KYUNGPOOK Math. J. 46(2006), 307-313

On Finite Integrals Involving Jacobi Polynomials and the $\overline{H}$-function

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Abstract. In this paper, we first establish an interesting new finite integral whose integrand involves the product of a general class of polynomials introduced by Srivastava [13] and the generalized H-function ([9], [10]) having general argument. Next, we present five special cases of our main integral which are also quite general in nature and of interest by themselves. The first three integrals involve the product of $\overline{H}$-function with Jacobi polynomial, the product of two Jacobi polynomials and the product of two general binomial factors respectively. The fourth integral involves product of Jacobi polynomial and well known Fox's H-function and the last integral involves product of a Jacobi polynomial and '**g**' function connected with a certain class of Feynman integral which may have practical applications.

1. Introduction

A function more general than well known Fox H-function was introduced by Inayat-Hussain ([9], [10]). This function has been put on a firm footing by Buschman and Srivastava [3]. Recently, a finite integral involving the $\overline{H}$-function was evaluated by Gupta and Soni [7]. This function will be defined and represented in following manner [3]

$$(1.1)\qquad \overline{H}^{M,N}_{P,Q}[z] = \overline{H}^{M,N}_{P,Q}\left[z\middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N}, & (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, & (b_j,\beta_j;B_j)_{M+1,Q} \end{matrix}\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \overline{\phi}(\xi) z^{\xi} d\xi,$$

where

$$(1.2)\qquad \overline{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}$$

which contains fractional powers of some of the gamma functions. Here, and through out the paper a_j $(j=1,\cdots,P)$ and b_j $(j=1,\cdots,Q)$ are complex parameters,

Received December 5, 2002, and, in revised form, September 12, 2005.
2000 Mathematics Subject Classification: 33C45, 33C60, 33E99.
Key words and phrases: Jacobi polynomials, Fox's H-function, Feynman integrals, $\overline{H}$-function

$\alpha_j \geq 0$ $(j = 1, \cdots, P)$, $\beta_j \geq 0$ $(j = 1, \cdots, Q)$ (not all zero simultaneously) and the exponents A_j $(j = 1, \cdots, N)$ and B_j $(j = M+1, \cdots, Q)$ can take on non-integer values. The contour in (1.1) is imaginary axis $\mathrm{Re}(\xi) = 0$. It is suitably indented in order to avoid the singularities of the gamma functions and to keep those singularities on appropriate sides. Again, for A_j $(j = 1, \cdots, N)$ not an integer, the poles of the gamma functions of the numerator in (1.2) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j - \beta_j\xi)$ $(j = 1, \cdots, M)$ and $\Gamma(1 - a_j + \alpha_j\xi)$ $(j = 1, \cdots, N)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

Evidently, when the exponents A_j and B_j are all positive integers, the $\overline{H}$-function reduces to the well-known Fox's H-function ([5], [14]). The basic properties and the following sufficient conditions for the absolute convergence of the defining integral for the $\overline{H}$-function have been given by Buschman and Srivastava [3].

$$\Omega \equiv \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} A_j \alpha_j - \sum_{j=M+1}^{Q} B_j \beta_j - \sum_{j=N+1}^{P} \alpha_j > 0 \tag{1.3}$$

and

$$|\arg(z)| < \frac{1}{2}\pi\Omega, \tag{1.4}$$

where Ω is given by (1.3).

The behaviour of the $\overline{H}$-function for small values of $|z|$ follows easily from a result recently given by Rathie [12, p.306, eq. (6.9)]. We have

$$\overline{H}_{P,Q}^{M,N}[z] = O\big(|z|^{\alpha}\big), \quad \alpha = \min_{1 \leq j \leq M}[\mathrm{Re}(b_j/\beta_j)], \quad |z| \to 0. \tag{1.5}$$

Investigations of the convergence conditions, all possible types of contours, type of critical points of the integrand of (1.1), etc. can be made by an interested reader by following analogous techniques given in the well known works of Braaksma [1], Hai and Yakubovich [8]. We however omit the details.

Also $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava [13, p.1, eq. (1)]

$$S_n^m[x] = \sum_{r=0}^{[n/m]} \frac{(-n)_{mr}}{r!} A_{n,r} x^r, \quad n = 0, 1, 2, \cdots, \tag{1.6}$$

where m is an arbitrary positive integer and the coefficients $A_{n,r}$ $(n, r \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,r}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Lagurre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others [15, pp.158-161].

The following known results [2, p.945, eq. (16)] and [4,p.172, eq.(29)] for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [11, p.254, eq.(1)], will be required in our investigation

$$\begin{aligned}(1.7)\qquad & P_\mu^{(\alpha,\beta)}(t+y)P_\mu^{(\alpha,\beta)}(t-y)\\ &= \frac{(-1)^\mu(1+\alpha)_\mu(1+\beta)_\mu}{(\mu!)^2}\sum_{n=0}^{\mu}\frac{(-\mu)_n(1+\alpha+\beta+\mu)_n}{(1+\alpha)_n(1+\beta)_n}P_n^{(\alpha,\beta)}(x)t^n\end{aligned}$$

$$(1.8)\qquad \frac{1}{y}(1-t+y)^{-\alpha}(1+t+y)^{-\beta} = 2^{-\alpha-\beta}\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)t^n,$$

where y denotes $(1-2xt+t^2)^{1/2}$ in both (1.7) and (1.8).

2. Main integral

$$\begin{aligned}(2.1)\quad & \int_{-1}^{1}(1-x)^\rho(1+x)^\sigma S_n^m[w(1-x)^u(1+x)^v]\\ & \overline{H}_{P,Q}^{M,N}\left[z(1-x)^h(1+x)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N}, & (a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M}, & (b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx\\ = \quad & 2^{\rho+\sigma+1}\sum_{r=0}^{[n/m]}\frac{(-n)_{mr}}{r!}A_{n,r}w^r2^{(u+v)r}\overline{H}_{P+2,Q+1}^{M,N+2}\\ & \left[z2^{h+k}\left|\begin{matrix}(-\sigma-vr,k;1),(-\rho-ur,h;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\rho-\sigma-(u+v)r-1,h+k;1)\end{matrix}\right.\right],\end{aligned}$$

where (i) $u \geq 0,\ v \geq 0;\ h \geq 0,\ k \geq 0$ (not both zero simultaneously),

(ii) $\mathrm{Re}(\rho) + h\min\limits_{1\leq j\leq M}[\mathrm{Re}(b_j/\beta_j)] > -1, \quad \mathrm{Re}(\sigma) + k\min\limits_{1\leq j\leq M}[\mathrm{Re}(b_j/\beta_j)] > -1.$

(iii) The $\overline{H}$-functions occurring in (2.1) satisfy conditions corresponding appropriately to those given by (1.3) and (1.4).

Proof. To establish the integral (2.1), we express the $\overline{H}$-function occurring in its left-hand side in terms of Mellin-Barnes contour integral given by (1.1), the general class of polynomials occurring therein the series form given by (1.6) and then interchange the order of summations and integration and the order of x-and ξ-integrals (which is permissible under the conditions stated with (2.1)) and evaluating the integral with the help of a modified form of the formula [6, p.314, eq.(3)], we easily arrive at the desired result (2.1) after a little simplification. □

3. Particular cases of main integral

[i] If we take $m = 1$; $w = \frac{1}{2}$; $u = 1$; $v = 0$ and $A_{n,r} = \binom{n+\alpha}{n}\dfrac{(\alpha+\beta+n+1)_r}{(\alpha+1)_r}$ in

(2.1), then the polynomial $S_n^1\left[\frac{1-x}{2}\right]$ occurring therein breaks up into the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [16, p.68, eq.(4.3.2)] and the integral (2.1) takes the following form after a little simplification

$$\begin{aligned}(3.1)\qquad &\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\overline{H}_{P,Q}^{M,N}\\ &\left[z(1-x)^h(1+x)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N}, & (a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M}, & (b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx\\ =\;& 2^{\rho+\sigma+1}\sum_{r=0}^{n}\frac{(-n)_r(\alpha+\beta+n+1)_r}{r!(\alpha+1)_r}\overline{H}_{P+2,Q+1}^{M,N+2}\\ &\left[z2^{h+k}\left|\begin{matrix}(-\sigma,k;1),(-\rho-r,h;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\rho-\sigma-r-1,h+k;1)\end{matrix}\right.\right]\end{aligned}$$

the conditions of validity of (3.1) can be easily obtained from those of (2.1).

[ii]

$$\begin{aligned}(3.2)\qquad &\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_\mu^{(\alpha,\beta)}(t+y)P_\mu^{(\alpha,\beta)}(t-y)\\ &\overline{H}_{P,Q}^{M,N}\left[z(1-x)^h(1+x)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N}, & (a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M}, & (b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx\\ =\;& 2^{\rho+\sigma+1}\frac{(-1)^{\mu}\Gamma(1+\alpha+\mu)\Gamma(1+\beta+\mu)}{(\mu!)^2}\\ &\sum_{n=0}^{\mu}\sum_{r=0}^{n}\frac{(-\mu)_n(1+\alpha+\beta+\mu)_n(-n)_r(\alpha+\beta+n+1)_r}{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)r!(\alpha+1)_r}t^n\,\overline{H}_{P+2,Q+1}^{M,N+2}\\ &\left[z2^{h+k}\left|\begin{matrix}(-\sigma,k;1),(-\rho-r,h;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\rho-\sigma-r-1,h+k;1)\end{matrix}\right.\right],\end{aligned}$$

where $y=(1-2xt+t^2)^{1/2}$ and the conditions of validity of (3.2) can be easily obtained from those of (2.1).

Proof. To establish (3.2), we first multiply both the sides of (1.7) by $(1-x)^{\rho}(1+x)^{\sigma}\overline{H}_{P,Q}^{M,N}[z(1-x)^h(1+x)^k]$ and integrate with respect to x between the limits -1 to 1, we get

$$(3.3)\qquad \int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_\mu^{(\alpha,\beta)}(t+y)P_\mu^{(\alpha,\beta)}(t-y)$$

$$\overline{H}_{P,Q}^{M,N}\left[z(1-x)^h(1+x)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N}, & (a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M}, & (b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$= \int_{-1}^{1}\frac{(-1)^{\mu}(1+\alpha)_{\mu}(1+\beta)_{\mu}}{(\mu!)^2}\sum_{n=0}^{\mu}\frac{(-\mu)_n(1+\alpha+\beta+\mu)_n}{(1+\alpha)_n(1+\beta)_n}P_n^{(\alpha,\beta)}(x)t^n$$

$$(1-x)^{\rho}(1+x)^{\sigma}\overline{H}_{P,Q}^{M,N}\left[z(1-x)^h(1+x)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx.$$

Now, interchanging the order of integration and summation in the right hand side of (3.3), which is justified due to the absolute convergent of the integral involved in the process and then evaluating the inner integral with the help of (3.1), we easily arrive at the required result (3.2).

[iii]

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}\frac{1}{y}(1-t+y)^{-\alpha}(1+t+y)^{-\beta} \tag{3.4}$$

$$\overline{H}_{P,Q}^{M,N}\left[z(1-x)^h(1+x)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N}, & (a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M}, & (b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$= 2^{-\alpha-\beta+\rho+\sigma+1}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(-n)_r(\alpha+\beta+n+1)_r}{r!(\alpha+1)_r}t^n$$

$$\overline{H}_{P+2,Q+1}^{M,N+2}\left[z2^{h+k}\left|\begin{matrix}(-\sigma,k;1),(-\rho-r,h;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\rho-\sigma-r-1,h+k;1)\end{matrix}\right.\right],$$

where $y=(1-2xt+t^2)^{1/2}$ and the conditions of validity of (3.4) can be easily obtained from those of (2.1).

The proof of the formulae (3.4) can be developed by preceding on similar lines with the help of the result (1.8).

[iv] If we take $A_j\ (j=1,\cdots,N)=B_j\ (j=M+1,\cdots,Q)=1$, in result (3.1), we arrive at the following integral which is also sufficiently general in nature

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x) \tag{3.5}$$

$$H_{P,Q}^{M,N}\left[z(1-x)^h(1+x)^k\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_P,\alpha_P)\\ (b_1,\beta_1)\cdots,(b_Q,\beta_Q)\end{matrix}\right.\right]dx$$

$$= 2^{\rho+\sigma+1}\sum_{r=0}^{n}\frac{(-n)_r(\alpha+\beta+n+1)_r}{r!(\alpha+1)_r}$$

$$H_{P+2,Q+1}^{M,N+2}\left[z2^{h+k}\left|\begin{matrix}(-\sigma,k),(-\rho-r,h),(a_1,\alpha_1),\cdots,(a_P,\alpha_P)\\ (b_1,\beta_1),\cdots,(b_Q,\beta_Q),(-\rho-\sigma-r-1,h+k)\end{matrix}\right.\right]$$

the conditions of validity of (3.5) follow directly from those given with (2.1).

Similarly, taking A_j $(j = 1, \cdots, N) = B_j$ $(j = M+1, \cdots, Q) = 1$, in results (3.2) and (3.4), we obtain corresponding integrals which are also sufficiently general in nature.

[v] Now we give an interesting special case of (3.1) involving '**g**' function connected with a certain class of Feynman integrals [7, p.98, eq.(1.3)]

$$\text{(3.6)} \qquad \int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x) g[\gamma, \eta, \tau, p; z(1-x)^h(1+x)^k]dx$$

$$= 2^{\rho+\sigma-p-1} \frac{K_{d-1}\Gamma(p+1)\Gamma(1/2+\tau/2)}{(-1)^p \pi^{1/2}\Gamma(\gamma)\Gamma(\gamma-\tau/2)} \sum_{r=0}^{n} \frac{(-n)_r(\alpha+\beta+n+1)_r}{r!(\alpha+1)_r} \overline{H}_{5,4}^{1,5} \left[-z2^{h+k} \left| \begin{matrix} (-\sigma, k; 1), (-\rho-r, h; 1), (1-\gamma, 1; 1), (1-\gamma+\frac{\tau}{2}, 1; 1), (1-\eta, 1; 1+p) \\ (0,1), (-\frac{\tau}{2}, 1; 1), (-\eta, 1; 1+p), (-\rho-\sigma-r-1, h+k; 1) \end{matrix} \right. \right],$$

where the **g**-function occurring in (3.6) is defined in the following manner[7, p.98, eq.(1.3)].

$$\text{(3.7)} \qquad g(\gamma, \eta, \tau, p; z) = \frac{K_{d-1}p!\Gamma(1+\tau/2)\mathbf{B}(1/2, 1/2+\tau/2)}{(-1)^p 2^{2+p}\pi\Gamma(\gamma)\Gamma(\gamma-\tau/2)} \int_{-i\infty}^{i\infty} \frac{d\xi}{2\pi i} \frac{(-z)^{\xi}\Gamma(-\xi)\Gamma(\gamma+\xi)\Gamma(\gamma-\tau/2+\xi)}{(\eta+\xi)^{1+p}\Gamma(1+\tau/2+\xi)}$$

$$= \frac{K_{d-1}\Gamma(p+1)\Gamma(1/2+\tau/2)}{(-1)^p 2^{2+p}\pi^{1/2}\Gamma(\gamma)\Gamma(\gamma-\tau/2)} \overline{H}_{3,3}^{1,3} \left[-z \left| \begin{matrix} (1-\gamma, 1; 1), (1-\gamma+\tau/2, 1; 1), (1-\eta, 1; 1+p) \\ (0,1), (-\tau/2, 1; 1), (-\eta, 1; 1+p) \end{matrix} \right. \right],$$

$K_d \equiv 2^{1-d}\pi^{-d/2}/\Gamma(d/2)$ [10, p.4121, eq.(5)] and conditions easily obtainable from (2.1) are satisfied. Specials cases of results (3.2) and (3.4) can also be given but we omit the details. A number of other integrals involving special cases of $\overline{H}$-function and $S_n^m[x]$ polynomials can also we obtained from our main integral but we don't record them here.

Acknowledgement. The author is thankful to Prof. H. M. Srivastava, University of Victoria, Canada and Prof. K. C. Gupta, Jaipur for their kind help and valuable suggestions in the preparation of this paper.

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