KYUNGPOOK Math. J. 46(2006), 367-376

Symmetry Properties of 3-dimensional D'Atri Spaces

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Abstract. We investigate semi-symmetry and pseudo-symmetry of some 3-dimensional Riemannian manifolds: the D'Atri spaces, the Thurston geometries as well as the η-Einstein manifolds. We prove that all these manifolds are pseudo-symmetric and that many of them are not semi-symmetric.

1. D'Atri space

A Riemannian manifold is a D'Atri space if its local geodesic symmetries are volume preserving, or equivalently are divergence preserving ([9], [16], [25]). O. Kowalski proved that a connected and complete D'Atri space of dimension 3 is isometric to one of the following manifolds: (i) Riemannian symmetric spaces: $\mathbb{R}^3$, $\mathrm{S}^3(c)$, $\mathrm{H}^3(-c)$, $\mathrm{S}^2\times\mathbb{R}$, $\mathrm{H}^2(-c)\times\mathbb{R}$, where c is a positive constant, or (ii) the group $SU(2)\equiv S^3$, the universal covering group of $SL(2,\mathbb{R})$, or the Heisenberg group H_3 of all real matrix of the form $\begin{pmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}$ with any left invariant metric. The metrics of 3-dimensional D'Atri spaces except for the metric of H^3 have the

Received February 17, 2005.
2000 Mathematics Subject Classification: 53B25, 53B30, 53C25.
Key words and phrases: Heisenberg space, D'Atri space, Thurston geometry, semi-symmetry, pseudo-symmetry, η-Einstein manifold, warped product.
Research supported by the Fund for Scientific Research-Flanders (F.W.O.) Belgium through project G.0366.98 for the first named author.

form ([8], [24]):

$$g = \frac{dx^2+dy^2}{(1+m(x^2+y^2))^2} + (dz + \frac{l}{2}\frac{ydx-xdy}{1+m(x^2+y^2)})^2, \quad l,\ m \in \mathbb{R},$$

which were studied already by L. Bianchi, E. Cartan and G. Vrançeanu ([3], [7], [8], [24]). According to [20] we have: if $m = 0$ and $l \neq 0$ then $M \equiv H_3$, or if $m > 0$ and $l \neq 0$ then $M \equiv SU(2)$, or if $m < 0$ and $l \neq 0$ then $M \equiv \widetilde{SL(2,\mathbb{R})}$, or if $m > 0$ and $l = 0$ then $M \equiv S^2 \times \mathbb{R}$, or if $m < 0$ and $l = 0$ then $M \equiv H^2 \times \mathbb{R}$, or if $4m - l^2 = 0$ then M is a real space forms of positive or zero curvature.
With respect to the above classification, we will investigate the property of pseudo-symmetry of H_3, $SU(2)$ and $\widetilde{SL(2,\mathbb{R})}$.

Let $X_1, \cdots, X_n$ be an orthonormal moving frame on a Riemanian manifold (M, g), $n = \dim M \geq 3$, and let ω^i and ω^i_j be the dual forms and the connection forms for this moving frame. Then the structure equations of (M, g) are given by:

$$d\omega^i = -\omega_j{}^i \wedge \omega^j, \quad d\omega_j{}^i = -\omega_k{}^i \wedge \omega_j{}^k + \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l,$$

where R^i_{jkl} are the local components of the Riemann curvature tensor of (M, g). The basis and the dual basis of 3-dimensional D'Atri space are given by:

$$X = (1+m(x^2+y^2))\,\frac{\partial}{\partial x} - \frac{l}{2}y\,\frac{\partial}{\partial z}\,, \quad Y = (1+m(x^2+y^2))\,\frac{\partial}{\partial y} + \frac{l}{2}x\,\frac{\partial}{\partial z}\,, \quad \xi = \frac{\partial}{\partial z}\,,$$

and

$$\omega^1 = \frac{dx}{1+m(x^2+y^2)}\,, \quad \omega^2 = \frac{dy}{1+m(x^2+y^2)}\,, \quad \omega^3 = dz + \frac{l}{2}\,\frac{ydx-xdy}{1+m(x^2+y^2)}\,.$$

The eigenvalues ρ_i, $i = 1, 2, 3$, of the Ricci tensor and the scalar curvature κ of (M, g) are the following

$$\rho_1 = \rho_2 = 4m - \frac{l^2}{2}\,, \quad \rho_3 = \frac{l^2}{2}, \quad \kappa = 8m - \frac{l^2}{2}\,. \tag{1}$$

Let (M, g), $n \geq 3$, be a semi-Riemannian manifold. We consider the endomorphisms $X \wedge_g Y$ and $\mathcal{R}(X, Y)$ of (M, g) defined by

$$\begin{aligned} X \wedge_g Y &= g(Y,Z)X - g(X,Z)Y\,, \\ \mathcal{R}(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,, \end{aligned}$$

where ∇ is the Levi-Civita connection, κ the scalar curvature and $\mathcal{S}$ the Ricci operator of (M, g). The Ricci tensor S and the Ricci operator $\mathcal{S}$ of (M, g) are related by $S(X, Y) = g(\mathcal{S}X, Y)$. The Riemann curvature tensor R and the tensor G of (M, g) are defined by

$$\begin{aligned} R(X_1, X_2, X_3, X_4) &= g(\mathcal{R}(X_1, X_2)X_3, X_4)\,, \\ G(X_1, X_2, X_3, X_4) &= g((X_1 \wedge_g X_2)X_3, X_4)\,, \end{aligned}$$

respectively. For a $(0,k)$-tensor T, $k \geq 1$, on M we define the $(0,k+2)$-tensors $R \cdot T$ and $Q(g,T)$ by

$$(R \cdot T)(X_1, X_2, \cdots, X_k; X, Y) = (\mathcal{R}(X,Y) \cdot T)(X_1, X_2, \cdots, X_k) =$$
$$-T(\mathcal{R}(X,Y)X_1, X_2, \cdots, X_k) - \cdots - T(X_1, X_2, \cdots, \mathcal{R}(X,Y)X_k)\,,$$

$$Q(g,T)(X_1, X_2, \cdots, X_k; X, Y) \;=\; ((X \wedge_g Y) \cdot T)(X_1, X_2, \cdots, X_k) =$$
$$-\quad T((X \wedge Y)X_1, X_2, \cdots, X_k) - \cdots - T(X_1, X_2, \cdots, X_{k-1}, (X \wedge_g Y)X_k).$$

If we set in the above formulas $T = R$, then we obtain the tensors: $R \cdot R$ and $Q(g,R)$.

A semi-Riemannian manifold (M,g), $n \geq 3$, is called semi- symmetric if $R \cdot R = 0$ on M. A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be pseudo-symmetric ([10], [12], [23], [26]) if at every point of M the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent. Thus we see that (M,g) is pseudo-symmetric if and only if

$$R \cdot R = L_R\, Q(g,R) \tag{2}$$

on $U_R = \{x \in M \,|\, R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, where L_R is some function on U_R. The condition (2) arose during the study on totally umbilical submanifolds of semi-symmetric manifolds as well as when considering geodesic mappings of semi-symmetric manifolds ([10], [23]). Every semi-symmetric manifold is pseudo-symmetric. The converse statement is not true (see e.g. [11]). A pseudo-symmetric space which is not semi-symmetric is said to be a proper pseudo-symmetric space. We will denote the class of space forms by R_0. Locally symmetric spaces ($\nabla R = 0$) form a generalization of the space forms. We will denote this class of manifolds by R_1. Similarly, the semi-symmetric manifolds form a generalization of the locally symmetric spaces. We will denote this class of manifolds by R_2. Finally, the pseudo-symmetric spaces form a generalization of the semi-symmetric manifolds. We will denote this class of manifolds by R_3. Thus we have $R_0 \subset R_1 \subset R_2 \subset R_3$. In addition, all inclusions being proper ones, provided that $n \geq 4$, ([2], [10], [12], [13], [23]). We recall that (M,g), $n \geq 3$, is said to be quasi-Einstein if at every $x \in M$ its Ricci tensor S has the form

$$S \;=\; \alpha\, g + \beta\, \omega \otimes \omega\,, \qquad \alpha\,, \beta \in \mathbb{R}\,, \qquad \omega \in T^{\star}_x M\,. \tag{3}$$

Theorem A ([14]). *A 3-dimensional semi-Riemannian manifold is pseudo-symmetric if and only if it is quasi-Einstein.*

Proposition B ([10], [12]). *For a 3-dimensional quasi-Einstein Riemannian manifold (M,g), for which $\rho_1 = \rho_2 \neq \rho_3$ on M, we have $R \cdot R = \frac{\rho_3}{2}\, Q(g,R)$.*

Consequently, we have

Theorem 1. *Every 3-dimensional D'Atri space is pseudo-symmetric.*

Proof. The eigenvalues of the Ricci tensor of D'Atri spaces are $\rho_1 = \rho_2 = 4m - \frac{l^2}{2}$, $\rho_3 = \frac{l^2}{2}$. Thus by Proposition B we have $R \cdot R = \frac{l^2}{4} Q(g, R)$. □

We note that for every D'Atri space we have $S = (4m - \frac{l^2}{2})\, g + (l^2 - 4m)\, \omega \otimes \omega$, where ω is a 1-form with the local components $\omega_h = \delta_h^1$.
Using the above results we obtain,

Proposition 2. *The D'Atri spaces: $S^2 \times \mathbb{R}, H^2 \times \mathbb{R}$, are non-Einstein semi-symmetric manifolds. Moreover, the D'Atri spaces H_3, $SU(2)$ and $\widetilde{SL}(2, \mathbb{R})$ are proper pseudo-symmetric spaces.*

Let M^{2n+1} be a $(2n + 1)$-dimensional differentiable manifold, and let ϕ, ξ and η be a tensor field of type $(1, 1)$, a vector field and a 1-form on M^{2n+1}, respectively. If the following conditions are satisfied: $\phi^2 X = -X + \eta(X)\xi$, $\phi\xi = 0$, $\eta(\phi X) = 0$, $\eta(\xi) = 1$, for any $X \in \chi(M^{2n+1})$, then M^{2n+1} admits an almost contact structure (ϕ, ξ, η) and is called an almost contact manifold. An almost contact structure on M^{2n+1} is said to be normal if the Nijenhuis tensor N_ϕ formed with ϕ, $N_\phi(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$, satisfies $N_\phi + 2d\eta \otimes \xi = 0$.

If a Riemannian metric g is given on M^{2n+1} such that $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, $\eta(X) = g(\xi, X)$, for any $X, Y \in \chi(M^{2n+1})$, then (ϕ, ξ, η, g) is called an almost contact metric structure and M^{2n+1} is called an almost contact metric manifold. If, in addition, $d\eta(X, Y) = g(X, \phi Y)$, for all $X, Y \in \chi(M^{2n+1})$, then an almost contact metric structure is called a contact metric structure. It is called a K-contact structure if the characteristic vector field ξ is a Killing vector field. The normal contact metric structure is called a Sasakian structure and a manifold with Sasakian structure is called a Sasakian manifold. Let M^{2n+1} be a contact metric manifold with contact metric structure (η, g, ξ, ϕ). M^{2n+1} is said to be η-Einstein if the Ricci tensor is of the form $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, where a and b are some functions on M^{2n+1}. It is known that if M^{2n+1} is a K-contact η-Einstein manifold, with $n > 1$, then the functions a and b are constant. Every K-contact three-manifold is η-Einstein. Every K-contact three-manifold is Sasakian, a Sasakian manifold of constant ϕ sectional curvature is an η-Einstein. For more details we refer to [4] and [19]. There are examples of non-Sasakian η-Einstein contact metric manifolds ([4], [5]).

Proposition C ([5]). *Let $M^3(\phi, \xi, \eta, g)$ be a contact metric manifold. Then any of the following three conditions is equivalent to each other:*

(i) *M^3 is η-Einstein,*

(ii) *$\mathcal{S}\phi = \phi\mathcal{S}$,*

(iii) *$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y)$.*

Theorem D ([5]). *Let M^3 be a contact metric manifold on which $S\phi = \phi S$. Then*

M^3 is either Sasakian, flat or of constant ξ-sectional curvature $K(X,\xi) = k < 1$ and constant ϕ-sectional curvature $K(X,\phi X) = -k$.

Using the above results we obtain,

Theorem 3. *Every 3-dimensional η- Einstein manifold is pseudo-symmetric. More precisely, $R\cdot R = k\,Q(g,R)$.*

Corollary 4. *Every 3-dimensional K-contact manifold is pseudo-symmetric, with $R\cdot R = Q(g,R)$.*

Corollary 5. *Every 3-dimensional Sasakian space form is pseudo-symmetric.*

Example 1.1 ([1]). Let (x,y,z) be a standard coordinates on $\mathbb{R}^3$ and let η be the 1-form $\eta = \frac{1}{2}(dz - ydx)$. We set $\xi = 2(\frac{\partial}{\partial z})$ and we define the matrix of ϕ by $\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & y & 0\end{pmatrix}$. In addition, we have $\eta(\xi) = 1$ and $\phi^2 = -I + \eta\otimes\xi$. So (ϕ,ξ,η) is an almost contact structure on $\mathbb{R}^3$. We define the metric tensor g by $g = \frac{1}{4}(dx^2 + dy^2) + \eta\otimes\eta$. The vector fields $X = 2\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)$, $Y = 2(\frac{\partial}{\partial y})$ and $\xi = 2(\frac{\partial}{\partial z})$ form an ϕ-orthonormal basis $\phi X = -Y$, $\phi Y = X$, $\phi\xi = 0$. Thus $(\mathbb{R}^3,\eta,\phi,\xi,g)$ is a Sasakian space form, denoted by $\mathbb{R}^3(-3)$. The Ricci curvatures are the following: $\rho_1 = \rho_2 = -1$, and $\rho_3 = 2$.

Consequently we have

Proposition 6. *The Sasakian space form $\mathbb{R}^3(-3)$ is a proper pseudo-symmetric manifold. More precisely, on $\mathbb{R}^3(-3)$ we have $R\cdot R = Q(g,R) \neq 0$.*

Theorem 7. *Let (M,g) be a 3-dimensional semi-Riemannian manifold and let X, Y, ξ be an orthonormal basis of T_xM, $x \in M$. If the following two conditions noted by* (**C**)*: $R(X,Y)X_i = \alpha\,(X\wedge_g Y)X_i$ and $R(X,\xi)X_i = \beta\,(X\wedge_g \xi)X_i$, are satisfied, where α and β are functions on M, $X_i \in \{X,Y,\xi\}$, then M is pseudo-symmetric.*

Proof. Any semi-Riemannian manifold (M,g), $n = 3$, satisfying the condition (**C**) is quasi-Einstein. Thus it is also pseudo-symmetric. □

We remark that the 3-dimensional Sasakian manifolds satisfy the condition (**C**).

In the next section we present examples of 3-dimensional manifolds satisfying the condition (**C**) with nonconstant functions α and β and examples of pseudo-symmetric manifolds which are not D'Atri spaces.

2. Warped products

Let $(M_1,\overline{g})$ and $(M_2,\widetilde{g})$, $\dim M_1 = p$, $\dim M_2 = n-p$, $1 \le p < n$, be semi-Riemannian manifolds covered by systems of charts $\{U;x^a\}$ and $\{V;y^\alpha\}$, respectively. Let F be a positive smooth function on M_1. The warped product

$M_1 \times_F M_2$ of $(M_1, \overline{g})$ and $(M_2, \widetilde{g})$ is the product manifold $M_1 \times M_2$ with the metric $g = \overline{g} \times_F \widetilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1) \pi_2^* \widetilde{g}$, where $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, being the natural projections ([18]). The local components g_{rs} of the metric $g = \overline{g} \times_F \widetilde{g}$, which may not vanish identically are the following: $g_{ab} = \overline{g}_{ab}$, $g_{\alpha\beta} = F \widetilde{g}_{\alpha\beta}$, where $a, b, c, d, e, f \in \{1, \cdots, p\}$, $\alpha, \beta, \gamma, \delta \in \{p+1, \cdots, n\}$ and $r, s, t, u, v, w \in \{1, \cdots, n\}$. The local components Γ^r_{st} of the Levi-Civita connection ∇ of $M_1 \times_F M_2$ are $\Gamma^a_{bc} = \overline{\Gamma}^a_{bc}, \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma}$, $\Gamma^a_{\alpha\beta} = -\frac{1}{2} \overline{g}^{ab} F_b \widetilde{g}_{\alpha\beta}, \Gamma^\alpha_{a\gamma} = \frac{1}{2F} F_a \delta^\alpha_\beta$ $\Gamma^a_{\alpha b} = \Gamma^\alpha_{ab} = 0$. Now the local components $R_{rstu} = g_{rw}(\partial_u \Gamma^w_{st} - \partial_t \Gamma^w_{su} + \Gamma^v_{st}\Gamma^w_{vu} - \Gamma^v_{su}\Gamma^w_{vt})$, $\partial_u = \frac{\partial}{\partial x^u}$, of the tensor R and the local components S_{ts} of the tensor S of $M_1 \times_F M_2$ which may not vanish identically are the following ([11]):

$$R_{abcd} = \overline{R}_{abcd}\,, \tag{4}$$

$$R_{\alpha ab\beta} = -\frac{1}{2} T_{ab} \widetilde{g}_{\alpha\beta}\,, \tag{5}$$

$$R_{\alpha\beta\gamma\delta} = F \widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4} \widetilde{G}_{\alpha\beta\gamma\delta}\,, \tag{6}$$

$$S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F} T_{ab}\,, \tag{7}$$

$$S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \frac{1}{2}\left(tr(T) + \frac{n-p-1}{2F} \Delta_1 F \right) \widetilde{g}_{\alpha\beta}\,, \tag{8}$$

$$T_{ab} = \nabla_b F_a - \frac{F_a F_b}{2F}\,,\ tr(T) = \overline{g}^{ab} T_{ab}\,,\ \Delta_1 F = \overline{g}^{ab} F_a F_b\,, \tag{9}$$

$$\nabla_b F_a = \partial_b F_a - \Gamma^d_{ab} F_d\,. \tag{10}$$

The scalar curvature κ of the manifold $M_1 \times_F M_2$ is given by

$$\kappa = \overline{\kappa} + \frac{1}{F}\widetilde{\kappa} - \frac{n-p}{F}\left(tr(T) + \frac{n-p-1}{4F} \Delta_1 F \right), \tag{11}$$

where $\overline{\kappa}$ and $\widetilde{\kappa}$ are the scalar curvature of $(M_1, \overline{g})$ and $(M_2, \widetilde{g})$, respectively.

Example 2.1. We consider the warped product $M_1 \times_F M_2$ of a 1-dimensional manifold $(M_1, \overline{g})$, $\overline{g}_{11} = \varepsilon = \pm 1$ and a 2-dimensional manifold $(M_2, \widetilde{g})$ with the warping function F. The local components of the Riemannian curvature tensor R and the Ricci tensor S of $M_1 \times_F M_2$ which may not vanish identically are the following

$$\begin{aligned} R_{1\alpha\beta 1} &= -\frac{1}{2} T_{11}\, \widetilde{g}_{\alpha\beta} = -\frac{tr(T)}{2F}\, \overline{g}_{11} g_{\alpha\beta} = -\frac{tr(T)}{2F}\, G_{1\alpha\beta 1}\,, \\ R_{\alpha\beta\gamma\delta} &= \frac{1}{F}(\frac{\widetilde{\kappa}}{2} - \frac{\triangle_1 F}{2F})\, G_{\alpha\beta\gamma\delta}\,, \\ S_{11} &= -\frac{tr(T)}{2F} g_{11}\,, \\ S_{\alpha\beta} &= \left(-\frac{tr(T)}{2F} + \frac{1}{2F}(\frac{\widetilde{\kappa}}{2} - \frac{\triangle_1 F}{2F}) \right) g_{\alpha\beta}\,, \end{aligned}$$

where $\triangle_1 F = \overline{g}^{11}(F_1)^2$, $tr(T) = \frac{1}{2F}\,\overline{g}^{11}T_{11}$, $T_{11} = \nabla_1 F_1 - \frac{1}{2F}\,(F_1)^2$, $F_1 = \frac{\partial F}{\partial x^1}$, and $\alpha, \beta, \gamma, \delta \in \{2, 3\}$. From the above formulas we get

$$S \;=\; \left(-\frac{tr(T)}{2F} + \frac{1}{2F}\,(\frac{\widetilde{\kappa}}{2} - \frac{\triangle_1 F}{2F})\right) g - \frac{\varepsilon}{2F}\,(\frac{\widetilde{\kappa}}{2} - \frac{\triangle_1 F}{2F})\,\omega \otimes \omega\,,$$

where $\omega_h = \delta^1_h$ are the local components of the 1-form ω. Thus $M_1 \times_F M_2$ is a quasi-Einstein manifold, and in a consequence of Theorem A, a pseudo-symmetric manifold.

Example 2.2. Let on the set $M_1 = \{(x^1, x^2) \in \mathbb{R}^2,\ x^1 \in (0, \frac{\pi}{2})\}$ be given a metric tensor $\overline{g}$, defined by $\overline{g}_{11} = a^2$, $\overline{g}_{12} = \overline{g}_{21} = 0$, $\overline{g}_{22} = a^2 \cos^2 x^1$, where $a = const. > 0$. It is easy to verify that the local components T_{ab}, $a, b \in \{1, 2\}$, of the tensor $T = \nabla^2 F - \frac{1}{2F}\,dF \otimes dF$, where F is defined by $F = F(x^1, x^2) = \cos^2 x^1$, are the following: $T_{11} = -\frac{2}{a^2}\cos^2 x^1\,\overline{g}_{11}$, $T_{12} = T_{21} = 0$, $T_{22} = \frac{2}{a^2}\sin^2 x^1\,\overline{g}_{22}$. Furthermore, we also have $\frac{1}{a^2} = \frac{\overline{\kappa}}{2}$. The local components of the curvature tensor R of $M_1 \times_F M_2$, which may not vanish identically, are the following $R_{abcd} = \frac{\overline{\kappa}}{2}\,G_{abcd}$, $R_{3ab3} = -\frac{1}{2F}\,T_{ab}g_{33}$. Using now the above relations we find

$$\begin{aligned}
R_{3113} &= -\frac{1}{2F}\,T_{11}g_{33} = -\frac{1}{2F}(-\frac{2}{a^2})F\,\overline{g}_{11}g_{33} = \frac{\overline{\kappa}}{2}\,G_{3113}\,,\\
R_{3123} &= -\frac{1}{2F}\,T_{12}g_{33} = 0,\\
R_{3223} &= -\frac{1}{2F}\,T_{22}g_{33} = -\frac{1}{2F}(\frac{2}{a^2})\sin^2 x^1\,\overline{g}_{22}g_{33} = -\frac{1-F}{F}\frac{\overline{\kappa}}{2}\,G_{3223}\,,\\
S_{11} &= \frac{\overline{\kappa}}{2}\overline{g}_{11} - \frac{1}{2F}\,T_{11} = \frac{2}{a^2}\,\overline{g}_{11} = \frac{2}{a^2}\,g_{11}\,,\\
S_{22} &= \frac{\overline{\kappa}}{2}\overline{g}_{22} - \frac{1}{2F}\,T_{22} = \frac{1}{a^2}\,(1 - \tan^2 x^1)\,\overline{g}_{22} = \frac{1}{a^2}\,(1 - \tan^2 x^1)\,g_{22}\,,\\
S_{33} &= \frac{tr(T)}{2}\widetilde{g}_{33} = \frac{1}{a^2}\,(1 - \tan^2 x^1)\,g_{33}\,.
\end{aligned}$$

From the above formulas we get

$$S \;=\; \frac{1}{a^2}\,(1 - \tan^2 x^1)\,g + (\frac{2}{a^2} - \frac{1}{a^2}\,(1 - \tan^2 x^1)a^2)\,\omega \otimes \omega\,,$$

where $\omega_h = \delta^1_h$ are the local components of a 1-form w. Thus $M_1 \times_F M_2$ is a quasi-Einstein manifold, and in a consequence of Theorem A, a pseudo-symmetric manifold.

3. 3-dimensional Thurston geometries

A model geometry (G, M) is a manifold M together with a Lie group G of diffeomorphisms of M such that ([22]): M is connected and simply connected, G

acts transitively on M, with compact point stabilizers, G is not contained in any larger group of diffeomorphisms of M with compact stabilizers of points, and there exists at least one compact manifold of type (modeled on) (G, M).

W. M. Thurston classified the 3-dimensional geometries which are $\mathbb{R}^3$, $\mathrm{S}^3(c)$, $\mathrm{H}^3(-c)$, $\mathrm{S}^2 \times \mathbb{R}$, $\mathrm{H}^2(-c) \times \mathbb{R}$, $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$, H_3 and the Lie group Sol ([21], [22]). The Lie group Sol is considered as $\mathbb{R}^3$ endowed with the left invariant metric $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$. Geometric properties of the eight 3-dimensional Thurston geometries were studied, among others, in [6] and [17].

It is known that the Lie group Sol admit the following family of metrics

$$g = g[\mu_1, \mu_2, \mu_3] = e^{-2\mu_1 z}dx^2 + e^{-2\mu_2 z}dy^2 + \mu_3^2 dz^2 \,,$$

where μ_1, μ_2 and μ_3 are real constants and μ_3 is positive. Recently, these metrics have been studied by J. Inoguchi ([15]). The Riemannian curvature R and the Ricci curvatures and the scalar curvature of such metrics are the following ([15]):

$$R_{1212} = -\frac{\mu_1\mu_2}{\mu_3^2},\ R_{1313} = -\frac{\mu_1^2}{\mu_3^2},\ R_{2323} = -\frac{\mu_2^2}{\mu_3^2}\,,$$

$$\rho_1 = -\frac{\mu_1(\mu_1+\mu_2)}{\mu_3^2},\ \rho_2 = -\frac{\mu_2(\mu_1+\mu_2)}{\mu_3^2},\ \rho_3 = -\frac{\mu_1^2+\mu_2^2}{\mu_3^2}, \kappa = -\frac{2}{\mu_3^2}(\mu_1^2+\mu_2^2+\mu_1\mu_2)\,,$$

respectively. We have,

Proposition 8. *The Riemannian manifold* $M^3 = (\mathbb{R}^3, g[\mu_1, \mu_2, \mu_3])$ *is pseudo-symmetric if and only if* $\mu_1 = 0$ *or* $\mu_2 = 0$ *or* $\mu_1 = \pm\mu_2$.

Proof. Our assertion is an immediate consequence of Proposition B and the relation

$$(\rho_1 - \rho_2)(\rho_1 - \rho_3)(\rho_2 - \rho_3) \ = \ \mu_1\mu_2(\mu_1 + \mu_3)(\mu_1 - \mu_2)^3\,,$$

which holds on M^3.
From the last proposition it follows that the hyperbolic 3-space $M^3 = H^3(-c^2)$ ($\mu_1 = \mu_2 = c \neq 0$), the 4-symmetric space M^3 ($\mu_1 + \mu_2 = 0$ and $|\mu_i| = \frac{1}{2}$), and the warped products $(N^2, dx^2 + \mu_3^2 dz^2) \times_{e^{-2\mu_2 z}} \mathbb{R}$ and $(N^2, dy^2 + \mu_3^2 dz^2) \times_{e^{+2\mu_2 z}} \mathbb{R}$ ($\mu_1\mu_2 = 0$) are pseudo-symmetric manifolds. □
Thus we have,

Corollary 9. *Every* 3*-dimensional Thurston's geometry is pseudo-symmetric.*

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