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Sensitivity Analysis for Generalized Nonlinear Implicit Quasi-variational Inclusions

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Abstract. In this paper, by using the concept of the resolvent operator, we study the behavior and sensitivity analysis of the solution set for a new class of parametric generalized nonlinear implicit quasi-variational inclusion problem in $L_p(p \geq 2)$ spaces. The results presented in this paper are new and generalize many known results in this field.

1. Introduction

In recent years, variational inequalities have been extended and generalized in different directions using novel and innovative techniques both for its own sake and for its applications. A useful and important generalization of variational inequalities is called variational inclusions. In 1994, using the concept and technique of resolvent operators, Hassouni and Moudafi[8] introduced and studied a class of mixed type variational inequalities with single-valued mappings which was called variational inclusions. Adly[1], Ding[6], Huang[9], Kazmi[10], Noor[16] and Noor, Noor and Rassias[17] have obtained some important extensions and generalizations of the results in [8] from various different directions.

Sensitivity analysis of solutions of variational inequalities with single-valued mappings have been studied by many authors via quite different techniques. By using the projection method, Dafermos[5], Yen[20], Mukherjee and Verma[13], Noor[15], and Pan[18] studied the sensitivity analysis of solutions of some variational inequalities with single-valued mappings in finite-dimensional spaces and Hilbert spaces.

In 1999, Ding and Luo[7] studied the behavior and sensitivity analysis of the solution set for a class of parametric generalized quasi-variational inequalities with set-valued mapping by using the projection method of Dafermos[5] in a Hilbert space. The projection method cannot be used to study the behavior and sensitivity analysis of the solution set for variational inequalities with the nonlinear term. By using the concept of the resolvent operator, Park and Jeong[19] dealt with the sensitivity analysis of the solution set for a class of parametric generalized mixed

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variational inequalities with set-valued mappings.

In this paper, we study the behavior and sensitivity analysis of the solution set for a class of parametric generalized nonlinear implicit quasi-variational inclusion problem in $L_P(p \geq 2)$ spaces. The results presented in this paper generalize, improve and unify the corresponding results of Dafermos[5], Yen[20], Mukherjee and Verma[13], Noor[15], Ding and Luo[7] and Park and Jeong[19].

2. Preliminaries

Let E be a real Banach space, E^* be the topological dual space of E, $CB(E)$ be a family of nonempty bounded closed subsets of E, $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\},$$

where $d(x,B) = \inf_{y\in B}\|x-y\|$, $d(A,y) = \inf_{x\in A}\|x-y\|$, $\langle\cdot,\cdot\rangle$ be the dual pair between E and E^*, $D(T)$ denotes the domain of T and $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|\}, \quad \forall x \in E.$$

Let $M : D(M) \subset E \to 2^E$ be a set-valued mapping. The mapping M is said to be accretive([2]) if for any $x, y \in D(M)$, $u \in M(x)$, $v \in M(y)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0.$$

The mapping M is said to be m-accretive if M is accretive and $(I+\rho M)(D(M)) = E$ for every $\rho > 0$, where I is the identity mapping.

If $M : D(M) \subset E \to 2^E$ is an m-accretive mapping, then for a constant $\rho > 0$, the resolvent operator associated with M is defined by

$$R_\rho^M(u) = (I+\rho M)^{-1}(u), \quad \forall u \in D(M),$$

where I is the identity operator. It is well known that R_ρ^M is a single-valued and nonexpansive mapping([2]).

We consider now the parametric generalized nonlinear implicit quasi-variational inclusion problem in Banach spaces. To this end, let Ω be a nonempty open subset of E in which the parameter λ takes values, $N : E\times E\times\Omega \to E$, $m : E\times\Omega \to E$ be single-valued mappings and $A, B, C, D, G : E\times\Omega \to CB(E)$ be set-valued mappings. Let $M : E\times E\times\Omega \to 2^E$ be a set-valued mapping such that for each given $(z,\lambda) \in E\times\Omega$, $M(\cdot, z, \lambda) : E \to 2^E$ is an m-accretive mapping with $(G(E,\lambda) - m(E,\lambda)) \cap \mathrm{dom}M(\cdot,z,\lambda) \neq \phi$. For each fixed $\lambda \in \Omega$, the parametric generalized nonlinear implicit quasi-variational inclusion problem in Banach spaces(PGNIQVIP) consists of finding $x \in E$, $u \in A(x,\lambda)$, $v \in B(x,\lambda)$, $w \in C(x,\lambda)$, $z \in D(x,\lambda)$, $s \in G(x,\lambda)$ such that

$$0 \in M(s - m(w,\lambda), z, \lambda) + N(u,v,\lambda). \tag{2.1}$$

Special Cases

I. Let $E = H$ be a Hilbert space, Ω be a nonempty open subset of H in which the parameter λ takes values. Let $\phi : H \times H \times \Omega \to R \cup \{+\infty\}$ be a functional such that for each $(z,\lambda) \in H \times \Omega$, $\partial\phi(\cdot, z, \lambda)$ denotes the subdifferential of a proper convex lower semicontinuous function ϕ with $G(H,\lambda) \cap \mathrm{dom}(\partial\phi(\cdot,z,\lambda)) \neq \phi$. Let $M(\cdot,z,\lambda) = \partial\phi(\cdot,z,\lambda)$ for all $(z,\lambda) \in H \times \Omega$ and $m(x,\lambda) = 0$ for all $(x,\lambda) \in H \times \Omega$. Then problem (2.1) is equivalent to finding $x \in H$, $u \in A(x,\lambda)$, $v \in B(x,\lambda)$, $z \in D(x,\lambda)$, $s \in G(x,\lambda)$ such that

$$\langle N(u,v,\lambda), y - s\rangle \geq \phi(s,z,\lambda) - \phi(y,z,\lambda), \quad y \in H. \tag{2.2}$$

II. If $N(u,v,\lambda) = u - v$, $G = g : E \times \Omega \to E$ is a single-valued mapping, and $\phi(\cdot,z,\lambda) = \phi(\cdot)$ for each $(z,\lambda) \in H \times \Omega$, then problem (2.2) reduces to the problem of finding $x \in H$, $u \in A(x,\lambda)$, $v \in B(x,\lambda)$ such that

$$\langle u - v, y - g(x,\lambda)\rangle \geq \phi(g(x,\lambda)) - \phi(y), \quad \forall y \in H. \tag{2.3}$$

Problem (2.3) is known as the parametric generalized mixed variational inequality problem and has been studied by Park and Jeong[19].

III. Let $K : H \times \Omega \to 2^H$ be a set-valued mapping with nonempty closed convex values and for each fixed $\lambda \in \Omega$, $\phi(\cdot) = I_{K(\cdot,\lambda)}(\cdot)$ is the indicator function of $K(\cdot,\lambda)$. Then problem (2.3) reduces to the problem of finding $x \in H$, $u \in A(x,\lambda)$, $v \in B(x,\lambda)$ such that $g(x,\lambda) \in K(x,\lambda)$ and

$$\langle u - v, y - g(x,\lambda)\rangle \geq 0, \quad \forall y \in K(x,\lambda). \tag{2.4}$$

Problem (2.4) is known as the parametric generalized quasi-variational inequality problem and has been studied by Ding and Luo[7].

Summing up the above arguments, it shows that for a suitable choice of the mappings A, B, C, D, G, N, m, M, we can obtain a number of known and new classes of parametric variational inequalities, parametric variational inclusions and the corresponding optimization problems from the parametric generalized nonlinear implicit quasi-variational inclusion problem in Banach spaces (2.1).

Definition 2.1. A set-valued mapping $G : E \times \Omega \to 2^E$ is said to be

(i) δ-strongly accretive, $\delta \in (0,1)$, if for any $x, y \in E$, $\lambda \in \Omega$, there exists $j(x-y) \in J(x-y)$ such that for any $u \in G(x,\lambda)$, $v \in G(y,\lambda)$,

$$\langle u - v, j(x-y)\rangle \geq \delta\|x-y\|^2,$$

(ii) λ_G-Lipschitz continuous if there exists a constant $\lambda_G > 0$ such that

$$H(G(x,\lambda), G(y,\lambda)) \le \lambda_G \|x-y\|, \quad \forall (x,y,\lambda) \in E \times E \times \Omega.$$

Definition 2.2. Let $A : E \times \Omega \to CB(E)$ be a set-valued mapping and $N : E \times E \times \Omega \to E$ be a single-valued mapping. Then

(i) N is said to be α-strongly monotone with respect to A in the first argument if there exists a constant $\alpha > 0$ such that

$$\begin{aligned}&\langle N(u_1,v,\lambda) - N(u_2,v,\lambda), x-y\rangle \ge \alpha\|x-y\|^2,\\ &\forall (x,y,v,\lambda) \in E\times E\times E\times\Omega, \quad u_1 \in A(x,\lambda),\ u_2 \in A(y,\lambda).\end{aligned}$$

(ii) N is said to be β-Lipschitz continuous in the first argument if there exists a cnstant $\beta > 0$ such that

$$\|N(u_1,v,\lambda) - N(u_2,v,\lambda)\| \le \beta\|u_1-u_2\|, \quad \forall (u_1,u_2,v,\lambda) \in E\times E\times E\times\Omega.$$

In a similar way, we can define the ξ-Lipschitz continuity of $N(u,v,\lambda)$ in the second argument.

Lemma 2.1 (Chidume [3], [4]). *Let $E = L_p$(or l_p), $2 \le p < \infty$. For any $x, y \in E$, we have*

$$\|x+y\|^2 \le (p-1)\|x\|^2 + \|y\|^2 + 2\langle x, j(y)\rangle, \quad \forall j \in J(x+y).$$

For the rest of this paper, the single-valued duality map is denoted by j.

Lemma 2.2 (Lim [12]). *Let (X,d) be a complete metric space and $T_1, T_2 : X \to CB(X)$ be two set-valued contractive mapping with same contractive constant $\theta \in (0,1)$, i.e.,*

$$H(T_i(x), T_i(y)) \le \theta d(x,y), \quad \forall x,y \in X, i = 1,2.$$

Then

$$H(F(T_1), F(T_2)) \le \frac{1}{1-\theta}\sup_{x\in X} H(T_1(x), T_2(x)),$$

where $F(T_1)$ and $F(T_2)$ are fixed point sets of T_1, T_2, respectively.

3. Sensitivity analysis of solution set

We first transfer the (PGNIQVIP)(2.1) into a parametric fixed point problem.

Theorem 3.1. *Fixed $\lambda \in \Omega$, $x \in E$, $u \in A(x,\lambda)$, $v \in B(x,\lambda)$, $w \in C(x,\lambda)$,*

$z \in D(x,\lambda)$, $s \in G(x,\lambda)$ is a solution of $(PGNIQVIP)(2.1)$ if and only if for some given $\rho > 0$, the set-valued mapping $F : E \times \Omega \to 2^E$ defined by

(3.1)
$$F(x,\lambda) = \bigcup_{\substack{u\in A(x,\lambda),v\in B(x,\lambda),w\in C(c,\lambda)\\ z\in D(x,\lambda),s\in G(x,\lambda)}} \big[x - s + m(w,\lambda) + R_{M(\cdot,z,\lambda)}(s - m(w,\lambda) - \rho N(u,v,\lambda))\big]$$

has a fixed point x.

Proof. For each fixed $\lambda \in \Omega$, let (x,u,v,w,z,s) be a solution of (PGNIQVIP) (2.1). Then $x \in E$, $u \in A(x,\lambda)$, $v \in B(x,\lambda)$, $w \in C(x,\lambda)$, $z \in D(x,\lambda)$, $s \in G(x,\lambda)$ such that

$$0 \in M(s - m(w,\lambda), z, \lambda) + N(u,v,\lambda).$$

The relation holds if and only if

$$s - m(w,\lambda) - \rho N(u,v,\lambda) \in (I + \rho M(\cdot,z,\lambda))(s - m(w,\lambda)).$$

That is,

$$R_{M(\cdot,z,\lambda)}[s - m(w,\lambda) - \rho N(u,v,\lambda)] = s - m(w,\lambda).$$

The equality holds if and only if

$$\begin{aligned} x &= x - s + m(w,\lambda) + R_{M(\cdot,z,\lambda)}[s - m(w,\lambda) - \rho N(u,v,\lambda)] \\ &\in \bigcup_{\substack{u\in A(x,\lambda),v\in B(x,\lambda),w\in C(x,\lambda)\\ z\in D(x,\lambda),s\in G(x,\lambda)}} [x - s + m(w,\lambda) + R_{M(\cdot,z,\lambda)}(s - m(w,\lambda) - \rho N(u,v,\lambda))] \\ &= F(x,\lambda). \end{aligned}$$

□

Theorem 3.2. *Let $E = L_p$ (or l_p), $2 \le p < \infty$. Let $A, B, C, D, G : E\times\Omega \to CB(E)$ be set-valued mappings such that A, B, C, D and G are Lipschitz continuous with constants $\lambda_A, \lambda_B, \lambda_C, \lambda_D, \lambda_G$, respectively, and $G : E \times \Omega \to CB(E)$ be δ-strongly accretive. Let $N : E \times E \times \Omega \to E$ be α-strongly monotone with respect to A in the first argument, β-Lipschitz continuous in the first argument and ξ-Lipschitz continuous in the second argument. Let $m : E \times \Omega \to E$ be η-Lipschitz continuous. Let $M : E\times E\times\Omega \to 2^E$ be such that for each fixed $(z,\lambda) \in E\times\Omega$, $M(\cdot,z,\lambda) : E \to 2^E$ is an m-accretive mapping satisfying $G(E,\lambda) - m(E,\lambda) \cap domM(\cdot,z,\lambda) \neq \phi$. Suppose that for any $(x,y,z,\lambda) \in E \times E \times E \times \Omega$,*

$$\|R_{M(\cdot,x,\lambda)}(z) - R_{M(\cdot,y,\lambda)}(z)\| \le \mu\|x - y\| \tag{3.2}$$

and there exists a constant $\rho > 0$ such that

$$\begin{aligned} &k = 2\sqrt{1 - 2\delta + (p-1)\lambda_G^2} + (\mu\lambda_D + 2\eta\lambda_C)(1+\varepsilon), \\ &\sqrt{p-1}\lambda_A\beta > \xi\lambda_B, \quad k + \rho\xi(1+\varepsilon)\lambda_B < 1, \\ &\alpha > (1-k)\xi\lambda_B + \sqrt{[(p-1)\lambda_A^2\beta^2 - \xi^2\lambda_B^2](2k - k^2)}, \end{aligned}$$

$$\left|\rho - \frac{\alpha-(1-k)\xi\lambda_B(1+\varepsilon)}{[(p-1)\beta^2\lambda_A^2-\xi^2\lambda_B^2](1+\varepsilon)^2}\right| < \frac{\sqrt{[\alpha-(1-k)\xi\lambda_B]^2-[(p-1)\lambda_A^2\beta^2-\xi^2\lambda_B^2](2k-k^2)}}{(p-1)\beta^2\lambda_A^2-\xi^2\lambda_B^2}. \tag{3.3}$$

Then

(1) *the set-valued mapping* $F: E\times\Omega\to 2^E$ *defined by* (3.1) *is a uniform* θ*-*H*-set-valued contractive mapping with respect to* $\lambda\in\Omega$*, where* $\theta = k+t(\rho)<1$, $t(\rho)=\sqrt{1-2\rho\alpha+(p-1)\rho^2\beta^2(1+\varepsilon)^2\lambda_A^2}+\rho\xi(1+\varepsilon)\lambda_B$.

(2) *for each* $\lambda\in\Omega$*, the* (*PGNIQVIP*)(2.1) *has nonempty solution set* $S(\lambda)$ *and* $S(\lambda)$ *is a closed subset in* E.

Proof. (1). By the definition of F, for any $(x,\lambda),(y,\lambda)\in E\times\Omega$, $a\in F(x,\lambda)$, there exist $u_1\in A(x,\lambda)$, $v_1\in B(x,\lambda)$, $w_1\in C(x,\lambda)$, $z_1\in D(x,\lambda)$, $s_1\in G(x,\lambda)$ such that

$$a = x-s_1+m(w_1,\lambda)+R_{M(\cdot,z_1,\lambda)}(s_1-m(w_1,\lambda)-\rho N(u_1,v_1,\lambda)).$$

Note that $A(y,\lambda),B(y,\lambda),C(y,\lambda),D(y,\lambda),G(y,\lambda)\in CB(E)$, there exist $u_2\in A(y,\lambda)$, $v_2\in B(y,\lambda)$, $w_2\in C(y,\lambda)$, $z_2\in D(y,\lambda)$ and $s_2\in G(y,\lambda)$ such that

$$\begin{aligned}
\|u_1-u_2\| &\leq (1+\varepsilon)H(A(x,\lambda),A(y,\lambda)),\\
\|v_1-v_2\| &\leq (1+\varepsilon)H(B(x,\lambda),B(y,\lambda)),\\
\|w_1-w_2\| &\leq (1+\varepsilon)H(C(x,\lambda),C(y,\lambda)),\\
\|z_1-z_2\| &\leq (1+\varepsilon)H(D(x,\lambda),D(y,\lambda)),\\
\|s_1-s_2\| &\leq (1+\varepsilon)H(G(x,\lambda),G(y,\lambda)).
\end{aligned}$$

Let

$$b = y-s_2+m(w_2,\lambda)+R_{M(\cdot,z_2,\lambda)}(s_2-m(w_2,\lambda)-\rho N(u_2,v_2,\lambda)).$$

Then we have $b\in F(y,\lambda)$. It follows that

$$\begin{aligned}
\|a-b\| \leq\ & \|x-y-(s_1-s_2)\|+\|m(w_1,\lambda)-m(w_2,\lambda)\|\\
&+\|R_{M(\cdot,z,\lambda)}(s_1-m(w_1,\lambda)-\rho N(u_1,v_1,\lambda))\\
&-R_{M(\cdot,z_2,\lambda)}(s_2-m(w_2,\lambda)-\rho N(u_2,v_2,\lambda))\|.
\end{aligned} \tag{3.4}$$

Since G is δ-strongly accretive and λ_G-Lipschitz continuous, we have

$$\begin{aligned}
&\|x-y-(s_1-s_2)\|^2\\
\leq\ & (p-1)\|s_1-s_2\|^2+\|x-y\|^2-2\langle s_1-s_2, j(x-y)\rangle\\
\leq\ & [(p-1)\lambda_G^2+1-2\delta]\|x-y\|^2.
\end{aligned} \tag{3.5}$$

By the Lipschitz continuity of $R_{M(\cdot,z_1,\lambda)}$ and condition (3.2), we have

$$
\begin{aligned}
(3.6)\quad & \|R_{M(\cdot,z_1,\lambda)}(s_1 - m(w_1,\lambda) - \rho N(u_1,v_1,\lambda)) \\
& \quad - R_{M(\cdot,z_2,\lambda)}(s_2 - m(w_2,\lambda) - \rho N(u_2,v_2,\lambda))\| \\
\leq\ & \|R_{M(\cdot,z_1,\lambda)}(s_1 - m(w_1,\lambda) - \rho N(u_1,v_1,\lambda)) \\
& \quad - R_{M(\cdot,z_1,\lambda)}(s_2 - m(w_2,\lambda) - \rho N(u_2,v_2,\lambda))\| \\
& \quad + \|R_{M(\cdot,z_1,\lambda)}(s_2 - m(w_2,\lambda) - \rho N(u_2,v_2,\lambda)) \\
& \quad - R_{M(\cdot,z_2,\lambda)}(s_2 - m(w_2,\lambda) - \rho N(u_2,v_2,\lambda))\| \\
\leq\ & \|s_1 - m(w_1,\lambda) - \rho N(u_1,v_1,\lambda) - (s_2 - m(w_2,\lambda) - \rho N(u_2,v_2,\lambda))\| \\
& \quad + \mu\|z_1 - z_2\| \\
\leq\ & \|x - y - (s_1 - s_2)\| + \|x - y - \rho(N(u_1,v_1,\lambda) - N(u_2,v_1,\lambda))\| \\
& \quad + \rho\|N(u_2,v_1,\lambda) - N(u_2,v_2,\lambda)\| + \|m(w_1,\lambda) - m(w_2,\lambda)\| + \mu\|z_1 - z_2\|.
\end{aligned}
$$

Since $N(u,v,\lambda)$ is α-strongly accretive with respect to A and β-Lipschitz continuous in the first argument and A is λ_A-Lipschitz continuous, we have

$$
\begin{aligned}
(3.7)\quad & \|x - y - \rho(N(u_1,v_1,\lambda) - N(u_2,v_1,\lambda))\|^2 \\
\leq\ & (p-1)\rho^2\|N(u_1,v_1,\lambda) - N(u_2,v_1,\lambda)\|^2 + \|x-y\|^2 \\
& \quad - 2\rho\langle N(u_1,v_1,\lambda) - N(u_2,v_1,\lambda), j(x-y)\rangle \\
\leq\ & (p-1)\rho^2\beta^2(1+\varepsilon)^2[H(A(x,\lambda),A(y,\lambda))]^2 + \|x-y\|^2 - 2\rho\alpha\|x-y\|^2 \\
=\ & [(p-1)\rho^2\beta^2(1+\varepsilon)^2\lambda_A^2 + 1 - 2\rho\alpha]\|x-y\|^2.
\end{aligned}
$$

Using ξ-Lipschitz continuity of $N(u,v,\lambda)$ in the second argument, λ_B-Lipschitz continuity of B, λ_D-Lipschitz continuity of D, η-Lipschitz continuity of m and λ_G-Lipschitz continuity of C, we have

$$
\begin{aligned}
(3.8)\quad \|N(u_2,v_1,\lambda) - N(u_2,v_2,\lambda)\| &\leq \xi\|v_1 - v_2\| \\
&\leq \xi(1+\varepsilon)H(B(x,\lambda),B(y,\lambda)) \\
&\leq \xi(1+\varepsilon)\lambda_B\|x-y\|,
\end{aligned}
$$

$$
\begin{aligned}
(3.9)\quad \|z_1 - z_2\| &\leq (1+\varepsilon)H(D(x,\lambda),D(y,\lambda)) \\
&\leq (1+\varepsilon)\lambda_D\|x-y\|,
\end{aligned}
$$

$$
\begin{aligned}
(3.10)\quad \|m(w_1,\lambda) - m(w_2,\lambda)\| &\leq \eta\|w_1 - w_2\| \\
&\leq \eta(1+\varepsilon)H(C(x,\lambda),C(y,\lambda)) \\
&\leq \eta(1+\varepsilon)\lambda_C\|x-y\|.
\end{aligned}
$$

By (3.4)-(3.10), we obtain

$$\begin{aligned}\|a-b\| &\leq [2\sqrt{1-2\delta+(p-1)\lambda_G^2}+\sqrt{1-2\rho\alpha+(p-1)\rho^2\beta^2(1+\varepsilon)^2\lambda_A^2} \\ &\quad +\rho\xi(1+\varepsilon)\lambda_B+\mu(1+\varepsilon)\lambda_D+2\eta(1+\varepsilon)\lambda_C]\|x-y\| \\ &= (k+t(\rho))\|x-y\| \\ &= \theta\|x-y\|,\end{aligned}$$

where

$$\begin{aligned}k &= 2\sqrt{1-2\delta+(p-1)\lambda_G^2}+\mu(1+\varepsilon)\lambda_D+2\eta(1+\varepsilon)\lambda_C, \\ t(\rho) &= \sqrt{1-2\rho\alpha+(p-1)\rho^2\beta^2(1+\varepsilon)^2\lambda_A^2}+\rho\xi(1+\varepsilon)\lambda_B,\end{aligned}$$

and $\theta = k+t(\rho)$. It follows from condition (3.3) that $\theta < 1$. Hence, we have

$$\begin{aligned}d(a, F(y,\lambda) &= \inf_{b\in F(y,\lambda)} \|a-b\| \\ &\leq \theta\|x-y\|.\end{aligned}$$

Since $a \in F(x,\lambda)$ is arbitrary, we obtain

$$\sup_{a\in F(x,\lambda)} d(a, F(y,\lambda)) \leq \theta\|x-y\|.$$

By using same argument, we can prove

$$\sup_{b\in F(y,\lambda)} d(F(x,\lambda), b) \leq \theta\|x-y\|.$$

By the definition of the Hausdorff metric H on $CB(E)$, we obtain that for all $(x,y,\lambda) \in E\times E\times\Omega$,

$$H(F(x,\lambda), F(y,\lambda)) \leq \theta\|x-y\|,$$

i.e., $F(x,\lambda)$ is a set-valued contractive mapping which is uniform with respect to $\lambda\in\Omega$.

(2). Since $F(x,\lambda)$ is a uniform θ-H-contractive mapping with respect to $\lambda\in\Omega$, by the Nadler fixed point theorem[14], $F(x,\lambda)$ has a fixed point x for each $\lambda\in\Omega$. By Theorem 3.1, $S(\lambda)\neq\phi$.

For each $\lambda\in\Omega$, let $\{x_n\}\subset S(\lambda)$ and $x_n\to x_0$ as $n\to\infty$. Then we have $x_n\in F(x_n,\lambda)$, $n=1,2,\cdots$. By (1), we have

$$H(F(x_n,\lambda), F(x_0,\lambda)) \leq \theta\|x_n-x_0\|.$$

It follows that

$$\begin{aligned}d(x_0, F(x_0,\lambda)) &\leq \|x_0-x_n\|+d(x_n, F(x_n,\lambda))+H(F(x_n,\lambda), F(x_0,\lambda)) \\ &\leq (1+\theta)\|x_n-x_0\|\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Hence, we have $x_0 \in F(x_0, \lambda)$ and $x_0 \in S(\lambda)$. Therefore, $S(\lambda)$ is a nonempty closed subset of E.

Theorem 3.3. *Under the hypotheses of Theorem 3.2, further assume*

(i) *for any $x \in E$, the mappings $\lambda \mapsto A(x,\lambda)$, $\lambda \mapsto B(x,\lambda)$, $\lambda \mapsto C(x,\lambda)$, $\lambda \mapsto D(x,\lambda)$, $\lambda \mapsto G(x,\lambda)$ and $\lambda \mapsto m(x,\lambda)$ are Lipschitz continuous with Lipschitz constants l_A, l_B, l_C, l_D, l_G and l_m, respectively.*

(ii) *for any $u, v, z, w \in E$, $\lambda \mapsto N(u,v,\lambda)$ and $\lambda \mapsto R_{M(\cdot,z,\lambda)}(w)$ are Lipschitz continuous with Lipschitz constants l_N and l_R, respectively.*

Then the solution set $S(\lambda)$ of the $(PGNIQVIP)$(2.1) is a Lipschitz continuous mapping from Ω to E.

Proof. For each $\lambda, \bar{\lambda} \in \Omega$, by Theorem 3.2, $S(\lambda)$ and $S(\bar{\lambda})$ are both nonempty closed subsets. Also, $F(x,\lambda)$ and $F(x,\bar{\lambda})$ are both set-valued contractive mappings with same constant $\theta \in (0,1)$. By Lemma 2.2, we obtain

$$H(S(\lambda), S(\bar{\lambda})) \leq \frac{1}{1-\theta} \sup_{x \in E} H(F(x,\lambda), F(x,\bar{\lambda})).$$

Taking any $a \in F(x,\lambda)$, there exist $u \in A(x,\lambda)$, $v \in B(x,\lambda)$, $w \in C(x,\lambda)$, $z \in D(x,\lambda)$ and $s \in G(x,\lambda)$ such that

$$a = x - s + m(w,\lambda) + R_{M(\cdot,z,\lambda)}(s - m(w,\lambda) - \rho N(u,v,\lambda)).$$

Since $A(x,\lambda) \in CB(E)$ and $A(x,\bar{\lambda}) \in CB(E)$, there exists $\bar{u} \in A(x,\bar{\lambda})$ such that

$$\|u - \bar{u}\| \leq (1+\varepsilon) H(A(x,\lambda), A(x,\bar{\lambda})).$$

Similarly, there exist $\bar{v} \in B(x,\bar{\lambda})$, $\bar{w} \in C(x,\bar{\lambda})$, $\bar{z} \in D(x,\bar{\lambda})$ and $\bar{s} \in G(x,\bar{\lambda})$ such that

$$\begin{aligned}
\|v - \bar{v}\| &\leq (1+\varepsilon) H(B(x,\lambda), B(x,\bar{\lambda})),\\
\|w - \bar{w}\| &\leq (1+\varepsilon) H(C(x,\lambda), C(x,\bar{\lambda})),\\
\|z - \bar{z}\| &\leq (1+\varepsilon) H(D(x,\lambda), D(x,\bar{\lambda})),\\
\|s - \bar{s}\| &\leq (1+\varepsilon) H(G(x,\lambda), G(x,\bar{\lambda})).
\end{aligned}$$

Let

$$b = x - \bar{s} + m(\bar{w},\bar{\lambda}) + R_{M(\cdot,\bar{z},\bar{\lambda})}(\bar{s} - m(\bar{w},\bar{\lambda}) - \rho N(\bar{u},\bar{v},\bar{\lambda})).$$

Then $b \in F(x, \bar{\lambda})$. It follows that

$$
\begin{aligned}
(3.11)\ \|a-b\| &= \|s-\bar{s}\| + \|m(w,\lambda) - m(\bar{w},\bar{\lambda})\| \\
&\quad + \|R_{M(\cdot,z,\lambda)}(s - m(w,\lambda) - \rho N(u,v,\lambda) \\
&\quad - R_{M(\cdot,\bar{z},\bar{\lambda})}(\bar{s} - m(\bar{w},\bar{\lambda}) - \rho N(\bar{u},\bar{v},\bar{\lambda}))\| \\
&\leq \|s-\bar{s}\| + \|m(w,\lambda) - m(\bar{w},\bar{\lambda})\| \\
&\quad + \|R_{M(\cdot,z,\lambda)}(s - m(w,\lambda) \\
&\quad - \rho N(u,v,\lambda)) - R_{M(\cdot,z,\lambda)}(\bar{s} - m(\bar{w},\bar{\lambda}) - \rho N(\bar{u},\bar{v},\bar{\lambda}))\| \\
&\quad + \|R_{M(\cdot,z,\lambda)}(\bar{s} - m(\bar{w},\bar{\lambda}) - \rho N(\bar{u},\bar{v},\bar{\lambda})) \\
&\quad - R_{M(\cdot,\bar{z},\lambda)}(\bar{s} - m(\bar{w},\bar{\lambda}) - \rho N(\bar{u},\bar{v},\bar{\lambda}))\| \\
&\quad + \|R_{M(\cdot,\bar{z},\lambda)}(\bar{s} - m(\bar{w},\bar{\lambda}) - \rho N(\bar{u},\bar{v},\bar{\lambda})) \\
&\quad - R_{M(\cdot,\bar{z},\bar{\lambda})}(\bar{s} - m(\bar{w},\bar{\lambda}) - \rho N(\bar{u},\bar{v},\bar{\lambda}))\| \\
&\leq 2\|s-\bar{s}\| + 2\|m(w,\lambda) - m(\bar{w},\lambda)\| \\
&\quad + \rho\|N(u,v,\lambda) - N(\bar{u},\bar{v},\bar{\lambda})\| + \mu\|z-\bar{z}\| + l_R\|\lambda-\bar{\lambda}\|.
\end{aligned}
$$

By Lipschitz continuity of G, m, C in $\lambda \in \Omega$, N and D, we have

$$
\begin{aligned}
(3.12)\qquad \|s-\bar{s}\| &\leq (1+\varepsilon)H(G(x,\lambda), G(x,\bar{\lambda}) \\
&\leq (1+\varepsilon)l_G\|\lambda-\bar{\lambda}\|,
\end{aligned}
$$

$$
\begin{aligned}
(3.13)\|m(w,\lambda) - m(\bar{w},\bar{\lambda})\| &\leq \|m(w,\lambda) - m(\bar{w},\lambda)\| + \|m(\bar{w},\lambda) - m(\bar{w},\bar{\lambda})\| \\
&\leq \eta\|w-\bar{w}\| + l_m\|\lambda-\bar{\lambda}\| \\
&\leq \eta H(C(x,\lambda), C(x,\bar{\lambda})) + l_m\|\lambda-\bar{\lambda}\| \\
&\leq [\eta(1+\varepsilon)l_C + l_m]\|\lambda-\bar{\lambda}\|,
\end{aligned}
$$

$$
\begin{aligned}
(3.14)\qquad &\|N(u,v,\lambda) - N(\bar{u},\bar{v},\bar{\lambda})\| \\
\leq\ &\|N(u,v,\lambda) - N(\bar{u},v,\lambda)\| \\
&+ \|N(\bar{u},v,\lambda) - N(\bar{u},\bar{v},\lambda)\| + \|N(\bar{u},\bar{v},\lambda) - N(\bar{u},\bar{v},\bar{\lambda})\| \\
\leq\ &\beta\|u-\bar{u}\| + \xi\|v-\bar{v}\| + l_N\|\lambda-\bar{\lambda}\| \\
\leq\ &\beta(1+\varepsilon)H(A(x,\lambda), A(x,\bar{\lambda})) + \xi(1+\varepsilon)H(B(x,\lambda), B(x,\bar{\lambda})) \\
&+ l_N\|\lambda-\bar{\lambda}\| \\
\leq\ &[\beta(1+\varepsilon)l_A + \xi(1+\varepsilon)l_B + l_N]\|\lambda-\bar{\lambda}\|,
\end{aligned}
$$

$$
\begin{aligned}
(3.15)\qquad \|z-\bar{z}\| &\leq (1+\varepsilon)H(D(x,\lambda), D(x,\bar{\lambda})) \\
&\leq (1+\varepsilon)l_D\|\lambda-\bar{\lambda}\|.
\end{aligned}
$$

It follows from (3.11)-(3.15) that

$$
\begin{aligned}
\|a-b\| &\leq [2\{(1+\varepsilon)l_G + \eta(1+\varepsilon)l_C + l_m\} + \rho\{(1+\varepsilon)\beta l_A + (1+\varepsilon)\xi l_B + l_N\} \\
&\quad + \mu(1+\varepsilon)l_D + l_J]\|\lambda-\bar{\lambda}\| \\
&= M\|\lambda-\bar{\lambda}\|,
\end{aligned}
$$

where

$$\begin{aligned} M &= 2\{(1+\varepsilon)l_G + \eta(1+\varepsilon)l_C + l_m\} + \rho\{(1+\varepsilon)\beta l_A + (1+\varepsilon)\xi l_B + l_N\} \\ &\quad + \mu(1+\varepsilon)l_D + l_J. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \sup_{a\in F(x,\lambda)} d(a, F(x,\bar{\lambda})) &\leq M\|\lambda - \bar{\lambda}\|, \\ \sup_{b\in F(x,\bar{\lambda})} d(F(x,\lambda), b) &\leq M\|\lambda - \bar{\lambda}\|. \end{aligned}$$

It follows that

$$H(F(x,\lambda), F(x,\bar{\lambda})) \leq M\|\lambda - \bar{\lambda}\|.$$

By Lemma 2.2, we obtain

$$H(S(\lambda), S(\bar{\lambda})) \leq \frac{M}{1-\theta}\|\lambda - \bar{\lambda}\|.$$

This proves that $S(\lambda)$ is Lipschitz continuous in $\lambda \in \Omega$. □

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