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Best Approximation Result in Locally Convex Space

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Abstract. A fixed point theorem of Singh and Singh [10] is generalized to locally convex spaces and the new result is applied to extend a result on invariant approximation of Jungck and Sessa [5].

1. Introduction

During the last four decades several interesting and valuable results as application of fixed point theorems were studied extensively in the field of best approximation theory. An excellent reference can be seen in [14].

In 1963, Meinardus [7] was the first who observed the general principle and employed a fixed point theorem to establish the existence of an invariant approximation. Afterwards in 1969, Brosowski [1] obtained the following generalization of Meinardus's result.

Theorem 1.1. *Let X be a normed space and $T : X \to X$ be a linear and nonexpansive operator. Let M be a $T-$invariant subset of X and $x_0 \in F(T)$. If D, the set of best approximations of x_0 in M, is nonempty compact and convex, then there exists a y in D which is also a fixed point of T.*

Using a fixed point theorem, Subrahmanyam [15] obtained the following generalization of the above mentioned theorem of Meinardus [7].

Theorem 1.2. *Let X be a normed space. If $T : X \to X$ is a nonexpansive operator with a fixed point x_0, leaving a finite dimensional subspace M of X invariant, then there exists a best approximation of x_0 in M which is also a fixed point of T.*

In 1979, Singh [11] observed that the linearity of mapping T and the convexity of the set D of best approximation of x_0 in Theorem 1.1, can be relaxed and proved the following extension of it.

Theorem 1.3. *Let X be a normed space, $T : X \to X$ be a nonexpansive mapping, M be a $T-$invariant subset of X and $x_0 \in F(T)$. If D is nonempty compact and*

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starshaped, then there exists a best approximation of x_0 in M which is also a fixed point of T.

In a subsequent paper, Singh [12] also observed that only the nonexpansiveness of T on $D' = D \cup \{x_0\}$ is necessary for the validity of Theorem 1.3. Further in 1982, Hicks and Humpheries [3] had shown that Theorem 1.3 remain true, if $T : M \mapsto M$ is replaced by $T : \partial M \mapsto M$, where ∂M, denotes the boundary of M. Furthermore, Sahab, Khan and Sessa [9] generalized the result of Hicks and Humpheries [3] and Theorem 1.3 using two mappings, one linear and other nonexpansive for commuting mappings and established the following result of common fixed point for best approximation in setup of normed linear space. They took this idea from Park [8].

Theorem 1.4. *Let I and T be self maps of X with $x_0 \in F(I) \cap F(T)$, $M \subset X$ with $T : \partial M \mapsto M$, and $p \in F(I)$. If D, the set of best approximation is compact and p-starshaped, $I(D) = D$, I is continuous and linear on D, I and T are commuting on D and T is I-nonexpansive on $D \cup \{x_0\}$, then I and T have a common fixed point in D.*

In another paper, Jungck and Sessa [5] further weakened the hypothesis of Sahab, Khan and Sessa [9] by replacing the condition of linearity by affineness to prove the existence of best approximation in normed linear space. However, they used weak continuity of the mapping for such purpose in the second result. For this, they used the result due to Jungck [4].

In this paper, we first derive a common fixed point result in locally convex space which generalize the result of Singh and Singh [10] which was generalization of Jungck [4]. This new result is used to prove another fixed point result for best approximation. By doing so, we infact, extend and improve the result of Jungck and Sessa [5] for generalized contraction mappings. For this purpose, we use the concept given by Köthe [6] and Tarafdar [16]. In this way, we give new direction to the line of investigation given by Brosowski [1].

2. Preliminaries

To prove our results, we need the following:

Definition 2.1 ([6]). In the sequel (E, τ) will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is said to be an associated family of seminorms for τ if the family $\{\gamma U : \gamma > 0\}$, where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighbourhood of zero for τ. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family with the property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that if given a locally convex space (E, τ), there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined of E such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$. A subset M of E is τ-bounded if and only if each p_α is bounded on M.

The following construction will be crucial. Suppose that M is a τ-bounded subset of E. For this set M, we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$ where $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Clearly, $B = \bigcap_\alpha \lambda_\alpha U_\alpha$ is τ-bounded, τ-closed, absolutely convex and contains M. The linear span E_B of B in E is $\bigcup_{n=1}^{\infty} nB$. The Minkowski functional of B is a norm $\|.\|_B$ on E_B. Thus, $(E_B, \|.\|_B)$ is a normed space with B as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$.

Definition 2.2. Let I and T be self maps on M. The map T is called

(i) $A^*(\tau)$-nonexpansive if for all $x, y \in M$

$$p_\alpha(Tx - Ty) \leq p_\alpha(x - y),$$

for each $p_\alpha \in A^*(\tau)$.

(ii) $A^*(\tau)$-$I-$nonexpansive if for all $x, y \in M$

$$p_\alpha(Tx - Ty) \leq p_\alpha(Ix - Iy),$$

for each $p_\alpha \in A^*(\tau)$.

For simplicity, we shall call $A^*(\tau)$-nonexpansive ($A^*(\tau)-I$- nonexpansive) maps to be nonexpansive ($I-$nonexpansive).

Definition 2.3. Let $x_0 \in M$. We denote by $P_M(x_0)$ the set of best $M-$approximation to x_0, i.e., if $P_M(x_0) = \{y \in M : p_\alpha(y - x_0) = d_{p_\alpha}(x_0, M)$ for all $p_\alpha \in A^*(\tau)\}$, where

$$d_{p_\alpha}(x_0, M) = \inf\{p_\alpha(x_0 - z) : z \in M\}.$$

Definition 2.4. The map $T : M \to E$ is said to be demiclosed at 0 if for every net $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$.

Throughout, this paper $F(T)$(resp. $F(I)$) denotes the fixed point set of mapping T(resp. $F(I)$).

We also use the following result due to Singh and Singh [10]:

Theorem 2.5 ([10]). *Let T, S and I be self maps of a complete metric space (X, d) such that $T(X) \cup S(X) \subseteq I(X), TI = IT, SI = IS$; I is continuous, and satisfying*

$$d(Tx, Sy) \leq N(x, y),$$

where

$$N(x, y) = h \max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Sy), \frac{1}{2}[d(Ix, Sy) + d(Iy, Tx)]\},$$

for all $x, y \in X$, where $h \in (0,1)$, then T, S and I have a unique common fixed point in X.

3. Main result

We use a technique of Tarafdar [16] to obtain the following common fixed point theorem which generalize Theorem 2.5.

Theorem 3.1. *Let M be a nonempty τ-bounded, τ-sequentially complete subset of a Hausdorff locally convex space (E,τ). Let T, S and I be self maps of M such that $T(X) \cup S(X) \subseteq I(X), TI = IT, SI = IS$; I is nonexpansive, and satisfying*

$$p_\alpha(Tx - Sy) \leq N(x,y), \tag{3.1}$$

where

$$\begin{aligned} N(x,y) \quad = \quad & h \max\{p_\alpha(Ix - Iy),\ p_\alpha(Ix - Tx),\ p_\alpha(Iy - Sy), \\ & \frac{1}{2}[p_\alpha(Ix - Sy) + p_\alpha(Iy - Tx)]\} \end{aligned}$$

for all $x, y \in M$ and $p_\alpha \in A^*(\tau)$, where $h \in (0,1)$, then T, S and I have a unique common fixed point in M.

Proof. Since the norm topology on E_B has a base of neighbourhood of zero consisting of τ-closed sets and M is τ-sequentially complete, therefore, M is a $\|.\|_B$-sequentially complete subset of $(E_B, \|.\|_B)$ (Theorem 1.2, [16]). From (3.1) we obtain for $x, y \in M$,

$$\begin{aligned} \sup_\alpha p_\alpha\left(\frac{Tx - Sy}{\lambda_\alpha}\right) \quad \leq \quad & h \max\{\sup_\alpha p_\alpha\left(\frac{Ix - Iy}{\lambda_\alpha}\right), \sup_\alpha p_\alpha\left(\frac{Ix - Tx}{\lambda_\alpha}\right), \sup_\alpha p_\alpha\left(\frac{Iy - Sy}{\lambda_\alpha}\right), \\ & \frac{1}{2}[\sup_\alpha p_\alpha\left(\frac{Ix - Sy}{\lambda_\alpha}\right) + \sup_\alpha p_\alpha\left(\frac{Iy - Tx}{\lambda_\alpha}\right)]\}. \end{aligned}$$

Thus

$$\begin{aligned} \|Tx - Sy\|_B \quad \leq \quad & h \max\{\|Ix - Iy\|_B,\ \|Ix - Tx\|_B,\ \|Iy - Ty\|_B, \\ & \frac{1}{2}[\ \|Ix - Sy\|_B + \|Iy - Tx\|_B]\}. \end{aligned} \tag{3.2}$$

Note that, if I is nonexpansive on a τ-bounded, τ-sequentially complete subset M of E, then I is also nonexpansive with respect to $\|.\|_B$ and hence $\|.\|_B$-continuous ([6]). A comparison of our hypothesis with that of Theorem 2.5 tells that we can apply Theorem 2.5 to M as a subset of $(E_B, \|.\|_B)$ to conclude that there exists a unique $v \in M$ such that $v = Tv = Sv = Iv$. $\square$

Theorem 3.2. *Let M be a nonempty τ-bounded, τ-sequentially complete and $q-$starshaped subset of a Hausdorff locally convex space (E,τ). Let T, S and I be* *self-maps of M such that $TI = IT$, $SI = IS$ on M. Suppose that T, S are continuous, I is nonexpansive and affine, $I(M) = M$, $p \in F(I)$. If T, S and I satisfy the following*:

$$p_\alpha(Tx - Sy) \leq N(x, y), \tag{3.3}$$

where

$$N(x,y) = h\max\{p_\alpha(Ix-Iy), p_\alpha(Ix-Tx), p_\alpha(Iy-Sy), \frac{1}{2}[p_\alpha(Ix-Sy)+p_\alpha(Iy-Tx)]\}$$

for all $x, y \in M$ and $p_\alpha \in A^*(\tau)$, where $h \in (0,1)$, then T, S and I have a common fixed point provided one of the following conditions hold:

(i) *M is τ-sequentially compact*;

(ii) *T, S is a compact map*;

(iii) *M is weakly compact in (E, τ), I is weakly continuous and $I - T$ and $I - S$ are demiclosed at* 0.

Proof. Choose a monotonically nondecreasing sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $\limsup k_n = 1$. For each $n \in \mathbb{N}$, define $T_n, S_n : M \to M$ as follows:

$$T_n x = k_n T x + (1 - k_n)p, \quad S_n x = k_n S x + (1 - k_n)p. \tag{3.4}$$

Obviously, for each n, T_n and S_n map M into itself since M is $q-$starshaped.

As I is affine, I commutes with T and $p \in F(I)$, so

$$\begin{aligned} T_n I x &= k_n T I x + (1 - k_n)p \\ &= k_n I T x + (1 - k_n) I p \\ &= I(k_n T x + (1 - k_n)p) \\ &= I T_n x \end{aligned}$$

for each $x \in M$. Thus T_n and I are commutative for each n and $T_n(M) \subseteq M = I(M)$. Similarly, we can prove S_n and I are commutative for each n and $S_n(M) \subseteq M = I(M)$. Therefore $T_n(M) \cup S_n(M) \subseteq I(M)$.

For all $x, y \in M, p_\alpha \in A^*(\tau)$ and for all $j \geq n$, (n fixed), we obtain from (3.4) and (3.3) that

$$\begin{aligned} p_\alpha(T_n x - S_n y) &= k_n p_\alpha(Tx - Sy) \leq k_j p_\alpha(Tx - Sy) \\ &\leq p_\alpha(Tx - Sy) \\ &\leq h\max\{p_\alpha(Ix - Iy), p_\alpha(Ix - Tx), p_\alpha(Iy - Sy), \\ &\quad \frac{1}{2}[p_\alpha(Ix - Sy) + p_\alpha(Iy - Tx)]\} \end{aligned}$$

$$\begin{aligned}
&\leq h\max\{p_\alpha(Ix-Iy), p_\alpha(Ix-T_nx)+p_\alpha(T_nx-Tx),\\
&\quad p_\alpha(Iy-S_ny)+p_\alpha(S_ny-Sy),\\
&\quad \frac{1}{2}[p_\alpha(Ix-S_ny)+p_\alpha(S_ny-Sy)+p_\alpha(Iy-T_nx)+p_\alpha(T_nx-Tx)]\}\\
&\leq h\max\{p_\alpha(Ix-Iy), p_\alpha(Ix-T_nx)\\
&\quad +(1-k_n)p_\alpha(p-Tx), p_\alpha(Iy-S_ny)\\
&\quad +(1-k_n)p_\alpha(p-Sy), \frac{1}{2}[p_\alpha(Ix-S_ny)\\
&\quad +(1-k_n)p_\alpha(p-Sy)+p_\alpha(Iy-T_nx)+(1-k_n)p_\alpha(p-Tx)]\}.
\end{aligned}$$

Hence for all $j \geq n$, we have

$$\begin{aligned}
p_\alpha(T_nx-S_ny) \leq\ & h\max\{p_\alpha(Ix-Iy), p_\alpha(Ix-T_nx)\\
&+(1-k_j)p_\alpha(p-Tx), p_\alpha(Iy-S_ny)\\
&+(1-k_j)p_\alpha(p-Sy), \frac{1}{2}[p_\alpha(Ix-S_ny)\\
&+(1-k_j)p_\alpha(p-Sy)+p_\alpha(Iy-T_nx)\\
&+(1-k_j)p_\alpha(p-Tx)]\}.
\end{aligned}\tag{3.5}$$

As $\lim k_j = 1$, from (3.5), for every $n \in N$, we have

$$\begin{aligned}
p_\alpha(T_nx-S_ny) =\ & \lim_j p_\alpha(T_nx-T_ny)\\
\leq\ & \lim_j\{h\max\{p_\alpha(Ix-Iy), p_\alpha(Ix-T_nx)\\
&+(1-k_j)p_\alpha(p-Tx), p_\alpha(Iy-S_ny)\\
&+(1-k_j)p_\alpha(p-Sy), \frac{1}{2}[p_\alpha(Ix-S_ny)\\
&+(1-k_j)p_\alpha(p-Sy)+p_\alpha(Iy-T_nx)\\
&+(1-k_j)p_\alpha(p-Tx)]\}.
\end{aligned}\tag{3.6}$$

This implies that for every $n \in N$,

$$\begin{aligned}
p_\alpha(T_nx-S_ny) \leq\ & h\max\{p_\alpha(Ix-Iy),\ p_\alpha(Ix-T_nx),\ p_\alpha(Iy-S_ny),\\
&\frac{1}{2}[p_\alpha(Ix-S_ny)+p_\alpha(Iy-T_nx)]\},
\end{aligned}\tag{3.7}$$

for all $x, y \in M$, for all $p_\alpha \in A^*(\tau)$ and $0 < h < 1$.

Moreover, I being nonexpansive on M, implies that I is $\|.\|_B$-nonexpansive and, hence, $\|.\|_B$-continuous. Since the norm topology on E_B has a base of neighbourhood of zero consisting of τ-closed sets and M is τ-sequentially complete, therefore, M is a $\|.\|_B$-sequentially complete subset of $(E_B, \|.\|_B)$ (see proof of Theorem 1.2 in [16]). Thus from Theorem 3.1, for every $n \in N$, T_n, S_n and I have unique common fixed point x_n in M, i.e.,

$$x_n = T_nx_n = S_nx_n = Ix_n, \tag{3.8}$$

for each $n \in N$.

(i) As M is τ-sequentially compact and $\{x_n\}$ is a sequence in M, so $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $x_m \to y \in M$. As I and S, T are continuous and

$$\begin{aligned} x_m &= Ix_m = T_m x_m = k_m T x_m + (1-k_m)p, \\ x_m &= S_m x_m = k_m S x_m + (1-k_m)p, \end{aligned}$$

so it follows that $y = Ty = Sy = Iy$.

(ii) As T is compact and $\{x_n\}$ is bounded, so $\{Tx_n\}$ has a subsequence $\{Tx_m\}$ such that $\{Tx_m\} \to z \in M$. Now we have

$$x_m = T_m x_m = k_m T x_m + (1-k_m)p$$

Proceeding to the limit as $m \to \infty$ and using the continuity of I and T, we have $Iz = z = Tz$. Similarly, we can show $Sz = z$.

(iii) The sequence $\{x_n\}$ has a subsequence $\{x_m\}$ converges to $u \in M$. Since I is weakly continuous and so as in (i), we have $Iu = u$. Now,

$$x_m = Ix_m = T_m x_m = k_m T x_m + (1-k_m)p$$

implies that

$$Ix_m - Tx_m = (1-k_m)[p - Tx_m] \to 0$$

as $m \to \infty$. The demiclosedness of $I - T$ at 0 implies that $(I-T)u = 0$. Hence $Iu = u = Tu$. Similarly, we can show $Su = u = Iu$, when $I - S$ is demiclosed at 0. This completes the proof. □

An immediate consequence of the above theorem is as follows:

Corollary 3.3. *Let M be a nonempty τ-bounded, τ-sequentially complete and $q-$starshaped subset of a Hausdorff locally convex space (E, τ). Let T, S and I be self-maps of M such that $TI = IT, SI = IS$ on M. Suppose that T, S are continuous, I is nonexpansive and affine, $I(M) = M$, $p \in F(I)$. If T, S and I satisfy the following:*

$$p_\alpha(Tx - Sy) \le N(x, y), \tag{3.9}$$

where

$$\begin{aligned} N(x,y) &= h \max\{p_\alpha(Ix - Iy), p_\alpha(Ix - Tx), p_\alpha(Iy - Sy), \frac{1}{2}p_\alpha(Ix - Sy), \\ &\quad \frac{1}{2}p_\alpha(Iy - Tx)\} \end{aligned}$$

for all $x, y \in M$ and $p_\alpha \in A^*(\tau)$, where $h \in (0,1)$, then T, S and I have a common fixed point in M under each of the conditions (i)-(iii) of Theorem 3.2.

An application of Theorem 3.2, we prove the following more general result in best approximation theory.

Theorem 3.4. *Let T, S and I be self maps of a Hausdorff locally convex space (E, τ) and M a subset of E such that $T, S(\partial M) \subseteq M$, where ∂M stands for the boundary of M and $x_0 \in F(T) \cap F(S) \cap F(I)$. Suppose that T, S are continuous, I is nonexpansive and affine on $D = P_M(x_0)$. Further, suppose T, S and I satisfy* (3.3) *for each $x, y \in D, p_\alpha \in A^*(\tau)$ and $0 < h < 1$. If D is nonempty $q-$starshaped with $p \in F(I)$ and $I(D) = D$, then T, S and I have a common fixed point in D provided one of the following conditions hold:*

(i) *D is τ-sequentially compact;*

(ii) *T, S is a compact map;*

(iii) *D is weakly compact in (E, τ), I is weakly continuous and $I - T$ and $I - S$ are demiclosed at* 0.

Proof. First, we show that T and S are self map on D, i.e.,$T, S : D \mapsto D$. Let $y \in D$, then $Iy \in D$, since $I(D) = D$. Also, if $y \in \partial M$, then $Ty \in M$, since $T(\partial M) \subseteq M$. Now since $Tx_0 = Sx_0 = x_0 = Ix_0$, so for each $p_\alpha \in A^*(\tau)$, we have from (3.3)

$$p_\alpha(Ty - x_0) = p_\alpha(Ty - Sx_0) \leq N(y, x_0).$$

Now,$Ty \in M$ and $Iy \in D$, this imply that Ty is also closest to x_0, so $Ty \in D$. Similarly $Sy \in D$. Consequently T, S and I are self maps on D. The conditions of Theorem 3.2 ((i)-(iii)) are satisfied and, hence, there exists a $w \in D$ such that $Tw = Sw = w = Iw$. This completes the proof. □

An immediate consequence of the above theorem is as follows:

Corollary 3.5. *Let T, S and I be self maps of a Hausdorff locally convex space (E, τ) and M a subset of E such that $T, S(\partial M) \subseteq M$, where ∂M stands for the boundary of M and $x_0 \in F(T) \cap F(S) \cap F(I)$. Suppose that T, S are continuous, I is nonexpansive and affine on $D = P_M(x_0)$. Further, suppose T, S and I satisfy* (3.9) *for each $x, y \in D, p_\alpha \in A^*(\tau)$ and $0 < h < 1$. If D is nonempty q-starshaped with $p \in F(I)$ and $I(D) = D$, then T, S and I have a common fixed point in D under each of the conditions $(i) - (iii)$ of Theorem 3.4.*

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