KYUNGPOOK Math. J. 46(2006), 565-571

Explicit Formulas of the Generalized Inverse $A_{T,S}^{(2)}$ and Its Applications

Bin Deng
Department of Mathematics, East China Normal University, Shanghai, 200062, P. R. China
e-mail : dengbinboy@hotmail.com

Guo-Liang Chen
Department of Mathematics, East China Normal University, Shanghai, 200062, P. R. China
e-mail : glchen@math.ecnu.edu.cn

Abstract. In this paper, we present the explicit formula of the generalized inverse $A_{T,S}^{(2)}$, and we apply this result to solve restricted linear equation $Ax + y = b,\ x \in T,\ y \in S$ and $Ax + By = b,\ x \in T,\ y \in S$.

1. Introduction

In their seminal paper, [2] Bott and Duffin introduced and widely used an important tool called the "constrained inverse" of the matrix. This inverse is called Bott-Duffin inverse($A_{(T)}^{(-1)} = P_T(AP_T + P_{T^\perp})^{-1}$), Ben Israel and Greville in [1] have mentioned many properties and applications. Later, Y. Chen in his paper [5] defined the generalized Bott-Duffin inverse and gave some properties and applications, G. Chen, G. Liu, Y. Xue in papers [3], [4], [6] defined L-zero matrices in order to simplify the expression of the generalized Bott-Duffin inverse($A_{(T)}^{(+)} = P_T(AP_T + P_{T^\perp})^{+}$). In [10], we have discussed another constrained inverse $A_{T,S}^{(-1)}$, which is defined by $A_{T,S}^{(-1)} = P_{T,S}(AP_{T,S} + P_{S,T})^{-1}$ of a matrix $A \in C^{n\times n}$, where T and S are subspaces of C^n such that $T \oplus S = C^n$. Through considering the properties of this constrained inverse, we establish the relation between the common important generalized inverse and the inverse, see Lemma 4 and Lemma 5.

It is well known that many common important generalized inverse such as the Moore-Penrose inverse A^+, the Drazin inverse $A^{(d)}$, the Group inverse $A^{\#}$, the Boot-Duffin inverse $A_{(L)}^{(-1)}$ and so on, are all generalized inverse $A_{T,S}^{(2)}$, which is a $\{2\}$-inverse of A having the prescribed range T and null space S. In this paper, we

Received July 20, 2005, and, in revised form, September 5, 2005.
2000 Mathematics Subject Classification: 15A03, 15A09.
Key words and phrases: generalized inverse $A_{T,S}^{(2)}$, explicit formulas, restricted linear equation.

present the explicit formula of the generalized inverse $A^{(2)}_{T,S}$, i.e., we also establish the relation between the $A^{(2)}_{T,S}$ and the inverse, and we apply this result to solve restricted linear equations

$$Ax + y = b, \quad x \in T,\ y \in S \tag{1.1}$$

and

$$Ax + By = b, \quad x \in T,\ y \in S. \tag{1.2}$$

We adopt in this paper the same notations on generalized inverse of matrices as those in [1]. And throughout the article (if we don't mention specially). Let I be the identity (unit) matrix, e_i be the ith column of I. $A \in C_t^{m\times n}$ and let T be a subspace of C^n, S be a subspace of C^m, with $\dim(T) = r \le t$, and $\dim(S) = m - r$. Let $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r\}$ be the basis of T, and $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ be the basis of C^n. Let $\{\eta_{r+1}, \cdots, \eta_m\}$ be the basis of S, and $\{\eta_1, \cdots, \eta_m\}$ be the basis of C^m. Let

$$E_1 = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r), \quad E_2 = (\varepsilon_{r+1}, \varepsilon_{r+2}, \cdots, \varepsilon_n), \quad E = (E_1, E_2).$$

$$F_1 = (\eta_1, \eta_2, \cdots, \eta_r), \quad F_2 = (\eta_{r+1}, \eta_{r+2}, \cdots, \eta_m), \quad F = (F_1, F_2).$$

For any $A \in C^{m\times n}$, we denote by

$$\begin{aligned} R(A) &= \{y \in C^m : y = Ax \text{ for some } x \in C^n\} : \ \text{the Range of } A. \\ N(A) &= \{x \in C^n : Ax = 0\} : \ \text{the Null space of } A. \end{aligned}$$

Lemma 1 ([1]). *Let $A \in C_t^{m\times n}$ and let T be a subspace of C^n, let S be a subspace of C^m, $\dim(T) = r \le t$, $\dim(S) = m - r$. Then A has a $\{2\}$-inverse X such that $R(X) = T$, $N(X) = S$ if and only if one of the following conditions is satisfied*:

(1) $AT \oplus S = C^m$;

(2) $A^* S^\perp \oplus T^\perp = C^n$;

(3) $P_{S^\perp} AT = S^\perp$;

(4) $P_T A^* S^\perp = T$.

in which case X is unique.

Lemma 2 ([7], [8]). *Let $A \in C_r^{m\times n}$, T be a subspace of C^n, $b \in AT$, $T \cap N(A) = 0$. Then the unique solution of $Ax = b, (x \in T)$ is given by $x = A^{(2)}_{T,S} b$ for any subspace of S of C^m satisfying $AT \oplus S = C^m$.*

Lemma 3 ([9]). *Let $A \in C^{m\times n}$, $B \in C_m^{m\times m}$, and $C \in C_n^{n\times n}$. Then*:

$$R(AC) = R(A) = B^{-1} R(BA) \tag{1.3}$$

$$N(BA) = N(A) = C N(AC) \tag{1.4}$$

Lemma 4 ([10]). *If $A \in C^{m\times n}$, then*

$$(1)\, A^{+} = A^{*}(AA^{*} + P_{N(A^{*})})^{-1}$$

$$(2)\, A^{+} = (A^{*}A + P_{N(A)})^{-1}A^{*}.$$

Lemma 5 ([10]).

(1) *If $A \in C^{n\times n}$ and* ind*(A)=1, then*

$$A^{\#} = P_{R(A),N(A)}(A + P_{N(A),R(A)})^{-1}.$$

(2) *If $A \in C^{n\times n}$ and* $\mathrm{ind}(A) = k > 1$*, then $\forall l \geq k$*

$$A^{(d)} = P_{R(A^l),N(A^l)}(AP_{R(A^l),N(A^l)} + P_{N(A^l),R(A^l)})^{-1}.$$

2. Main results

Theorem 1. *Let $A \in C_t^{m\times n}$ and let T be a subspace of C^n, let S be a subspace of C^m,* $\dim(T) = r \leq t$*,* $\dim(S) = m - r$ *and $AT \oplus S = C^m$. Then:*

$$(2.1)\quad A^{(2)}_{T,S} = P_T E \begin{pmatrix} I \\ 0 \end{pmatrix} (AP_T E \begin{pmatrix} I \\ 0 \end{pmatrix} + P_S F)^{-1}, \begin{pmatrix} I \\ 0 \end{pmatrix} \in C^{n\times m}, \quad \text{if } m \leq n$$

$$(2.2)\quad A^{(2)}_{T,S} = P_T E \begin{pmatrix} I & 0 \end{pmatrix} (AP_T E \begin{pmatrix} I & 0 \end{pmatrix} + P_S F)^{-1}, \begin{pmatrix} I & 0 \end{pmatrix} \in C^{n\times m}, \quad \text{if } n \leq m$$

$$\begin{aligned}(2.3)\ A^{(2)}_{T,S} &= (E_1, 0)(AE_1, F_2)^{-1} \\ &= (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r, 0, \cdots, 0)(A\varepsilon_1, A\varepsilon_2, \cdots, A\varepsilon_r, \eta_{r+1}, \cdots, \eta_m)^{-1},\ (E_1, 0) \in C^{n\times m}.\end{aligned}$$

Proof. From $P_T E = P_T(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r, 0, \cdots, 0)$, it follows that $P_T = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r, 0, \cdots, 0)(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)^{-1} = (E_1, 0)E^{-1}$ and $P_S = (0, \cdots, 0, \eta_{r+1}, \cdots, \eta_m)(\eta_1, \cdots, \eta_m)^{-1} = (0, F_2)F^{-1}$. So $AP_T E \begin{pmatrix} I \\ 0 \end{pmatrix} + P_S F = (A\varepsilon_1, A\varepsilon_2, \cdots, A\varepsilon_r, \eta_{r+1}, \cdots, \eta_m)$. From $AT \oplus S = C^n$ and $Aspan\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r\} = AT$, we can easily get $(A\varepsilon_1, A\varepsilon_2, \cdots, A\varepsilon_r, \eta_{r+1}, \cdots, \eta_m)$ is nonsingular. Let

$$D = P_T E \begin{pmatrix} I \\ 0 \end{pmatrix} (AP_T E \begin{pmatrix} I \\ 0 \end{pmatrix} + P_S F)^{-1}.$$

Thus

$$\begin{aligned} D &= (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r, 0, \cdots, 0) \begin{pmatrix} I \\ 0 \end{pmatrix} (A(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r, 0, \cdots, 0) \begin{pmatrix} I \\ 0 \end{pmatrix} \\ &\quad + (0, \cdots, 0, \eta_{r+1}, \cdots, \eta_m))^{-1} = (E_1, 0)(AE_1, F_2)^{-1},\ (E_1, 0) \in C^{n\times m}.\end{aligned}$$

For (AE_1, F_2) is nonsingular, we can get $R(D) = R(E_1, 0) = T$. From Lemma 3, $N(D) = (AE_1, F_2)N(E_1, 0)$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in N(E_1, 0)$, then $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r, 0, \cdots, 0) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0 = x_1\varepsilon_1 + \cdots + x_r\varepsilon_r$. Since $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ are linear independence, we can get $x_1 = x_2 = \cdots = x_r = 0$. So we can take $e_{r+1}, \cdots, e_n$ as the basis of $N(E_1, 0)$. Then

$$N(D) = (AE_1, F_2)N(E_1, 0) = (AE_1, F_2)span\{e_{r+1}, \cdots, e_n\} = span\{\eta_{r+1}, \cdots, \eta_m\} = S.$$

$$\begin{aligned} DAD &= (E_1, 0)(AE_1, F_2)^{-1}A(E_1, 0)(AE_1, F_2)^{-1} \\ &= (E_1, 0)(AE_1, F_2)^{-1}((AE_1, F_2) - (0, F_2))(AE_1, F_2)^{-1} \\ &= (E_1, 0)(AE_1, F_2)^{-1} - (E_1, 0)(AE_1, F_2)^{-1}(0, F_2)(AE_1, F_2)^{-1} \\ &= D - D(0, F_2)(AE_1, F_2)^{-1}. \end{aligned}$$

From $N(D) = S$, it follows that $D(0, F_2)(AE_1, F_2)^{-1} = 0$, so $DAD = D$, $N(D) = S$ and $R(D) = T$. From Lemma 1(the uniqueness of $A^{(2)}_{T,S}$), we get (5) and (8). In an analogous manner, we can also get (6). □

Remark. Common important generalized inverse such as the Moore-Penrose inverse A^+, the Drazin inverse $A^{(d)}$, the Group inverse $A^{\#}$, the Boot-Duffin inverse $A^{(-1)}_{(L)}$ are all generalized inverse $A^{(2)}_{T,S}$, from (7) or (8), we can get explicit formulas of these important generalized inverse when we take different T and S.

In [1], it has discussed the solution of the equation $Ax + y = b,\ x \in L,\ y \in L^{\perp}$, similarly we can get next theorem.

Theorem 2. *Let $A \in C_t^{m\times n}$ and let T be a subspace of C^n, let S be a subspace of C^m, $\dim(T) = r \le t$, $\dim(S) = m - r$ and $AT \oplus S = C^m$. Then:*

$$Ax + y = b,\ x \in T,\ y \in S,$$

has for every b, the unique solution

$$x = A^{(2)}_{T,S}b, \tag{2.4}$$

$$y = (I - AA^{(2)}_{T,S})b. \tag{2.5}$$

Proof. Firstly, we will prove $Ax + y = b,\ x \in T,\ y \in S$ has solution is equivalent to that

$$(AP_TE\begin{pmatrix} I \\ 0 \end{pmatrix} + P_SF)z = b \tag{2.6}$$

has solution, when $m \le n$.

(Sufficiency) (2.6) has solution, then take $x = P_T E \begin{pmatrix} I \\ 0 \end{pmatrix} z \in T,\ \ y = P_S F \in S$.

(Necessity) $\forall z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix}$,

$$\begin{aligned}(AP_T E \begin{pmatrix} I \\ 0 \end{pmatrix} + P_S F)z &= A(E_1, 0) \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix} + (0, F_2) \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix} \\ &= z_1 A\varepsilon_1 + \cdots + z_r A\varepsilon_r + z_{r+1}\eta_{r+1} + \cdots + z_m \eta_m \\ &= A(z_1\varepsilon_1 + \cdots + z_r\varepsilon_r) + z_{r+1}\eta_{r+1} + \cdots + z_m\eta_m \\ &= Ax + y = b.\end{aligned}$$

Since $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ are the basis of T, $\eta_{r+1}, \cdots, \eta_m$ are the basis of S, $x \in T$, $y \in S$, we can solve z. From $AT \oplus S = C^n$, we have known $AP_T E \begin{pmatrix} I \\ 0 \end{pmatrix} + P_S F$ is nonsingular. So $Ax + y = b$ has solution $x = P_T E \begin{pmatrix} I \\ 0 \end{pmatrix} (AP_T E \begin{pmatrix} I \\ 0 \end{pmatrix} + P_S F)^{-1} b = A^{(2)}_{T,S} b$ (Theorem 1). When $n \le m$, we can get the conclusion similarly. □

Remark. Only $b \in AT$, $Ax = b$, $x \in T$ is consistent and has a solution. It is the case that Lemma 2 has discussed. But when $AT \oplus S = C^m$, $Ax + y = b$, $x \in T$, $y \in S$ is always consistent and has a unique solution.

Example.

$$Ax + y = b,\ x \in R(A^*),\ y \in N(A^*).$$

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

From Theorem 2, we know $x = A^+ b$.

Taking $\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$, $\varepsilon_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $\varepsilon_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\eta_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

Then $A(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \end{pmatrix}$. $(A\varepsilon_1, A\varepsilon_2, A\varepsilon_3, \eta_4)^{-1} = \frac{1}{4} \begin{pmatrix} 3 & 1 & -3 & 1 \\ 1 & 2 & -2 & 1 \\ -2 & -1 & 3 & 0 \\ 1 & 1 & 1 & -1 \end{pmatrix}$.

$$A^{+} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, 0)(A\varepsilon_1, A\varepsilon_2, A\varepsilon_3, \eta_4)^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & -2 & 1 \\ -2 & 1 & 1 & 0 \\ -2 & -1 & 3 & 0 \end{pmatrix}.$$

$$x = A^{+}b = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad y = b - Ax = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

Similar to Theorem 1 and Theorem 2, we can get

Theorem 3. *Let $A \in C_t^{m\times n}, B \in C^{m\times m}$, $m \le n$ and let T be a subspace of C^n, let S be a subspace of C^m, $\dim(T) = r \le t$, $\dim(S) = m - r$ and $AT \oplus BS = C^m$, then*

$$Ax + By = b, \quad x \in T,\ y \in S,$$

has for every b, the unique solution

$$x = P_T E \begin{pmatrix} I \\ 0 \end{pmatrix} (AP_T E \begin{pmatrix} I \\ 0 \end{pmatrix} + BP_S F)^{-1} b, \tag{2.7}$$

$$y = P_S F (AP_T E \begin{pmatrix} I \\ 0 \end{pmatrix} + BP_S F)^{-1} b. \tag{2.8}$$

Theorem 4. *Let $A \in C_t^{n\times n}, B \in C^{n\times n}$ and T, S be a subspace of C^n, $\dim(T) = r \le t$, $\dim(S) = n - r$, $AT \oplus BS = C^n$ and $T \oplus BS = C^n$, then*

$$P_T E(AP_T E + BP_S F)^{-1} = (AP_{T,BS})^{(2)}_{T,BS}. \tag{2.9}$$

Proof. Similar to Theorem 2, since $AT \oplus BS = C^n$, $AP_T E \begin{pmatrix} I \\ 0 \end{pmatrix} + BP_S F$ is nonsingular. Let $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r\}$ be the basis of T, $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ be the basis of C^n, $\{\eta_{r+1}, \cdots, \eta_n\}$ be the basis of S, and $\{\eta_1, \cdots, \eta_n\}$ be another basis of C^n. $E_1 = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r)$, $E_2 = (\varepsilon_{r+1}, \varepsilon_2, \cdots, \varepsilon_n)$, $E = (E_1, E_2)$. $F_1 = (\eta_1, \cdots, \eta_r)$, $F_2 = (\eta_{r+1}, \cdots, \eta_n)$, $F = (F_1, F_2)$. Let

$$\begin{aligned} D &= P_T E(AP_T E + BP_S F)^{-1} \\ &= (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r, 0, \cdots, 0)(A\varepsilon_1, A\varepsilon_2, \cdots, A\varepsilon_r, B\eta_{r+1}, \cdots, B\eta_n)^{-1}. \end{aligned}$$

So $R(D) = R(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r) = T$. From Lemma 3,

$$\begin{aligned} N(D) &= (A\varepsilon_1, A\varepsilon_2, \cdots, A\varepsilon_r, B\eta_{r+1}, \cdots, B\eta_n) N((\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r, 0, \cdots, 0)) \\ &= (A\varepsilon_1, A\varepsilon_2, \cdots, A\varepsilon_r, B\eta_{r+1}, \cdots, B\eta_n) span\{e_{r+1}, \cdots, e_n\} \\ &= span\{B\eta_{r+1}, \cdots, B\eta_n\} = BS. \end{aligned}$$

$$
\begin{aligned}
DAP_{T,BS}D &= P_TE(AP_TE+BP_SF)^{-1}AP_{T,BS}P_TE(AP_TE+BP_SF)^{-1}\\
&= P_TE(AP_TE+BP_SF)^{-1}AP_TE(AP_TE+BP_SF)^{-1}\\
&= P_TE(AP_TE+BP_SF)^{-1}(AP_TE+BP_SF-BP_SF)(AP_TE+BP_SF)^{-1}\\
&= D-DBP_SF(AP_TE+BP_SF)^{-1}.
\end{aligned}
$$

$R(BP_SF(AP_TE+BP_SF)^{-1}) = R(BP_SF) = span\{B\eta_{r+1},\cdots,B\eta_n\} = N(D)$. So $DBP_SF(AP_TE+BP_SF)^{-1}=0$, i.e., $DAP_{T,BS}D = D$. From the uniqueness of $A^{(2)}_{T,S}$, we can get the conclusion. □

Acknowledgements. This research was supported by National Natural Science Foundation of China (10371044) and Shanghai Priority Academic Discipline (04JC14031).

References

[1] Ben Israel A and Greville T. N. E., Generalized inverse:Theory and applications, 2nd Edition, Springer Verlag, New York, 2003.

[2] R. Bott and R. J. Duffin, *On the algebra of network*, Trans. Amer. Math. Soc., **74**(1953), 99-109.

[3] GuoLiang Chen, GuoMing Liu and YiFeng Xue, *Perturbation analysis of the generalized Bott-Duffin inverse of L-zero matrices*, Linear and multilinear algebra, **51**(2003), 11-20.

[4] GuoLiang Chen, GuoMing Liu and YiFeng Xue, *Perturbation theory for the generalized Bott-Duffin inverse and its applications,* Applied Mathematics and Computation, **129**(2002), 145-155.

[5] Y. Chen, *The generalized Bott-Duffin inverse and its applications*, Linear Algebra and its applitions, **134**(1990), 71-91

[6] YiFeng Xue and GuoLiang Chen, *The expression of the generalized Bott-Duffin inverse and its perturbation theory*, Applied Mathematics and Computation, **132**(2002), 437-444.

[7] Sheng Zhang, *A characterization and determinantal formula for the generalized inverse $A^{(2)}_{T,S}$ and the its applications*, Applied Mathematics and Computation, **125**(2002), 261-269.

[8] Yonglin Chen and Yongxin Yuan, *A oblique projection method for solving restricted linear equations*, J. Nanjing Normal University, **1**(1996), 1-4.

[9] Roger A. Horn and Charles R. Johnson, Matrix analysis, Cambridge [Cambridge shire], New York Cambridge University Press, 1985.

[10] Bin Deng and GuoLiang Chen, *Explicit formulas of the common generalized inverses*, Perturbation analysis and the iterative computing methods, to appear.