]]$.

Proof. For a block sequence $[[2,\cdots,2,1,\cdots,1,0,\cdots,0]]$, the length of sequence of 2's is called *h-length* of the block sequence. By Lemma 4.3, we see,

$$[[2,2,0,\cdots,0]]\ [[2,0,\bullet\bullet\bullet]]\ B_4 * [[0,2,\bullet\bullet\bullet]] = [[0,0,\bullet\bullet\bullet]]\ [[2,2,\bullet\bullet\bullet]],$$

where $\bullet\bullet\bullet$ indicate the part which are the same each other. This equation shows that, if h-length of a block sequence is at least two, then this is a product of B_4, block sequences whose h-length are shorter than this, and $[[2,2,0,\cdots,0]]$. By the induction on h-length, we show this lemma. □

The block sequence $[[2,2,0,\cdots,0]]$ is represented by $[1,2,3,4] = T_{b_4}\overline{T_{b'_4}}$, which is an element of $\mathcal{H}_g$. As a representative of $[[1,\cdots,\overset{2j+2}{1},0,\cdots,0]]$, we take $[2,3,6,7,\cdots,4l-2,4l-1,\cdots,4j+2,4j+3] = T_{d_j}\overline{T_{d'_j}}$, and as a representative of $[[2,1,\cdots,\overset{2k+3}{1},0,\cdots,0]]$, we take $[1,2,4,5,\cdots,4l,4l+1,\cdots,4k+4,4k+5] = T_{e_k}\overline{T_{e'_k}}$. Therefore, $\mathcal{H}_g \cup \{T_{d_j}\overline{T_{d'_j}}, T_{e_k}\overline{T_{e'_k}} \mid 1 \leq j \leq [\frac{g-1}{2}],\ 1 \leq k \leq [\frac{g-2}{2}]\}$ generates $\mathcal{HH}_g$. Takahashi [7] showed

Theorem 4.7 ([7]). *$\mathcal{H}_g$ is generated by* C_1, $C_2C_1^2C_2$, $C_2C_1C_3C_2$, $C_{2i}C_{2i-1}B_{2i}C_{2i}$, $C_{2i}C_{2i+1}B_{2i}C_{2i}$ $(2 \leq i \leq g-1)$, $C_{2g}C_{2g-1}C_{2g+1}C_{2g}$.

Theorem 2.4 follows from this theorem and the above observation.

Remark 4.8. McCullough and Miller [5] showed that $\mathcal{I}_2$ is not finitely generated. So, the author does not know whether $\mathcal{HH}_2$ is finitely generated or not.

Acknowledgements. A part of this work was done while the author stayed at Michigan State University as a visiting scholar sponsored by the Japanese Ministry of Education, Culture, Sports, Science and Technology. He is grateful to the Department of Mathematics, Michigan State University, especially to Professor Nikolai V. Ivanov, for their hospitality. The author would also like to thank the referee, whose comments improved the paper.

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