KYUNGPOOK Math. J. 46(2006), 377-387

L^p Boundedness for Singular Integral Operators with $L(\log^+ L)^2$ Kernels on Product Spaces

Hussain Al-Qassem and Mohammed Ali
Department of Mathematics, Yarmouk University, Irbid-Jordan
e-mail : husseink@yu.edu.jo *and* hamadneh2004@hotmail.com

Abstract. In this paper, we study the L^p mapping properties of singular integral operators related to homogeneous mappings on product spaces with kernels which belong to $L(\log^+ L)^2$. Our results extend as well as improve some known results on singular integrals.

1. Introduction

Let $n, m \geq 2$ and let $\mathbf{S}^{d-1}(d = n \text{ or } m)$ denote the unit sphere in $\mathbf{R}^d$ which is equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. For a nonzero point $x \in \mathbf{R}^d$, we let $x' = x/|x|$. Let $K_\Omega(\cdot, \cdot)$ be the singular kernel on $\mathbf{R}^n \times \mathbf{R}^m$ given by

$$K_\Omega(u, v) = \Omega(u', v')|u|^{-n}|v|^{-m}, \tag{1.1}$$

where $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies the cancellation conditions

$$\int_{\mathbf{S}^{n-1}} \Omega(u, \cdot)\, d\sigma(u) = \int_{\mathbf{S}^{m-1}} \Omega(\cdot, v)\, d\sigma(v) = 0. \tag{1.2}$$

For suitable mappings $\Gamma : \mathbf{R}^n \longrightarrow \mathbf{R}^N$ and $\Upsilon : \mathbf{R}^m \longrightarrow \mathbf{R}^M$, define the singular integral operator $T_{\Gamma,\Upsilon,\Omega}$ and its related maximal truncated operator $T^*_{\Gamma,\Upsilon,\Omega}$ on the product space $\mathbf{R}^n \times \mathbf{R}^m$ by

$$T_{\Gamma,\Upsilon,\Omega}f(x, y) = \text{p.v.}\int_{\mathbf{R}^n\times\mathbf{R}^m} f(x - \Gamma(u), y - \Upsilon(v))\, K_\Omega(u, v)\, du dv, \tag{1.3}$$

$$\begin{aligned} &T^*_{\Gamma,\Upsilon,\Omega}f(x, y) \\ &= \sup_{\varepsilon_1,\varepsilon_2>0}\left|\int_{\{|u|\geq\varepsilon_1,|v|\geq\varepsilon_2\}} f(x - \Gamma(u), y - \Upsilon(v))\, K_\Omega(u, v)\, du dv\right| \end{aligned} \tag{1.4}$$

Received March 3, 2005.
2000 Mathematics Subject Classification: 42B20, 42B15, 42B25.
Key words and phrases: singular integrals, oscillatory integrals, Fourier transform, product spaces, rough kernels.

for $f \in \mathcal{S}(\mathbf{R}^N \times \mathbf{R}^M)$.

When $(N, M) = (n, m)$ and $(\Gamma(x), \Upsilon(y)) = (x, y)$ for $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$, the operators $T_\Omega = T_{\Gamma,\Upsilon,\Omega}$ and $T^*_\Omega = T^*_{\Gamma,\Upsilon,\Omega}$ become the classical Calderón-Zygmund singular integral operator and its corresponding maximal truncated operator on the product space $\mathbf{R}^n \times \mathbf{R}^m$:

$$\begin{aligned} T_\Omega \quad &: \quad f \to \text{p.v.} \int_{\mathbf{R}^n \times \mathbf{R}^m} f(x-u, y-v) K_\Omega(u,v)\, dudv, \\ T^*_\Omega \quad &: \quad f \to \sup_{\varepsilon_1, \varepsilon_2 > 0} \left| \int_{\{|u| \geq \varepsilon_1, |v| \geq \varepsilon_2\}} f(x-u, y-v) K_\Omega(u,v)\, dudv \right|. \end{aligned}$$

The L^p boundedness of the operators T_Ω and T^*_Ω, under various conditions on Ω, has been investigated by many authors ([3], [7], [9], [11], [12]). For example, R. Fefferman and E. Stein proved in [12] that T_Ω and T^*_Ω are bounded on $L^p(\mathbf{R}^{n+m})$ for $1 < p < \infty$ if Ω satisfies certain Lipschitz conditions. Subsequently in [7], Duoandikoetxea established the L^p $(1 < p < \infty)$ boundedness of T_Ω under the weaker condition $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ (with $q > 1$) and then in Fan-Guo-Pan [9] for Ω belongs to the block space $B_q^{(0,1)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$ which contains $\bigcup_{d>1} L^d(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ as a proper subspace (for $p = 2$, it was proved by Jiang and Lu in [13]). In [3], Al-Qassem and Pan established the L^p $(1 < p < \infty)$ boundedness of the more general class of operators $T_{\Gamma,\Upsilon,\Omega}$ and $T^*_{\Gamma,\Upsilon,\Omega}$ if $\Omega \in B_q^{(0,1)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$ and Γ, Υ are polynomial mappings on $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively.

Very recently, Al-Salman-Al-Qassem-Pan [2] were able to show that the L^p $(1 < p < \infty)$ boundedness of T_Ω and T^*_Ω holds if $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Furthermore, the condition that $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ turns out to be the most desirable size condition for the L^p boundedness of T_Ω. This was made clear by the authors of [2], where it was shown that T_Ω may fail to be bounded on L^p for any p if the condition is replaced by the condition $\Omega \in L(\log^+ L)^{2-\varepsilon}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for any $\varepsilon > 0$.

In order to state our main result, we first give the following definition.

Definition. For $d = (d_1, \cdots, d_l) \in \mathbf{R}^l$, define the family of dilations $\{\delta_t\}_{t>0}$ on $\mathbf{R}^l$ by

$$\delta_t(x_1, \cdots, x_l) = \left(t^{d_1} x_1, \cdots,\ t^{d_l} x_l\right).$$

We say that $\Gamma : \mathbf{R}^n \longrightarrow \mathbf{R}^l$ is a (non-isotropic) homogeneous mapping of degree d if

$$\Gamma(tx) = \delta_t(\Gamma(x))$$

holds for all $x \in \mathbf{R}^n \backslash \{0\}$ and $t > 0$.

The main result in this paper is the following:

Theorem 1.1. *Let* $T_{\Gamma,\Upsilon,\Omega}$ *and* $T^*_{\Gamma,\Upsilon,\Omega}$ *given by* (1.3)-(1.4)*, respectively. Suppose that* $\Omega \in L(\log^+ L)^2 \left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ *and satisfies* (1.2)*. Let* $\Gamma : \mathbf{R}^n \longrightarrow \mathbf{R}^N$, $\Upsilon : \mathbf{R}^m \longrightarrow \mathbf{R}^M$ *be homogeneous mappings of degrees* $d = (d_1, \cdots, d_N)$ *and*

$h = (h_1, \cdots, h_M)$, respectively with d_l, $h_r \neq 0$ for $1 \leq l \leq N$ and $1 \leq r \leq M$. Assume that $\Gamma \mid \mathbf{S}^{n-1}$ and $\Upsilon \mid \mathbf{S}^{m-1}$ are real-analytic mappings. Then there exists a positive constant $C_p > 0$ such that

$$\|T_{\Gamma,\Upsilon,\Omega}(f)\|_p \leq C_p \|f\|_p \,, \tag{1.5}$$

and

$$\left\|T^*_{\Gamma,\Upsilon,\Omega}(f)\right\|_p \leq C_p \|f\|_p \tag{1.6}$$

for any $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$ with $1 < p < \infty$.

We point out that Theorem 1.1 extends and improves the corresponding results in [12] and [7]. Also, we point out that the one parameter case of Theorem 1.1 was studied by many authors (see for example [10], [6], [1]). The paper is organized as follows. In Section 2, a few lemmas will be recalled or proved. The proof of Theorem 1.1 can be found in Section 3.

Throughout this paper, the letter C will denote a positive constant whose value may change at each appearance but independent of the essential variables.

2. Preliminary results

Definition 2.1. For $\mu \in \mathbf{N} \cup \{0\}$ and $k \in \mathbf{Z}$, let $a_\mu = 2^{(\mu+1)}$ and $I_{k,\mu} = [a_\mu^k, a_\mu^{k+1})$. For suitable mappings $\Gamma : \mathbf{R}^n \backslash \{0\} \longrightarrow \mathbf{R}^N$ and $\Upsilon : \mathbf{R}^m \backslash \{0\} \longrightarrow \mathbf{R}^M$ and a suitable function $\Omega_\mu \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, we define the sequence of measures $\{\lambda_{k,j,\Gamma,\Upsilon,\mu} : k, j \in \mathbf{Z}\}$ and its corresponding maximal operator $\lambda^*_{\Gamma,\Upsilon,\mu}$ by

$$\begin{aligned} &\int_{\mathbf{R}^N \times \mathbf{R}^M} f \, d\lambda_{k,j,\Gamma,\Upsilon,\mu} \\ = \; &\int_{\{(u,v) \in \mathbf{R}^n \times \mathbf{R}^m : \, (|u|,|v|) \in I_{k,\mu} \times I_{j,\mu}\}} f(\Gamma(u), \Upsilon(v)) \, K_{\Omega_\mu}(u,v) \, dv du, \end{aligned}$$

$$\lambda^*_{\Gamma,\Upsilon,\mu}(f) = \sup_{k,j \in \mathbf{Z}} \left| |\lambda_{k,j,\Gamma,\Upsilon,\mu}| * f \right|.$$

We shall need the following two lemmas due to Ricci and Stein.

Lemma 2.2 ([15]). *Let $\gamma(t) = (a_1 t^{q_1}, \cdots, a_n t^{q_n})$, where a_l, $q_l \in \mathbf{R}$ for $1 \leq l \leq n$. Let $\mathcal{M}_\gamma$ be the maximal operator defined on $\mathbf{R}^n$ by*

$$\mathcal{M}_\gamma f(x) = \sup_{R>0} \frac{1}{R} \left| \int_0^R f(x - \gamma(t)) \, dt \right|$$

for $x \in \mathbf{R}^n$. Then, for $1 < p \leq \infty$, there exists a constant $C_p > 0$ such that

$$\left\|\mathcal{M}_\gamma f\right\|_p \leq C_p \|f\|_p$$

for all f in $L^p(\mathbf{R}^n)$. The constant C_p is independent of a_l for all $1 \leq l \leq n$.

Lemma 2.3. *Let $\gamma(t) = (a_1 t^{q_1}, \cdots, a_n t^{q_n})$, $\vartheta(s) = (b_1 s^{r_1}, \cdots, b_m s^{r_m})$, where a_l, q_l, b_s and $r_s \in \mathbf{R}$ for $1 \leq l \leq n$ and $1 \leq s \leq m$. Let $\mathcal{M}_{\gamma,\vartheta}$ be the maximal operator defined on $\mathbf{R}^n \times \mathbf{R}^m$ by*

$$\mathcal{M}_{\gamma,\vartheta} f(x,y) = \sup_{R_1,R_2>0} (R_1 R_2)^{-1} \left| \int_0^{R_1} \int_0^{R_2} f(x-\gamma(t), y-\vartheta(s))\, dtds \right|$$

for $(x,y) \in \mathbf{R}^n \times \mathbf{R}^m$. Then, for $1 < p \leq \infty$, there exists a positive constant C_p such that

$$\left\| \mathcal{M}_{\gamma,\vartheta} f \right\|_p \leq C_p \left\| f \right\|_p \tag{2.1}$$

for all f in $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. The constant C_p is independent of $a_l$and b_s for all $1 \leq l \leq n$ and $1 \leq s \leq m$.

The proof of this lemma follows easily by using Lemma 2.2 and the inequality $\mathcal{M}_{\gamma,\vartheta} f(x,y) \leq \mathcal{M}_{\vartheta} \circ \mathcal{M}_{\gamma} f(x,y)$, where "$\circ$" denotes the composition of operators.

We shall need the following lemma of van der Corput type proved by Ricci and Stein in [14].

Lemma 2.4. *Let $n \in \mathbf{N}, \mu_1, \cdots, \mu_n \in \mathbf{R}$ and $a_1, \cdots, a_n$ be distinct positive real numbers. Let $\varepsilon = \min\{1/a_1, 1/n\}$ and $\psi \in C^1([0,1])$.Then there exists a positive constant C independent of $\{\mu_j\}$ such that*

$$\left| \int_\alpha^\beta e^{i(\mu_1 t^{a_1} + \mu_2 t^{a_2} + \cdots + \mu_n t^{a_n})} \psi(t)\, dt \right| \leq C \left|\mu_1\right|^{-\varepsilon} \left(\left|\psi(\beta)\right| + \int_\alpha^\beta \left|\psi'(t)\right| dt \right),$$

holds for $0 \leq \alpha < \beta \leq 1$.

Now, in Lemma 2.4, if $1/2 \leq \alpha < \beta \leq 1$ and some of the a_j's are negative, we get the following lemma which can proved by using the arguments employed in the proof of lemma 3 in [14].

Lemma 2.5. *Let $n \in \mathbf{N}, \mu_1, \cdots, \mu_n \in \mathbf{R}$ and $a_1, \cdots, a_n$ be distinct nonzero real numbers. Then there exists a positive constant Cindependent of $\{\mu_j\}$ such that*

$$\left| \int_\alpha^\beta e^{i(\mu_1 t^{a_1} + \mu_2 t^{a_2} + \cdots + \mu_n t^{a_n})} \psi(t)\, dt \right| \leq C \left|\mu_1\right|^{-1/n} \left(\left|\psi(\beta)\right| + \int_\alpha^\beta \left|\psi'(t)\right| dt \right),$$

holds for $1/2 \leq \alpha < \beta \leq 1$ and $\psi \in C^1([1/2,1])$.

We shall need the following lemma from [6]:

Lemma 2.6. *For $j \in \{1,2\}$, let U_j be a domain in $\mathbf{R}^{n_j}$ and K_j a compact subset of U_j. Let $R(\cdot,\cdot)$ be a real-analytic function on $U_1 \times U_2$ such that $R(\cdot,y)$ is a nonzero* *function for every $y \in U_2$. Then there exists a positive constant $\delta = \delta(R, K_1, K_2)$ such that*

$$\sup_{y \in K_2} \int_{K_1} |R(x,y)|^{-\delta} \, dx < \infty.$$

By tracking the constants in the proof of Lemma 1 in [7], we have the following:

Lemma 2.7. *Let $A > 0$ and let $\{\lambda_{k,j}\}$ be a sequence of Borel measures on $\mathbf{R}^n \times \mathbf{R}^m$. Suppose that $\left\| \sup_{k,j \in \mathbf{Z}} \left| |\lambda_{k,j}| * f \right| \right\|_{q_0} \leq A \|f\|_{q_0}$ for some $q_0 > 1$and for every f in $L^{q_0}(\mathbf{R}^n \times \mathbf{R}^m)$. Then the inequality*

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\lambda_{k,j} * g_{k,j}|^2 \right)^{1/2} \right\|_{p_0} \leq \left(A \sup_{k,j \in \mathbf{Z}} \|\lambda_{k,j}\| \right)^{1/2} \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{1/2} \right\|_{p_0} \tag{2.2}$$

holds for $|1/p_0 - 1/2| = 1/(2q_0)$ and for arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$.

The following result follows directly from Lemma 2.7 and Theorem 16 due to Al-Qassem and Pan in [3] which is a generalization of a result of J. Duoandikoetxea [7].

Lemma 2.8. *Let M, $N \in \mathbf{N}$ and let $\{\lambda_{k,j}^{(l,s)} : k,\, j \in \mathbf{Z},\, 0 \leq l \leq N,\, 0 \leq s \leq M\}$ be a family of Borel measures on $\mathbf{R}^n \times \mathbf{R}^m$with $\lambda_{k,j}^{(l,M)} = 0$ and $\lambda_{k,j}^{(N,s)} = 0$ for k, $j \in \mathbf{Z}$. Let $\{a_l,\, b_s : 0 \leq l \leq N-1,\, 0 \leq s \leq M-1\} \subset [2, \infty)$, $\{b(l),\, d(s) : 0 \leq l \leq N-1,\, 0 \leq s \leq M-1\} \subset \mathbf{N}$, $\{\alpha_l, \beta_s : 0 \leq l \leq N-1,\ 0 \leq s \leq M-1\} \subseteq \mathbf{R}^+$, and let $L^{(l)} \in L(\mathbf{R}^n, \mathbf{R}^{b(l)})$ and $Q^{(s)} \in L(\mathbf{R}^m, \mathbf{R}^{d(s)})$ be for $0 \leq l \leq N-1$ and $0 \leq s \leq M-1$,where $L(\mathbf{R}^n, \mathbf{R}^N)$ denotes the space of linear transformations from $\mathbf{R}^n$ into $\mathbf{R}^N$. Suppose that for some $C > 0$ and $B > 1$, the following hold for k, $j \in \mathbf{Z}$, $0 \leq l \leq N-1$, $0 \leq s \leq M-1$ and $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^m$:*

(i) $\left| \lambda_{k,j}^{(l,s)} \right| \leq CB^2$;

(ii) $\left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) \right| \leq CB^2 \left| a_l^{kB} L^{(l)}(\xi) \right|^{-\frac{\alpha_l}{B}} \left| b_s^{jB} Q^{(s)}(\eta) \right|^{-\frac{\beta_s}{B}}$;

(iii) $\left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\lambda}_{k,j}^{(l+1,s)}(\xi, \eta) \right| \leq CB^2 \left| a_l^{kB} L^{(l)}(\xi) \right|^{\frac{\alpha_l}{B}} \left| b_s^{jB} Q^{(s)}(\eta) \right|^{-\frac{\beta_s}{B}}$;

(iv) $\left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\lambda}_{k,j}^{(l,s+1)}(\xi, \eta) \right| \leq CB^2 \left| a_l^{kB} L^{(l)}(\xi) \right|^{-\frac{\alpha_l}{B}} \left| b_s^{jB} Q^{(s)}(\eta) \right|^{\frac{\beta_s}{B}}$;

(v) $$\left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\lambda}_{k,j}^{(l+1,s)}(\xi, \eta) - \hat{\lambda}_{k,j}^{(l,s+1)}(\xi, \eta) + \hat{\lambda}_{k,j}^{(l+1,s+1)}(\xi, \eta) \right| \leq CB^2 \left| a_l^{kB} L^{(l)}(\xi) \right|^{\frac{\alpha_l}{B}} \left| b_s^{jB} Q^{(s)}(\eta) \right|^{\frac{\beta_s}{B}};$$

(vi) $\left| \hat{\lambda}_{k,j}^{(l,s+1)}(\xi, \eta) - \hat{\lambda}_{k,j}^{(l+1,s+1)}(\xi, \eta) \right| \leq CB^2 \left| a_l^{kB} L^{(l)}(\xi) \right|^{\frac{\alpha_l}{B}}$;

(vii) $\left|\hat{\lambda}_{k,j}^{(l+1,s)}(\xi,\eta)-\hat{\lambda}_{k,j}^{(l+1,s+1)}(\xi,\eta)\right|\leq CB^2\left|b_s^{jB}Q^{(s)}(\eta)\right|^{\frac{\beta_s}{B}}$;

(viii) $\left\|\sup_{k,j\in\mathbf{Z}}\left|\left|\lambda_{k,j}^{(l,s)}\right|*f\right|\right\|_p\leq CB^2\|f\|_p$ *for* $1<p<\infty$

and for every f *in* $L^p(\mathbf{R}^n\times\mathbf{R}^m)$. *Then for every* $1<p<\infty$, *there exists a positive constant* C_p *independent of* $\{L^{(l)},Q^{(s)}:0\leq l\leq N-1,\ 0\leq s\leq M-1\}$ *such that*

$$\left\|\sum_{k,j\in\mathbf{Z}}\lambda_{k,j}^{(0,0)}*f\right\|_p\leq C_pB^2\|f\|_p \tag{2.3}$$

hold for all f *in* $L^p(\mathbf{R}^n\times\mathbf{R}^m)$.

3. Proof of Theorem 1.1

Assume that $\Omega\in L(\log^+L)^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and satisfies (1.2). We decompose Ω as follows: For $\mu\in\mathbf{N}$, let $E_\mu=\{(x,y)\in\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}:2^\mu\leq|\Omega(x,y)|<2^{\mu+1}\}$, $b_\mu=\Omega\chi_{E_\mu}$ and $\lambda_\mu=\left\|\tilde{b}_\mu\right\|_1$. Let $\mathbf{D}=\left\{\mu\in\mathbf{N}:\lambda_\mu\geq 2^\mu\right\}$,

$$\begin{aligned}\Omega_\mu(x,y) &= \left(\lambda_\mu\right)^{-1}\Big(b_\mu(x,y)-\int_{\mathbf{S}^{n-1}}b_\mu(u,y)d\sigma(u)-\int_{\mathbf{S}^{m-1}}b_\mu(x,v)d\sigma(v)\\ &\quad+\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}b_\mu(u,v)d\sigma(u)d\sigma(v)\Big)\end{aligned}$$

for $\mu\in\mathbf{N}$ and $\Omega_0=\Omega-\sum_{\mu\in\mathbf{D}}\lambda_\mu\Omega_\mu$. Then it is easy to verify that

$$\int_{\mathbf{S}^{n-1}}\Omega_\mu(u,\cdot)\,d\sigma(u)=\int_{\mathbf{S}^{m-1}}\Omega_\mu(\cdot,v)\,d\sigma(v)=0, \tag{3.1}$$

$$\left\|\Omega_\mu\right\|_2 \leq 4(a_\mu)^2, \tag{3.2}$$

$$\left\|\Omega_\mu\right\|_1 \leq 4, \tag{3.3}$$

$$\Omega(x,y) = \sum_{\mu\in\mathbf{D}\cup\{0\}}\lambda_\mu\Omega_\mu(x,y), \tag{3.4}$$

for $\mu\in\mathbf{D}\cup\{0\}$, where we used $\lambda_0=1$. Thus

$$\|T_{\Gamma,\Upsilon,\Omega}(f)\|_p \leq \sum_{\mu\in\mathbf{D}\cup\{0\}}\left|\lambda_\mu\right|\left\|T_{\Gamma,\Upsilon,\Omega_\mu}(f)\right\|_p, \tag{3.5}$$

$$\left\|T^*_{\Gamma,\Upsilon,\Omega}(f)\right\|_p \leq \sum_{\mu\in\mathbf{D}\cup\{0\}}\left|\lambda_\mu\right|\left\|T^*_{\Gamma,\Upsilon,\Omega_\mu}(f)\right\|_p. \tag{3.6}$$

Therefore, to prove (1.5)-(1.6), it suffices to prove that

$$\left\|T_{\Gamma,\Upsilon,\Omega_\mu}(f)\right\|_p \leq C_p(\mu+1)^2\|f\|_p, \tag{3.7}$$

$$\left\|T^*_{\Gamma,\Upsilon,\Omega_\mu}(f)\right\|_p \leq C_p(\mu+1)^2\|f\|_p \tag{3.8}$$

for $1<p<\infty$ and $\mu\in\mathbf{D}\cup\{0\}$. Let us first prove (3.7). To this end, by assumptions $\Gamma=(\Gamma_1,\cdots,\Gamma_N):\mathbf{R}^n\to\mathbf{R}^N$ and $\Upsilon=(\Upsilon_1,\cdots,\Upsilon_M):\mathbf{R}^m\to\mathbf{R}^M$ are homogeneous mappings of degrees $d=(d_1,\cdots,d_N)$ and $h=(h_1,\cdots,h_M)$, respectively such that $\Gamma\mid\mathbf{S}^{n-1}$ and $\Upsilon\mid\mathbf{S}^{m-1}$ are real-analytic and d_l, $h_s\neq 0$ for $1\leq l\leq N$ and $1\leq s\leq M$. In view of Lemmas 2.4-2.5, we shall only prove our theorem for the case $d_1,\cdots,d_N,h_1,\cdots,h_M>0$. The argument for the other cases that some or all of the d_l's and h_r's are negative is similar and requires only minor modifications. If $\{\Gamma_l: d_l=d_1\}=\{0\}$ or $\{\Upsilon_s: h_s=h_1\}=\{0\}$, then $T_{\Gamma,\Upsilon,\Omega_\mu}$ is the zero operator and hence (3.7) holds trivially. Now, if $\{\Gamma_l: d_l=d_1\}\neq\{0\}$ and $\{\Upsilon_s: h_s=h_1\}\neq\{0\}$, by a simple reordering of the mappings $\Gamma_1,\cdots,\Gamma_N,\Upsilon_1,\cdots,\Upsilon_M$, if necessary, we may assume that there are z_1, $\tilde{z}_1$, w_1 and $\tilde{w}_1\in\mathbf{N}$ such that $z_1\leq\tilde{z}_1\leq N$, $\{l:1\leq l\leq N$ and $d_l=d_1\}=\{1,\cdots,\tilde{z}_1\}$, $w_1\leq\tilde{w}_1\leq M$, $\{s:1\leq s\leq M$ and $h_s=h_1\}=\{1,\cdots,\tilde{w}_1\}$, $\{\Gamma_1,\cdots,\Gamma_{z_1}\}$ forms a basis for $span\{\Gamma_1,\cdots,\Gamma_{\tilde{z}_1}\}$ and $\{\Upsilon_1,\cdots,\Upsilon_{w_1}\}$ forms a basis for $span\{\Upsilon_1,\cdots,\Upsilon_{\tilde{w}_1}\}$.

Let $\Phi_0=\Gamma$, $\Phi_1=(0,\cdots,0,\Gamma_{\tilde{z}_1+1},\cdots,\Gamma_N)$, $\Psi_0=\Upsilon$, $\Psi_1=(0,\cdots,0,\Upsilon_{\tilde{w}_1+1},\cdots,\Upsilon_M)$, and $\lambda^{(l,s)}_{k,j,\mu}=\lambda_{k,j,\Gamma_l,\Upsilon_s,\mu}$ for $l,s\in\{0,1\}$. Under the above assumptions, we have the following:

Lemma 3.1. *There exist $L\in L(\mathbf{R}^{\tilde{z}_1},\mathbf{R}^{z_1})$, $Q\in L(\mathbf{R}^{\tilde{w}_1},\mathbf{R}^{w_1})$ and positive constants α_0, β_0 and C such that*

$$\left|\lambda^{(l,s)}_{k,j,\mu}(\xi,\eta)\right| \leq C(\mu+1)^2 \textit{for } l,s\in\{0,1\}; \tag{3.9}$$

$$\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)\right| \leq C(\mu+1)^2\left|a_\mu^{kd_1}L^{(0)}(\xi)\right|^{-\frac{\alpha_0}{\mu+1}}\left|a_\mu^{jh_1}Q^{(0)}(\eta)\right|^{-\frac{\beta_0}{\mu+1}}; \tag{3.10}$$

$$\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(0,1)}_{k,j,\mu}(\xi,\eta)\right| \leq C(\mu+1)^2\left|a_\mu^{kd_1}L^{(0)}(\xi)\right|^{-\frac{\alpha_0}{\mu+1}}\left|a_\mu^{jh_1}Q^{(0)}(\eta)\right|^{\frac{\beta_0}{\mu+1}}; \tag{3.11}$$

$$\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(1,0)}_{k,j,\mu}(\xi,\eta)\right| \leq C(\mu+1)^2\left|a_\mu^{kd_1}L^{(0)}(\xi)\right|^{\frac{\alpha_0}{\mu+1}}\left|a_\mu^{jh_1}Q^{(0)}(\eta)\right|^{-\frac{\beta_0}{\mu+1}}; \tag{3.12}$$

$$\left|\hat{\lambda}^{(0,1)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(1,1)}_{k,j,\mu}(\xi,\eta)\right| \leq C(\mu+1)^2\left|a_\mu^{kd_1}L^{(0)}(\xi)\right|^{\frac{\alpha_0}{\mu+1}}; \tag{3.13}$$

$$\left|\hat{\lambda}^{(1,0)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(1,1)}_{k,j,\mu}(\xi,\eta)\right| \leq C(\mu+1)^2\left|a_\mu^{jh_1}Q^{(0)}(\eta)\right|^{\frac{\beta_0}{\mu+1}}; \tag{3.14}$$

$$\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(0,1)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(1,0)}_{k,j,\mu}(\xi,\eta)+\hat{\lambda}^{(1,1)}_{k,j,\mu}(\xi,\eta)\right|$$
$$\leq C(\mu+1)^2\left|a_{\mu}^{kd_1}L^{(0)}(\xi)\right|^{\frac{\alpha_0}{\mu+1}}\left|a_{\mu}^{jh_1}Q^{(0)}(\eta)\right|^{\frac{\beta_0}{\mu+1}}, \tag{3.15}$$

for all $(\xi,\eta)\in\mathbf{R}^N\times\mathbf{R}^M$, *where* $L^{(0)}(\xi)=L(\Pi_{\tilde{z}_1}\xi)$, $Q^{(0)}(\eta)=Q(\Pi_{\tilde{w}_1}\eta)$, $\Pi_{\tilde{z}_1}\xi=(\xi_1,\cdots,\xi_{\tilde{z}_1})$ *and* $\Pi_{\tilde{w}_1}\eta=(\eta_1,\cdots,\eta_{\tilde{w}_1})$.

Proof. First, it is easy to verify that (3.9) holds trivially. Now, we prove (3.10). By assumptions, there exist two linear transformations $L=(L_1,\cdots,L_{z_1})\in L(\mathbf{R}^{\tilde{z}_1},\mathbf{R}^{z_1})$ and $Q=(Q_1,\cdots,Q_{w_1})\in L(\mathbf{R}^{\tilde{w}_1},\mathbf{R}^{w_1})$ such that

$$\sum_{l=1}^{\tilde{z}_1}\xi_l\Gamma_l(x)=\sum_{l=1}^{z_1}L_l(\Pi_{\tilde{z}_1}\xi)\Gamma_l(x)\text{ and }\sum_{s=1}^{\tilde{w}_1}\eta_s\Upsilon_s(y)=\sum_{s=1}^{w_1}Q_s(\Pi_{\tilde{w}_1}\eta)\Upsilon_s(y). \tag{3.16}$$

Thus we have

$$\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)\right|\leq C(\mu+1)\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\left|\Omega_{\mu}(x,y)\right|\left|\int_{1/a_{\mu}}^{1}e^{-iY_{\xi,k}(t,x)}\frac{dt}{t}\right|d\sigma(x)d\sigma(y),$$

where

$$Y_{\xi,k}(t,x)=\left(\sum_{l=1}^{\tilde{z}_1}\xi_l\Gamma_l(x)\right)t^{d_1}a_{\mu}^{(k+1)d_1}+\sum_{s=\tilde{z}_1+1}^{N}\xi_s\Gamma_s(x)t^{d_s}a_{\mu}^{(k+1)d_s}. \tag{3.17}$$

Define $R:\mathbf{S}^{n-1}\times\mathbf{S}^{z_1-1}\to\mathbf{R}$ by

$$R(x,u)=\sum_{l=1}^{z_1}u_l\Gamma_l(x),$$

where $x\in\mathbf{S}^{n-1}$ and $u=(u_1,\cdots,u_{z_1})\in\mathbf{S}^{z_1-1}$. Since $\{\Gamma_1,\cdots,\Gamma_{z_1}\}$ is linearly independent, $R(\cdot,u)$ is a nonzero function for every $u\in\mathbf{S}^{z_1-1}$. By Lemma 2.6, there exists a $\delta_1>0$ such that

$$\sup_{u\in\mathbf{S}^{z_1-1}}\int_{\mathbf{S}^{n-1}}|R(x,u)|^{-\delta_1}d\sigma(x)<\infty. \tag{3.18}$$

By letting $\varepsilon=\min\{1/d_1,1/N,\delta_1/2\}$,(3.2), (3.18) and Hölder's inequality, we get

$$\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)\right|\leq C(\mu+1)\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\left|\Omega_{\mu}(x,y)\right|\left|\sum_{l=1}^{z_1}L_l(\Pi_{\tilde{z}_1}\xi)\Gamma_l(x)\right|^{-\varepsilon}d\sigma(x)d\sigma(y)$$

$$\leq C(\mu+1)a_{\mu}^{-\varepsilon d_1}\left\|\Omega_{\mu}\right\|_{L^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\left|a_{\mu}^{kd_1}L(\Pi_{\tilde{z}_1}\xi)\right|^{-\epsilon}$$
$$\leq C(\mu+1)a_{\mu}^{-\varepsilon d_1}(a_{\mu})^2\left|a_{\mu}^{kd_1}L(\Pi_{\tilde{z}_1}\xi)\right|^{-\epsilon}. \tag{3.19}$$

Thus, by combining the last estimate with the trivial estimate $\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)\right| \le C(\mu+1)^2$, we get

$$\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)\right| \le C(\mu+1)^2 \left|a_\mu^{kd_1} L(\Pi_{\tilde{z}_1}\xi)\right|^{-\epsilon/(\mu+1)}. \tag{3.20}$$

Similarly, we have

$$\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)\right| \le C(\mu+1)^2 \left|a_\mu^{jh_1} Q(\Pi_{\tilde{w}_1}\eta)\right|^{-\beta/(\mu+1)}. \tag{3.21}$$

Combining the last two estimates yields (3.10). Next, we prove (3.11).

$$\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(0,1)}_{k,j,\mu}(\xi,\eta)\right| \le \int_{1/a_\mu}^1 \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \left|\Omega_\mu(x,y)\right| \times$$
$$\left|\int_{1/a_\mu}^1 e^{-iY_{\xi,k}(t,x)}\frac{dt}{t}\right| \left|e^{-i\eta\cdot\Psi_0(a_\mu^{j+1}sy)} - e^{-i\eta\cdot\Psi_1(a_\mu^{j+1}sy)}\right| d\sigma(x)d\sigma(y)\frac{ds}{s},$$

where $Y_{\xi,k}(t,x)$ is given by (3.17). By a similar argument as that employed in the proof of (3.10) we get

$$\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(0,1)}_{k,j,\mu}(\xi,\eta)\right| \le C(\mu+1)a_\mu^{-\varepsilon d_1}\left\|\Omega_\mu\right\|_2 \left|a_\mu^{kd_1}L^{(0)}(\xi)\right|^{-\varepsilon} \left|a_\mu^{(j+1)h_1}Q^{(0)}(\eta)\right|$$

which when combined with the trivial estimate $\left|\hat{\lambda}^{(0,0)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(0,1)}_{k,j,\mu}(\xi,\eta)\right| \le C(\mu+1)^2$ yields (3.11). Similarly, we get (3.12)-(3.15). The proof of the lemma is complete. □

Similarly, by using (3.1)-(3.3) we can find additional mappings $\Phi_2,\cdots,\Phi_K$ from $\mathbf{R}^n\setminus\{0\}$ to $\mathbf{R}^N$, $\Psi_2,\cdots,\Psi_J$ from $\mathbf{R}^m\setminus\{0\}$ to $\mathbf{R}^M$, $\{\alpha_l,\beta_s : 1\le l\le K-1, 1\le s\le J-1\}\subset(0,\infty)$, appropriate linear transformations $\{L^{(l)}, Q^{(s)} : 1\le l\le K-1, 1\le s\le J-1\}$, two sets of distinct real numbers $\{d_{u_l} : 1\le l\le K-1\}$, $\{h_{v_s} : 1\le s\le J-1\}$ with $\{d_{u_l} : 1\le l\le K-1\}=\{d_l : 2\le l\le N\}\backslash\{d_1\}$, $\{h_{vs} : 1\le s\le J-1\}=\{h_s : 2\le s\le M\}\backslash\{h_1\}$ and a finite family of measures $\{\lambda^{(l,s)}_{k,j,\mu} : 2\le l\le K, 2\le s\le J\}$ with the following properties:

$$\begin{aligned}
\Phi_K &= (0,\cdots,0),\ \Psi_J=(0,\cdots,0);\\
\lambda^{(l,s)}_{k,j,\mu}(\xi,\eta) &= \lambda_{k,j,\mu,\Phi_l,\Psi_s} \text{ for } 2\le l\le K \text{ and } 2\le s\le J;\\
\lambda^{(K,s)}_{k,j,\mu} &= \lambda^{(l,J)}_{k,j,\mu}=0 \text{ for } 2\le l\le K \text{ and } 2\le s\le J;
\end{aligned}$$

$$\left|\lambda^{(l,s)}_{k,j,\mu}(\xi,\eta)\right| \le C(\mu+1)^2; \tag{3.22}$$

$$\left|\hat{\lambda}^{(l,s)}_{k,j,\mu}(\xi,\eta)\right| \le C(\mu+1)^2 \left|a_\mu^{kd_{u_l}}L^{(l)}(\xi)\right|^{-\frac{\alpha_l}{\mu+1}} \left|a_\mu^{jh_{vs}}Q^{(s)}(\eta)\right|^{-\frac{\beta_s}{\mu+1}} \tag{3.23}$$

(3.24)
$$\left|\hat{\lambda}^{(l,s)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(l+1,s)}_{k,j,\mu}(\xi,\eta)\right|\leq C(\mu+1)^2\left|a_{\mu}^{kd_{u_l}}L^{(l)}(\xi)\right|^{\frac{\alpha_l}{\mu+1}}\left|a_{\mu}^{jh_{v_s}}Q^{(s)}(\eta)\right|^{-\frac{\beta_s}{\mu+1}};$$

(3.25)
$$\left|\hat{\lambda}^{(l,s)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(l,s+1)}_{k,j,\mu}(\xi,\eta)\right|\leq C(\mu+1)^2\left|a_{\mu}^{kd_{u_l}}L^{(l)}(\xi)\right|^{-\frac{\alpha_l}{\mu+1}}\left|a_{\mu}^{jh_{v_s}}Q^{(s)}(\eta)\right|^{\frac{\beta_s}{\mu+1}};$$

$$\left|\hat{\lambda}^{(l,s+1)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(l+1,s+1)}_{k,j,\mu}(\xi,\eta)\right|\leq C(\mu+1)^2\left|a_{\mu}^{kd_{u_l}}L^{(l)}(\xi)\right|^{\frac{\alpha_l}{\mu+1}}; \tag{3.26}$$

$$\left|\hat{\lambda}^{(l,s)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(l+1,s+1)}_{k,j,\mu}(\xi,\eta)\right|\leq C(\mu+1)^2\left|a_{\mu}^{jh_{v_s}}Q^{(s)}(\eta)\right|^{\frac{\beta_s}{\mu+1}}; \tag{3.27}$$

$$\begin{aligned}&\left|\hat{\lambda}^{(l,s)}_{k,j,\mu}(\xi,\eta)-\hat{\lambda}^{(l,s+1)}_{k,j,\mu}-\hat{\lambda}^{(l+1,s)}_{k,j,\mu}+\hat{\lambda}^{(l+1,s+1)}_{k,j,\mu}(\xi,\eta)\right|\\ \leq\ & C(\mu+1)^2\left|a_{\mu}^{kd_{u_l}}L^{(l)}(\xi)\right|^{\frac{\alpha_l}{\mu+1}}\left|a_{\mu}^{jh_{v_s}}Q^{(s)}(\eta)\right|^{\frac{\beta_s}{\mu+1}}\end{aligned} \tag{3.28}$$

for $1\leq l\leq K-1$ and $1\leq s\leq J-1$. By (3.3) and Lemma 2.3, we immediately get

$$\left\|\sup_{k,j\in Z}\left\|\lambda^{(l,s)}_{k,j,\mu}*f\right\|\right\|_p\leq C_p(\mu+1)^2\left\|f\right\|_p \tag{3.29}$$

for $1<p<\infty$, $0\leq l\leq K-1$ and $0\leq s\leq J-1$. By (3.9)-(3.15), (3.22)-(3.28), Lemma 2.8, we have

$$\left\|T_{\Gamma,\Upsilon,\Omega_\mu}(f)\right\|_p=\left\|\sum_{k,j\in\mathbf{Z}}\lambda^{(0,0)}_{k,j,\mu}*f\right\|_p\leq C_p\,(\mu+1)^2\left\|f\right\|_p \tag{3.30}$$

for $1<p<\infty$ and $f\in L^p(\mathbf{R}^n\times\mathbf{R}^m)$ which completes the proof of (3.7).

We can construct a proof of (3.8) by using the above estimates and the techniques developed in [3]. We omit the details.

References

[1] A. Al-Salman, H. Al-Qassem and Y. Pan, *Singular integrals associated to homogeneous mappings with rough kernels*, Hokkaido Mathematical Journal, **33**(2004), 551-569.

[2] A. Al-Salman, H. Al-Qassem and Y. Pan, *Singular Integrals on Product Domains*, Indiana Univ. Math. J., **55(1)**(2006), 369-387.

[3] H. Al-Qassem and Y. Pan, *L^p boundedness for singular integrals with rough kernels on product domains*, Hokkaido Math. J., **31**(2002), 555-613.

[4] A. Al-Salman and Y. Pan, *Singular integrals with rough kernels in $Llog^+L(\mathbf{S}^{n-1})$*, J. London Math. Soc., **66(2)**(2002), 153-174.

[5] Calderón, A. P. and Zygmund, A., *On singular integrals*, Amer. J. Math., **78**(1956), 289-309.

[6] L. Cheng, *Singular integrals related to homogeneous mappings*, Michigan Math. J., **47(2)**(2000), 407-416.

[7] J. Duoandikoetxea, *Multiple singular integrals and maximal functions along hypersurfaces*, Ann. Ins. Fourier (Grenoble), **36**(1986), 185-206.

[8] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal functions and singular integral operators via Fourier transform estimates*, Invent. Math., **84**(1986), 541-561.

[9] D. Fan, K. Guo and Y. Pan, *Singular integrals with rough kernels on product spaces*, Hokkaido Math. J., **28**(1999), 435-460.

[10] D. Fan, K. Guo and Y. Pan, *L^p estimates for singular integrals associated to homogeneous surfaces*, J. Reine Angew. Math., **542**(2002), 1-22.

[11] R. Fefferman, *Singular integrals on product domains*, Bull. Amer. Math. Soc., **4**(1981), 195-201.

[12] R. Fefferman and E. M. Stein, *Singular integrals on product spaces*, Adv. in Math., **45**(1982), 117-143.

[13] Y. Jiang and S. Lu, *A class of singular integral operators with rough kernels on product domains*, Hokkaido Math. J., **24**(1995), 1-7.

[14] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals I: Oscillatory integrals*, Jour. Func. Anal., **73**(1987), 179-194.

[15] F. Ricci and E. M. Stein, *Multiparameter singular integrals and maximal functions*, Ann. Inst. Fourier, **42**(1992), 637-670.

[16] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, NJ, 1970.

[17] E. M. Stein, Harmonic analysis real-variable methods, orthogonality and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.