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Discrete Group Method for Nonlinear Heat Equation

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Abstract. In the category of the group theoretic methods using invertible discrete group transformation, we give a useful relation between Emden-Fowler equations and nonlinear heat equation. In this paper, by means of appropriate transformations of discrete group analysis, the nonlinear hate equation transformed into the class of the Emden-Fowler equations. This approach shows that, under the group action, the solution of reference equation can be transformed into the solution of the transformed equation.

1. Introduction

It is well known that the theory of differential equations takes a central place among possible instruments for the modeling of different processes and phenomena. The classical concepts of groups introduced by S. Lie and A. Bäcklund, which constitute the foundation of modern group analysis are responsible for out standing achievements in the theory of partial differential equations. However, similar approach based on a search for continuous transformation groups, which map the equation under investigation into itself (i.e., exactly into the same equation), proved to be ineffective for solving ordinary differential equations. The latter circumstance is accounted for by the fact that the issue of searching for a continuous group of transformations for an ordinary differential equation is as complicated as the problem of its integration.

In this paper, we analyze the relation between the classical Emden-Fowler equations and nonlinear equation of heat conduction problem with variable transfer coefficients. Also, we shows that, the nonlinear heat equation by using the effect of series of discrete group transformations, transformed to the classical Emden-Fowler equation. On the other hands, all the transformations applied for the conversion of heat equation are invertible and this invertibility clearly allows us to avoid some lengthy computations for the conversion of the initial and boundary condition. Also, under the discrete group transformations, the solution of the transformed equation can be converted into the solution of the reference equation [5], [6].

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2. Preliminaries

In this section, we shall state some concepts and definitions about discrete group analysis.

Definition 1. The class of classical Emden-Fowler equations is written as:

$$y''_{xx} = Ax^n y^m, \quad a = (n, m, 0),$$

where it is determined by a three dimensional parameter vector $a = (n, m, 0) \in R^3$.

Let D be a class of ODE and

$$D(x, y, a) = 0,$$

be an equation in this class, where a is a vector parameters.
We shall seek the transformations F_i that are closed in the class $D(x, y, a) = 0$, i.e., they change only the vector a:

$$F_i : D(x, y, a) \to D(t, u, b_i).$$

If each F_i has an inverses, then the collection $\{F_i\}$ defines a D.T.G (Discrete Transformation Group) on the class $D(x, y, a) = 0$.

Definition 2. An RF-Pair is an operation of consecutive raising and lowering the order of equation.

Now, we define the following R-operations and F-operations:

i) Termwise m-fold differentiation of the original equation, type RD^m.

ii) Termwise one or two-fold differentiation of original equation with respect to the independent variable, type accordingly RX or RX^2.

iii) The equation is an exact derivative of the m th order: termwise integration m times, type FI^m.

iv) The equation is autonomous, i.e., it does not conclude an independent variable in an explicit form, type FX

$$FX : y'_x = u(y), \qquad y''_{xx} = uu'_y.$$

v) The equation is homogeneous in the extended sense, type FU : the transformation $x = e^t$, $y = ue^{kt}$, with an appropriate choice of k, leads to an autonomous form followed by a transformation FX.

If an $R(F)$-operating $RZ^m(FZ^m)$ is inverted, it is denoted by $RZ^{-m}(FZ^{-m})$. The RF-pair will be written in a contracted form by means of an ordered pair, the second letters used in the designation of the operation symbol, the left letter corresponding to the transformation performed first $RF(D,X) \equiv (FX) \otimes (RD)$.

Definition 3. Abel equation of the second kind, which is applied in many problems of mechanics, physics and other sciences is written as:

$$yy'_x = F_1(x)y + F_0(y). \tag{1}$$

This equation is called degenerate if $F_0 = 0$ or $F_1 = 0$. It is obvious that degenerate Abel equations are easily integrable, therefore, we assume that: $F_0 \neq 0$ and $F_1 \neq 0$.

Furthermore, it is important that the more general Able equation:

$$(\phi_1(x)z + \phi_0(x))z'_x = \psi_2(x)z^2 + \psi_1(x)z + \psi_0(x),$$

can be reduced to the Abel equation using the following transformation:

$$y = (\frac{\phi_0}{\phi_1} + z)E, \qquad E = \exp(-\int (\frac{\psi_2}{\phi_1})dx),$$

where

$$F_1 = (\frac{d}{dx}(\frac{\phi_0}{\phi_1}) + \frac{\psi_1}{\phi_1} - 2\frac{\phi_0\psi_2}{\phi_1^2})E, \qquad F_0 = (\frac{\psi_0}{\phi_1} - \frac{\phi_0\psi_1}{\phi_1^2} + \frac{\phi_0^2\psi_2}{\phi_1^3})E^2.$$

(For further details see [6]).
If a new variable $\tau = \tau(x)$ and a function φ (which is in the general case is given in the parametric form) are introduced:

$$\tau = \int F_1(x)dx, \qquad \varphi = \frac{F_0(x)}{F_1(x)},$$

where transformations above are called the canonical transformations of Abel equations of the second kind, then the original equation (1) may be written in the canonical form:

$$yy'_\tau - y = \varphi(\tau).$$

3. Analysis of the method

This section presents analysis of the relation between the classical Emden-Fowler equation and nonlinear equations of heat conduction. First of all, we shall consider the problem of a nonstationary heat exchange between a wall and an immobile medium [6, pp. 171-172]. The thermal conductivity λ, density ρ, and specific heat c_p are assumed to arbitrarily depend on the temperature T. We assume that at initial time $t = 0$, the temperature throughout the medium was uniform and equal T_0, and at $t > 0$, the temperature T_s on the wall surface is maintained constant

$(T_s \neq T_0)$. The nonlinear problem under investigation is described by the following equation with initial and boundary conditions :

$$c_p(T)\rho(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial X}\lambda(T)\frac{\partial T}{\partial X}, \tag{2}$$

$$T(X,0) = T_0, \qquad T(0,t) = T_s, \qquad \lim_{X\to\infty} T(X,t) = T_0, \tag{3}$$

where X is the distance from the wall.

Following procedures in [6], we introduce dimensionless variables:

$$x = \frac{X}{L}, \qquad \tau = (\frac{\lambda}{\rho c_p})_0\frac{t}{L^2}, \qquad y = \frac{\int_T^{T_0}\rho c_p dT}{\int_{T_s}^{T_0}\rho c_p dT}, \qquad f = \frac{\lambda}{\rho c_p}(\frac{\rho c_p}{\lambda})_0, \tag{4}$$

where L is a constant chosen as the length scale, the index "0" signifies that the corresponding quantity is taken at $T = T_0$.

The problem (2)-(3) using (4) is written in the following form:

$$\frac{\partial y}{\partial \tau} = \frac{\partial f(y)}{\partial \tau}\frac{\partial y}{\partial x}, \tag{5}$$

$$y(X,0) = 0, \qquad y(X,\tau) = 1, \qquad \lim_{X\to\infty} y(X,\tau) = 0. \tag{6}$$

Note that it is generally assumed that the product $\rho(T).c_p(T) = constant$. So, in this case formulas (4) are significantly simplified as follows:

$$x = \frac{X}{L}, \qquad \tau = \frac{\lambda(T_0)t}{\rho c_p L^2}, \qquad y = \frac{T_0 - T}{T_0 - T_s}, \qquad f = \frac{\lambda(T)}{\lambda(T_0)},$$

and by employment of a new self-similarity variable

$$z = \frac{x}{\sqrt{\tau}},$$

equation (5)-(6) reduces to a two-point boundary value problem for an ordinary differential equation of second order of the following type:

$$2\big(f(y)y'_z\big)'_z + zy'_z = 0, \tag{7}$$

$$z = 0, \quad y = 1; \qquad z \to \infty, \quad y \to 0.$$

This boundary value problem via substitution:

$$u = f(y)y'_z,$$

is transformed to:

$$2uu''_{yy} = -f(y). \tag{8}$$

Let $f = (by + a)^n$, and setting $\overline{y} = by + a$ in (8), we obtain the Emden-Fowler equation $(n, -1, 0)$. Therefore, it follows that the boundary value problems (7) and (8) are integrable by quadratures with $n = -1, -2$, (for further details see [6], [7]).

In what follows, we shall consider the following nonlinear heat conduction problems with variable transfer coefficients which is the problem of Inertial Confinement Fusion (ICF)[2], [3]:

$$\nabla.(K(T)\nabla T) = \frac{\partial T}{\partial t}, \tag{9}$$

and our aim is to convert this problem into the classical Emden-Fowler equations using a series of transformations. In this equation x is the distance, t is the time and $K = k_0 T^{\frac{5}{2}}$, where k_0 is constant.

In the spherical coordinates, let $T(r, \theta, \phi, t) = T(r, t)$, it is evident that $\nabla_r = \widehat{r}\frac{\partial}{\partial r}$, then equation (9) reduces as follows:

$$\frac{\partial K}{\partial r}\widehat{r}.\frac{\partial T}{\partial r}\widehat{r} + \frac{K}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial T}{\partial r}) = \frac{\partial T}{\partial t}. \tag{10}$$

By substituting:

$$\frac{\partial K}{\partial r} = \frac{5}{2}k_0 T^{\frac{3}{2}}\frac{\partial T}{\partial r},$$

into (10), we obtain the following equation:

$$\frac{5}{2}k_0 T^{\frac{3}{2}}(\frac{\partial T}{\partial r})^2 + \frac{2k_0}{r}T^{\frac{5}{2}}\frac{\partial T}{\partial r} + k_0 T^{\frac{5}{2}}\frac{\partial^2 T}{\partial r^2} = \frac{\partial T}{\partial t}. \tag{11}$$

Application of the separation of variables:

$$T(r, t) = R(r)P(t),$$

gives:

$$\frac{\partial T}{\partial P} = RP'_t, \qquad \frac{\partial T}{\partial r} = PR'_r, \qquad \frac{\partial^2 T}{\partial r^2} = PR''_{rr},$$

and hence equation(11) reduces to:

$$\frac{5k_0}{2}R^{\frac{1}{2}}R'^2_r + 2k_0 r^{-1}R^{\frac{3}{2}}R'_r + k_0 R^{\frac{3}{2}}R''_{rr} = P^{\frac{-7}{2}}P'_t = \lambda^2,$$

where λ is the separation parameter. As a result of this decomposition, we obtain the following defining system:

$$P^{-\frac{7}{2}}P'_t = \lambda^2, \tag{12}$$

$$\frac{5k_0}{2}R^{\frac{1}{2}}R'^2_r + 2k_0 r^{-1}R^{\frac{3}{2}}R'_r + k_0 R^{\frac{3}{2}}R''_{rr} = \lambda^2. \tag{13}$$

Note that the first equation of this system yields the function $P(t)$, to the following form :

$$-\frac{2}{5}P^{-\frac{5}{2}}+c_1=\lambda^2 t. \tag{14}$$

We shall now apply the following discrete group transformation (DGT), as a consequence of which the equation (13) is transformed to the classical Emden-Fowler equation. Having applied the operation $FU: r=e^s, R=ue^{ks}$, on the equation (13) it leads to the autonomous equation:

$$k_0u^{\frac{3}{2}}u''_{ss}+k_0(7k+1)u^{\frac{3}{2}}u'_s+\frac{5k_0}{2}u^{\frac{1}{2}}u'^2_s+k_0k(\frac{7k}{2}+1)u^{\frac{5}{2}}-\lambda^2=0, \tag{15}$$

where $k=\frac{4}{5}$ and using the raising order operation $FX: u'_s=w(u),\quad u''_{ss}=ww'_u$, equation (15) reduces to the following form :

$$ww'_u=-\frac{5}{2}u^{-1}w^2-(7k+1)w+\frac{\lambda^2}{k_0}u^{-\frac{3}{2}}-k(\frac{7k}{2}+1)u, \tag{16}$$

and applying the transformation :

$$\nu=(\frac{\phi_0}{\phi_1}+w)E,\qquad E=\exp(-\int(\frac{\psi_2}{\phi_1}du)),$$

where

$$\phi_0=0,\quad \phi_1=1,\quad \psi_0=-\frac{\lambda^2}{k_0}u^{-\frac{3}{2}}-k(\frac{7k}{2}+1)u,\quad \psi_1=-(7k+1),\qquad \psi_2=-\frac{5}{2}u^{-1},$$

the equation (16) is reduced to:

$$\nu\nu'_u=-c_1(7k+1)u^{\frac{5}{2}}\nu+c_1^2\frac{\lambda^2}{k_0}u^{\frac{7}{2}}-c_1^2k(\frac{7k}{2}+1)u^6. \tag{17}$$

where

$$F_0(u)=c_1^2\frac{\lambda^2}{k_0}u^{\frac{7}{2}}-c_1^2k(\frac{7k}{2}+1)u^6,\quad F_1=-c_1(7k+1)u^{\frac{5}{2}}\nu.$$

Substituting:

$$\tau(u)=\int F_1(u)du=-c_1(7k+1)\int u^{\frac{5}{2}}du=c_2-c_1(7k+1)u^{\frac{7}{2}},$$

and

$$\varphi(\tau)=\frac{F_0(u)}{F_1(u)}=\frac{k(\frac{7k}{2}+1)c_1}{7k+1}u^{\frac{7}{2}}-\frac{\lambda^2c_1}{k_0(7k+1)}u=c_3(c_2-\tau)-c_4(c_2-\tau)^{\frac{2}{7}},$$

where c_i's are constants, in the equation (17) we obtain that:

$$\nu\nu'_\tau-\nu=c_3(c_2-\tau)+c_4(c_2-\tau)^{\frac{2}{7}}, \tag{18}$$

where

$$c_3 = \frac{7k(\frac{7k}{2}+1)}{2(7k+1)^2}, \qquad c_4 = -\frac{\lambda^2 c_1}{k_0(7k+1)}(\frac{7}{2c_1(7k+1)})^{\frac{8}{7}}.$$

Substituting : $c_2 - \tau = -q$, equation (18) may be rewritten in the canonical form:

$$\nu\nu'_q - \nu = c_4 q^{\frac{2}{7}} - c_3 q.$$

Now, let

$$c_3 = \frac{(n+2)(n+\frac{9}{7})}{(2n+\frac{23}{7})^2}, \qquad c_4 = A(-\frac{35}{(14n+23)})^2, \tag{19}$$

then, the transformation:

$$z = -\frac{(14n+23)}{5}\frac{\nu}{q} + \frac{7(n+2)}{5}, \qquad f = Aq^{-\frac{5}{7}},$$

reduces equation (19) to the following form:

$$(z - z^2 + f)f'_z = [-\frac{5}{7}z + n + 2]f.$$

(For further details see [6], [7]).
Through the substitution:

$$f = Ax^{n+2}y^{-\frac{5}{7}}, \qquad z = \frac{x}{y}y'_x,$$

we obtain the classical Emden-Fowler equation:

$$y''_{xx} = Ax^n y^{\frac{2}{7}},$$

where A depends on λ and λ has been defined in (14) as a separation parameter. Note that n is also a constant depending on the initial and boundary conditions. For some values of n, this equation is integrable [6], [7] as a consequence of which, we have been able to construct some solution for the ICF problem.

Note that, all the RF-pair operations and transformations applied for the conversion of heat equation are invertible and this invertibility clearly allows us to avoid some lengthy computations for the conversion of the initial and boundary condition. Also, under the discrete group transformations, the solution of the transformed equation can be converted into the solution of the reference equation.

4. Conclusion

By introducing the discrete group transformation, the nonlinear heat conduction problem with variable transfer coefficient in applied physics, can be transformed into the classical Emden-Fowler equation which maybe integrated using classical methods. This approach shows that, under the discrete group transformation, the solution of transformed equation can be converted into the solution of the reference equation.

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