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The Uniform Convergence of a Sequence of Weighted Bounded Exponentially Convex Functions on Foundation Semigroups

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Abstract. In the present paper we shall prove that on a foundation $*$-semigroup S with an identity and with a locally bounded Borel measurable weight function ω, the pointwise convergence and the uniform convergence of a sequence of ω-bounded exponentially convex functions on S which are also continuous at the identity are equivalent.

1. Introduction

In [6] Okb El-Bab proved that if S is a foundation topological $*$-semigroup with an identity e and with a Borel measurable weight function ω such that $0 < \omega \leq 1$ and $1/\omega$ is locally bounded (i.e., bounded on compact subsets of S) and if $P_e(S,\omega)$ is the set of ω-bounded Borel measurable exponentially convex functions on S which are continuous at e. Then a sequence (ϕ_n) in $P_e(S,\omega)$, converges pointwise on S to a function $\phi \in P_e(S,\omega)$ if and only if (ϕ_n) converges to ϕ uniformly on compact subsets of S. He proved also that this result remains valid for any Borel measurable weight function ω such that ω and $1/\omega$ are locally bounded.

2. Preliminaries

A topological semigroup S is called a $*$-semigroup if there is a continuous mapping $* : S \to S$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$. A locally bounded (i.e bounded on compact subsets of S) mapping $\omega : S \to \mathbb{R}^+$ ($\mathbb{R}^+$ denote the set of positive real numbers) is called a weight function on S if $\omega(x^*) = \omega(x)$ and $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in S$. A function $f : S \to \mathbb{R}$ is called ω-bounded if there is a positive number K such that $|f(x)| \leq K\omega(x)(x \in S)$. A real valued function ϕ on S is called exponentially convex if it satisfies

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \phi(x_i x_j) \geq 0$$

for all $\{x_1, \cdots, x_n\}$ from S and $\{c_1, \cdots, c_n\}$ from $\mathbb{R}$. We denote by $P_e(S,\omega)(P(S,\omega),$

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respectively) the set of ω-bounded, Borel measurable, continuous at e and exponentially convex function on S (the set of ω-bounded continuous exponentially convex function on S, respectively). A $*$-representation of S by bounded operators on a Hilbert space H is a homomorphism: $x \to \pi(x)$ of S into $L(H)$, the space of all bounded linear operators on H, such that $\pi(x^*) = (\pi(x))^*$ for all $x \in S$ and $\pi(e)$ is the identity operator on H. A representation π is called cyclic if there is a (cyclic) vector $\xi \in H$ such that the set $\{\pi(x)\xi : x \in S\}$ is dense in H, and π is called ω-bounded if there is a positive number K such that $\|\pi(x)\| \leq K\omega(x)(x \in S)$. Note that a $*$-representation π is ω-bounded if and only if $\|\pi(x)\| \leq \omega(x)(x \in S)$. For further information see [2], [4], [5].

Recall that (see for example, [1]) $\tilde{L}(S)$ or $M_a(S)$ denotes the set of all measures $\mu \in M(S)$ (the convolution measure algebra of bounded complex measure on S with the total variation norm $\| \, . \, \|$), for which the mapping $x \to \delta_x * |\mu|$ and $x \to |\mu| * \delta_x$ (where δ_x denotes the point mass at x for $x \in S$) from S into $M(S)$ are weakly continuous. If ω is a locally bounded Borel measurable weight function on S, then we denote by $M_a^k(S,\omega)$ the set of all complex regular measures μ on S such that $\omega\mu \in M_a^k(S)$, where $M_a^k(S)$ denote the set of all measures in $M_a(S)$ with compact support. We observe that $M_a^k(S,\omega)$ with convolution

$$(\mu * \nu)(f) = \int f(xy)d\mu(x)d\nu(y) \quad (f \in C_c(S)),$$

where $C_c(S)$ denote the space of all continuous complex valued function on S with compact support.

A semigroup S is called foundation if $U(\text{supp}(\mu) : \mu \in M_a(S))$ is dense in S. It is well known that $M_a(S)$ is a two sided closed L-ideal of $M(S)$ and if S is also foundation semigroup with identity, then both mapping $x \to \delta_x * \mu$ and $x \to \mu * \delta_x (\mu \in M_a(S))$ from S into $M_a(S)$ are norm continuous (see [7]). We observe that if S is a foundation semigroup with identity and with a locally bounded Borel measurable weight function ω, then both the mappings $x \to \delta_x * \mu$ and $x \to \mu * \delta_x$ $(\mu \in M_a^k(S,\omega))$ from S into $M_a^k(S,\omega)$ are $\|.\|_\omega$ norm continuous, where $\|\mu\|_\omega = \|\omega\mu\|$ for every $\mu \in M_a^k(S,\omega)$.
Now we introduce two new topologies $\tau_{\mathcal{U}}$ and τ_J on $P(S,\omega)$.

3. The $\tau_{\mathcal{U}}$-topology and the τ_J-topology on $P(S,\omega)$

The following two definitions are needed for the proof of the main result.

Definition 3.1. For each compact subset F of S, positive numbers α, β, and $\phi_0 \in P(S,\omega)$ of a foundation $*$-semigroup S with an identity e and with a locally bounded Borel measurable weight function ω we define,

$$\mathcal{U}_{F;\alpha,\beta}(\phi_0) = \{\phi \in P(S,\omega) : |\phi(x) - \phi_0(x)| < \alpha$$

and

$$|\phi(x^2) - \phi_0(x^2)| < \beta \quad \text{for all} \quad x \in F\}. \tag{1}$$

The family of the sets of the form (1) define a base for a topology on $P(S,\omega)$ which is denoted by $\tau_{\mathcal{U}}$.

Definition 3.2. For $\mu_1,\cdots,\mu_m \in M_a^k(S,\omega)$, positive real numbers α,β,γ, and $\phi_0 \in P(S,\omega)$ let

$$(2)\ J_{\mu_1,\cdots,\mu_m;\alpha,\beta,\gamma}(\phi_0) = \Big\{\phi \in P(S,\omega) : \Big|\int_S [\phi(y)-\phi_0(y)]d\mu_j(y)\Big| < \alpha,\ \Big|\int_S [\phi(y^2)-\phi_0(y^2)]d\mu_j(y)\Big| < \beta, \text{ for } j=1,\cdots,m \text{ and } |\phi(e)-\phi_0(e)| < \gamma\Big\}$$

the family of the sets of the form (2) define a base for a topology τ_J on $P(S,\omega)$.

Lemma 3.1. *Let S be a $*$-semigroup (not necessarily topological) with an identity and with a weight ω. Then every ω-bounded exponentially convex function ϕ on S satisfies the following inequality*

$$(3)\qquad |\phi(x)-\phi(xy)|^2 \le \phi(e)\omega^2(x)[\phi(e)-2\phi(y)+\phi(y^2)] \quad (x,y \in S).$$

Proof. Since ϕ is ω-bounded, from [3] it follows that there exists a ω-bounded cyclic $*$-representation π of S by bounded operators on a Hilbert space H with a cyclic vector ξ. Such that $\|\xi\|^2=\phi(e)$ and $\phi(x)=(\pi(x)\xi,\xi) \quad (x\in S)$.
For every $x,y\in S$, we have

$$\begin{aligned}
|\phi(x)-\phi(yx)|^2 &= |(\pi(x)\xi,\xi)-(\pi(yx)\xi,\xi)|^2\\
&= |(\pi(x)\xi,\xi-\pi(y)\xi)|^2\\
&\le \|\pi(x)\xi\|^2\|\xi-\pi(y)\xi\|^2\\
&= (\pi(x)\xi,\pi(x)\xi)[\|\xi\|^2-2\phi(y)+\phi(y^2)]\\
&= \phi(x^2)[\phi(e)-2\phi(y)+\phi(y^2)]\\
&\le \phi(e)\omega^2(x)[\phi(e)-2\phi(y)+\phi(y^2)].
\end{aligned}$$

□

Lemma 3.2. *Let S be a foundation $*$-semigroup with identity and with a locally bounded Borel measurable weight function ω. Then $P_e(S,\omega)=P(S,\omega)$.*

Proof. Let $\phi \in P_e(S,\omega)$. Take a fixed $x_0 \in S$ and let W be a fixed compact neighborhood of x_0. Since ω is locally bounded, there exists a positive real number M such that $\omega(x)\le M$ for all $x\in W$. Given $\varepsilon>0$, by the continuity of ϕ at e there exists a neighborhood U of e such that

$$[\phi(e)-2\phi(u)-\phi(u^2)]^{\frac{1}{2}} < \frac{\varepsilon}{2M[(\phi(e))^{\frac{1}{2}}+1]} \quad (u\in U).$$

Let

$$W_1=[U^{-1}(Ux)\cap(xU)U^{-1}]\cap W$$

define a neighborhood of e. Let $z \in W_1$, then $uz = vx$ for some $u, v \in U$, so by (3)

$$\begin{aligned} |\phi(z) - \phi(x)| &\leq |\phi(z) - \phi(ux)| + |\phi(vx) - \phi(x)| \\ &\leq (\phi(e))^{\frac{1}{2}}\omega(z)([\phi(e) - 2\phi(u) + \phi(u^2)])^{\frac{1}{2}} \\ &\quad + (\phi(e))^{\frac{1}{2}}\omega(x)([\phi(e) - 2\phi(v) + \phi(v^2)])^{\frac{1}{2}} \\ &\leq 2M(\phi(e))^{\frac{1}{2}}\frac{\varepsilon}{2M[(\phi(e))^{\frac{1}{2}} + 1]} < \varepsilon \end{aligned}$$

so $\phi \in P(S, \omega)$ and the proof is complete. □

The following theorem is the main result of this paper and it generalizes. Theorem 1 of [6]. Note that $P_e(S, \omega) = P(S, \omega)$, by Lemma 3.2.

Theorem 3.1. *Let S be a foundation $*$-semigroup with identity and with a locally bounded Borel measurable weight function ω. Then the $\tau_{\mathcal{U}}$-topology and the τ_J-topology are identical on $P(S, \omega)$.*

Proof. Take ϕ_0 fixed in $P(S, \omega)$. Let $J_{\mu_1,\cdots,\mu_n;\beta,\gamma,\lambda}(\phi_0)$ be an arbitrary basic τ_J-neighborhood of ϕ_0. Choose a positive number η such that $\eta \leq \lambda$ and

$$2\eta + \eta \max\{\|\mu_1\|, \cdots, \|\mu_n\|\} < \min(\beta, \gamma).$$

Choose a compact set F_0 such that $e \in F_0$ with

$$\int_{S\setminus F_0} (\omega(y))^2 d|\mu_j|(y) < \eta, \text{ and } \int_{S\setminus F_0} \omega(y) d|\mu_j|(y) < \eta (j = 1, \cdots, m).$$

Then it is clear that

$$\mathcal{U}_{F_0,\eta,\eta}(\phi_0) \subset J_{\mu_1,\cdots,\mu_m;\beta,\gamma,\lambda}(\phi_0).$$

Conversely, suppose that $\mathcal{U}_{F,\alpha_0,\beta_0}(\phi_0)$ is an arbitrary $\tau_{\mathcal{U}}$-neighborhood of ϕ_0. Let $\beta = \min\{\alpha_0, \beta_0\}$ and M be a positive number such that $\omega(x) \leq M$ for all $x \in F$. Put

$$\begin{aligned} \gamma &= \min\{\frac{\beta^2}{81M^4(1 + \phi_0(e))}, \frac{\beta^2}{81M^2(1 + \phi_0(e))}\} \\ \delta &= \min\{\frac{\beta}{6\{\phi_0(e) + 1)}, 1\}. \end{aligned}$$

By the continuity of ϕ_0 at e there exists a compact neighborhood U of e such that for all $y \in U$

(4) $$|\phi_0(y) - \phi_0(e)| < \gamma \text{ and } |\phi_0(y^2) - \phi_0(e)| < \gamma.$$

Now choose a positive measure $\mu \in M_a^k(S, \omega)$ such that $\mu(S) = 1$ and $e \in supp(\mu) \subseteq U$.

By the $\|.\|_\omega$ norm continuity of the mapping $x \to \delta_x * \mu$ from S into $M_a^k(S, \omega)$ and the compactness of F we can find a finite subset $\{x_1, \cdots, x_n\}$ of F such that the set $\{\delta_x * \mu : x \in F\}$ can be covered by $\{N_{x_1}, \cdots, N_{x_n}\}$, where

$$N_{x_i} = \{\lambda \in M_a^k(S, \omega) : \|\lambda - \delta_{x_i} * \mu\|_\omega < \delta\} \text{ for } i = 1, \cdots, n.$$

Again by the $\|.\|_\omega$-norm continuity of the mapping $x \to \delta_{x^2} * \mu$ from S into $M_a^k(S, \omega)$, we can find $s_1, s_2, \cdots, s_\ell \in S$ such that the set $\{\delta_{x^2} * \mu : x \in F\}$ can be covered by $\{N'_{s_1}, \cdots, N'_{s_\ell}\}$, where

$$N'_{s_j} = \{\lambda \in M_a^k(S, \omega) : \|\lambda - \delta_{s_j^2} * \mu\|_\omega < \delta\}(j = 1, \cdots, \ell).$$

Put

$$z_i = x_i, i = 1, \cdots, n, z_{n+j} = s_j s_j^* = s_j^2 \text{ for } 1 \le j \le \ell.$$

Put $p = n + \ell$ and let $\mu_k = \delta_{x_k} * \mu(k = 1, 2, \cdots, p)$. We shall prove that

$$J_{\mu_1, \mu_2, \cdots, mp; \delta, \delta, \delta}(\phi_0) \cap J_{\mu, \gamma, \gamma, \gamma}(\phi_0) \subseteq \mathcal{U}_{F; \beta, \gamma}(\phi_0).$$

To prove this we choose $\phi \in J_{\mu_1, \cdots, \mu_p, \delta, \delta, \delta}(\phi_0)$. Let x be fixed but arbitrary element in F. We have $\|\delta_x * \mu - \delta_{x_j} * \mu\|_\omega < \delta$ and $\|\delta_{x*x} - \delta_{x_q * x_q} * \mu\|_\omega < \delta$ for some j and $q \in \{1, 2, \cdots, p\}$. Therefore

$$\begin{aligned} (5) \qquad & |\delta_x * \mu(\phi) - \delta_x * \mu(\phi_0)| \\ = \; & |\int [\phi(y) - \phi_0(y)] d\delta_x * \mu(y)| \\ \le \; & |\int \phi(y) d(\delta_x * \mu - \delta_{x_j} * \mu)(u)| \\ & + |\int [\phi(y) - \phi_0(y)] d\mu_j(y)| + |\int \phi_0(y) d(\delta_{x_j} * \mu - \delta_x * \mu)(y)| \\ \le \; & \phi(e)\|\delta_x * \mu - \delta_{x_j} * \mu\|_\omega + \delta + \phi_0(e)\|\delta_{x_j} * \delta_x * \mu\|_\omega \\ < \; & \delta(\phi(e) + \phi_0(e) + 1) \; < \; \frac{\beta}{3}. \end{aligned}$$

(In the above we have used [3]). Similarly by using the inequality $\|\delta_{x^2} * \mu - \delta_{x_q^2} * \mu\|_\omega < \delta$, we can prove that

$$(6) \qquad |\delta_{x^2} * \mu(\phi) - \delta_{x^2}(\phi_0)| < \frac{\beta}{3}.$$

Suppose now that

$$\phi \in J_{\mu; \gamma, \gamma, \gamma}(\phi_0).$$

Then for every $x \in F$ by (3) and the Holder inequality we have

$$\begin{aligned} |\delta_x * \mu(\phi) - \phi(x)| &\leq |\int_S \phi(xy)d\mu(y) - \int_S \phi(x)d\mu(y)| \\ &\leq \int_S |\phi(xy) - \phi(x)|d\mu(y) \\ &\leq \omega(x)(\phi(e))^{\frac{1}{2}}(\int_U |\phi(e) - 2\phi(y) + \phi(y^2)]d\mu(y))^{\frac{1}{2}} \\ &\leq M\phi(e)^{\frac{1}{2}}(\int_U [\phi(e) - 2\phi(y) + \phi(y^2)]d\mu(y))^{\frac{1}{2}} \end{aligned}$$

by (4) we obtain

$$\begin{aligned} &\int_S [\phi(e) - 2\phi(y) + \phi(y^2)]d\mu(y) \\ \leq\ & 2|\int_U [\phi(e) - \phi(y)]d\mu(y)| + |\int_U [\phi(y^2) - \phi(e)]d\mu(y)| \\ \leq\ & 2[\int_U [\phi(e) - \phi_0(e)]d\mu(y) + \int_U |\phi_0(e) - \phi_0(y)|d\mu(y) \\ & + \int_U |\phi_0(y) - \phi(y)|d\mu(y)] + \int_U |\phi(y^2) - \phi_0(y^2)|d\mu(y) \\ & + \int_U |\phi_0(y^2) - \phi_0(e)|d\mu(y) + \int_U |\phi_0(e) - \phi(e)|d\mu(y) \ < \ 9\gamma \end{aligned}$$

so for every $x \in F$

$$|\delta_x * \mu(\phi) - \phi(x)| \leq 3M(\phi(e)\gamma)^{\frac{1}{2}} < \frac{\beta}{3}. \tag{7}$$

Similarly for every $x \in F$

$$\begin{aligned} |\delta_{x^2} * \mu(\phi) - \phi(x^2)| &\leq \omega(x^2)\phi(e)^{\frac{1}{2}}(\int_U |\phi(e) - 2\phi(y) + \phi(y^2)]d\mu(y))^{\frac{1}{2}} \\ &< 3M^2(\phi(e)\gamma)^{\frac{1}{2}} < \frac{\beta}{3}. \end{aligned} \tag{8}$$

Finally, for every $\phi \in J_{\mu_1,\cdots,\mu_p,\delta,\delta,\delta}(\phi_0) \cap J_{\mu,\gamma,\gamma,\gamma}(\phi_0)$ and every $x \in F$ from (5) and (7).

We have

$$\begin{aligned} |\phi(x) - \phi_0(x)| &\leq |\phi(x) - \delta_x * \mu(\phi)| \\ &\quad + |\delta_x * \mu(\phi) - \delta_x * \mu(\phi_0)| + |\delta_x * \mu(\phi_0) - \delta_x * \mu(\phi)| \\ &< \frac{\beta}{3} + \frac{\beta}{3} + \frac{\beta}{3} = \beta \end{aligned}$$

similarly for every $x \in F$ from (6) and (8) we have

$$|\phi(x^2) - \phi_0(x^2) < \beta.$$

That is $\phi \in \mathcal{U}_{F,\beta,\beta}(\phi_0)$. The proof is now complete. Since $\mathcal{U}_{F,\beta,\beta} \subset \mathcal{U}_{F;\alpha_0,\beta_0}$. □

Theorem 3.2. *Let S be a foundation topological $*$-semigroup with an identity and with a locally bounded measurable ω. Then a sequence $\{\phi_n\}$ of ω-bounded continuous exponential convex function on S converges pointwise to a continuous function ϕ if and only if $\{\phi_n\}$ converges to ϕ in the topology of uniform convergence on compact subset of S.*

Proof. It is known that the pointwise limit of a sequence of exponentially convex function is an exponentially convex. Suppose that $\{\phi_n\}$ converges to ϕ pointwise on S and ϕ is also continuous. Then $\phi \in P(S, \omega)$.
From the Lebesgue dominated convergence Theorem it follows that $\phi_n \to \phi$ in the τ_J-topology. So by Theorem 3.1, $\phi_n \to \phi$ in the $\tau_{\mathcal{U}}$-topology. The converse is obvious. □

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