KYUNGPOOK Math. J. 46(2006), 553-563

Generalized Weyl's Theorem for Some Classes of Operators

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Abstract. Let A be a bounded linear operator acting on a Hilbert space H. The B-Weyl spectrum of A is the set $\sigma_{Bw}(A)$ of all $\lambda \in \mathbb{C}$ such that $A-\lambda I$ is not a B-Fredholm operator of index 0. Let $E(A)$ be the set of all isolated eigenvalues of A. Recently in [6] Berkani showed that if A is a hyponormal operator, then A satisfies generalized Weyl's theorem $\sigma_{Bw}(A) = \sigma(A) \setminus E(A)$, and the B-Weyl spectrum $\sigma_{Bw}(A)$ of A satisfies the spectral mapping theorem. In [51], H. Weyl proved that weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators [12], and to several classes of operators including semi-normal operators ([9], [10]). Recently W. Y. Lee [35] showed that Weyl's theorem holds for algebraically hyponormal operators. R. Curto and Y. M. Han [14] have extended Lee's results to algebraically paranormal operators. In [19] the authors showed that Weyl's theorem holds for algebraically p-hyponormal operators. As Berkani has shown in [5], if the generalized Weyl's theorem holds for A, then so does Weyl's theorem. In this paper all the above results are generalized by proving that generalized Weyl's theorem holds for the case where A is an algebraically (p,k)-quasihyponormal or an algebarically paranormal operator which includes all the above mentioned operators.

1. Introduction

Let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on infinite dimensional separable Hilbert space H. If $A \in B(H)$ we shall write $N(A)$ and $R(T)$ for the null space and the range of A, respectively. Also, let $\alpha(A) := dimN(A)$, $\beta(A) := dimN(A^*)$, and let $\sigma(A)$, $\sigma_a(A)$ and $\pi_0(A)$ denote the spectrum, approximate point spectrum and point spectrum of A, respectively. An operator $A \in B(H)$ is called Fredholm if it has a closed range, finite dimensional null space, and its range has finite co-dimension. The index of a Fredholm operator is given by

$$I(A) := \alpha(A) - \beta(A).$$

A is called Weyl if it is of index zero, and Browder if it is Fredholm of finite

Received July 19, 2005.
2000 Mathematics Subject Classification: 47A10, 47A12, 47B20.
Key words and phrases: hyponormal operator, (p,k)-quasihyponormal operator, paranormal operator Generalized Weyl's theorem, Browder's theorem.

ascent and descent, equivalently ([26], Theorem 7.9.3) if A is Fredholm and $A-\lambda$ is invertible for sufficiently small $|\lambda|>0$, $\lambda\in\mathbb{C}$. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ of A are defined by

$$\sigma_e(A)=\{\lambda\in\mathbb{C}:\ A-\lambda \text{ is not Fredholm}\},$$

$$\sigma_w(A)=\{\lambda\in\mathbb{C}:\ A-\lambda \text{ is not Weyl}\},$$

$$\sigma_b(A)=\{\lambda\in\mathbb{C}:\ A-\lambda \text{ is not Browder}\},$$

respectively (see [25], [26]). Evidently

$$\sigma_e(A)\subseteq\sigma_w(A)\subseteq\sigma_b(A)=\sigma_e(A)\cup acc\sigma(A),$$

where we write $accK$ for the accumulation points of $K\subseteq\mathbb{C}$. If we write $isoK=K\setminus accK$, then we let

$$\pi_{00}(A):=\{\lambda\in iso\sigma A:\ 0<\alpha(A-\lambda)<\infty\},$$

$$p_{00}(A):=\sigma(A)\setminus\sigma_b(A).$$

We say that Weyl's theorem holds for A if

$$\sigma(A)\setminus\sigma_w(A)=\pi_{00}(A).$$

More generally, Berkani in [5] says that the generalized Weyl's theorem holds for A provided

$$\sigma(A)\setminus\sigma_{Bw}(A)=E(A),$$

where $E(A)$ and $\sigma_{Bw}(A)$ denote the isolated point of the spectrum which are eigenvalues (no restriction on multiplicity) and the set of complex numbers λ for which $A-\lambda I$ fails to be Weyl, respectively. Let X be a Banach space. An operator $A\in B(X)$ is called B-Fredholm by Berkani [5] if there exists $n\in\mathbb{N}$ for which the induced operator

$$A_n: A^n(X)\to A^n(X)$$

is Fredholm in the usual sense, and B-Weyl if in addition A_n has index zero. Note that, if the generalized Weyl's theorem holds for A, then so does Weyl's theorem [5].

For any operator A in $B(H)$ set, as usual, $|A|=(A^*A)^{\frac{1}{2}}$ and $[A^*,A]=A^*A-AA^*=|\,A\,|^2-|\,A^*\,|^2$ (the self commutator of A), and consider the following standard definitions: A is normal if $A^*A=AA^*$, hyponormal if $A^*A-AA^*\geq 0$, p-hyponormal $(0<p\leq 1)$ if $(|A|^{2p}-|A^*|^{2p})\geq 0$.

A is said to be p-quasihyponormal if $A^*((A^*A)^p-(AA^*)^p)A\geq 0$ $(0<p\leq 1)$, (p,k)-quasihyponormal if $A^{*k}((A^*A)^p-(AA^*)^p)A^k\geq 0$ $(0<p\leq 1, k\in\mathbb{N})$, if $p=1$, k=1 and $p=k=1$, then A is k-quasihyponormal, p-quasihyponormal and quasihyponormal respectively. A is said to be normaloid if $\|A\|=r(A)$ (the spectral

radius of A). Let (pH), (HN), $Q(p)$, $(Q(p,k))$ and (NL) denote the classes consisting of hyponormal, p-hyponormal, p-quasihyponormal, (p,k)-quasihyponormal, and normaloid operators. These classes are related by proper inclusion:

$$(HN) \subset (pH) \subset (Q(p)) \subset (Q(p,k)) \subset (NL)$$

(see [35]). Then a (p,k)-quasihyponormal operator is an extension of hyponormal, p-hyponormal, p-quasihyponormal and k- quasihyponormal. A 1-hyponormal operator is called hyponormal operator, which has been studied by many authors and it is known that hyponormal operators have many intersting properties similar to those of normal operators (see [52]). A. Aluthge, B. C. Gupta, A. C. Arora and P. Arora introduced p-hyponormal, p-quasihyponormal and k-quasihyponormal operators, rapectively (see [2], [3], [15]), and now it is known that these operators have many interesting properties (see [16], [36], [45], [49]). It is obvious that p-hyponormal operators are q-hyponormal for $0 < q \leq p$ by Lowner-Heinz's inequality (see [28], [38]). But $(p,1)$-quasihyponormal operators are not always $(q,1)$-quasihyponormal operators for $0 < q \leq p$ (see [50]). Also, it is obvious that (p,k)-quasihyponormal operators are $(p,k+1)$-quasihyponormal.

A is said to be algebraically (p,k)-quasihyponormal if there exists a nonconstant complex polynomial p such $p(A)$ is (p,k)-quasihyponormal.

An operator $A \in B(H)$ is said to be paranormal if

$$||Ax||^2 \leq ||A^2x||||x||$$

for all $x \in H$. We say that A is algebraically paranormal if there exists a nonconstant complex polynomial p such that $p(A)$ is paranormol. In general

$$\text{hyponormal} \subset \text{p} - \text{hyponormal} \subset \text{paranormal} \subset \text{Algebarically paranormal}.$$

A is said to be log-hyponormal if A is invertible and satisfies the following equality

$$log(A^*A) \geq log(AA^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal operators but the converse is not true [46]. However it is very interesting that we may regards log-hyponormal operators as 0-hyponormal operators [46], [47]. The idea of log-hyponormal operator is due to Ando [1] and the first paper in which log-hyponormality appeared is [23]. See [2], [46], [47], [49] for properties of log-hyponormal operators.

We say that an operator $A \in B(H)$ belongs to the class A if $|A^2| \geq |A|^2$. Class A was first introduced by Furuta-Ito-Yamazaki [29] as a subclass of paranormal operators which includes the classes of p-hyponormal and log-hyponormal operators. The following theorem is one of the results associated with a class A operator.

Theorem 1.1 ([29]).

(1) *Every log-hyponormal operator is a class A operator.*

(2) *Every class A operator is a paranormal operator.*

Recently in [6] Berkani showed that if A is a hyponormal operator, then A satisfies generalized Weyl's theorem $\sigma_{Bw}(A) = \sigma(A) \setminus E(A)$, and the B-Weyl spectrum $\sigma_{Bw}(A)$ of A satisfies the spectral mapping theorem. In [51], H. Weyl proved that weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators [12], and to several classes of operators including semi-normal operators ([9], [10]). Recently W. Y. Lee [35] showed that Weyl's theorem holds for algebraically hyponormal operators. R. Curto and Y. M. Han [14] have extended Lee's results to algebraically paranormal operators. In [19] the authors showed that Weyl's theorem holds for algebraically p-hyponormal operators. As Berkani has shown in [5], if the generalized Weyl's theorem holds for A, then so does Weyl's theorem.

In this paper all the above results are generalized by proving that generalized Weyl's theorem holds for the case where A is an algebraically (p, k)-quasihyponormal or an algebarically paranormal operator which includes all the above mentioned operators.

2. Main results

Before proving the following lemma, we need a notation and a definition.

We say that $A \in B(H)$ has the single valued extension property (SVEP) if for every open set $U \subseteq \mathbb{C}$ the only analytic function $f : U \to H$ which satisfies the equation $(A - \lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$.

Lemma 2.1 ([48]). *Let $A \in B(H)$ be a (p, k)-quasihyponormal. If $\lambda \in \sigma_p(A)$ and $\lambda \neq \{0\}$, then $\lambda \in \sigma_p(A^*)$.*

Lemma 2.2. *Let $A \in B(H)$ be a (p, k)-quasihyponormal operator. Then A has SVEP.*

Proof. If A is (p, k)-quasihyponormal, then it follows from ([48], Theorem 4) that

$$||A^k x||^2 \leq ||A^{k-1}x||||A^{k+1}x||,$$

for all unit vector $x \in H$. If $x \in A^{k+1}$, then

$$||A^k x||^2 \leq ||A^{k-1}x||||A^{k+1}x|| = 0$$

and $x \in N(A^k)$. Since the non-zero eigenvalues of a (p, k)-quasihyponormal operator are normal eigenvalues of A by Lemma 2.1. If $0 \neq \lambda \in \sigma_p(A)$ and $(A - \lambda)^{k+1}x = 0$, then

$$(A - \lambda)(A - \lambda)^k x = 0 = (A - \lambda)^*(A - \lambda)^k x$$

and

$$||(A - \lambda)^k x||^2 = ((A - \lambda)^*(A - \lambda)^k x, (A - \lambda)^{k-1}x) = 0.$$

Hence, if A is (p, k)-quasihyponormal, then $asc(A - \lambda) \leq k$.

For $\lambda = 0$, let $A^{k+1}x = 0$ for $x \in H$. Now using the Holder-McCarthy inequality [41] we get

$$||A^k x||^2 = \langle |A|^2 A^{k-1}x, A^{k-1}x\rangle \le \left\langle |A|^{2(p+1)} A^{k-1}x, A^{k-1}x\right\rangle^{\frac{1}{p+1}} ||A^{k-1}x||^{\frac{2p}{p+1}}$$

$$= \langle |A^*|^{2p} A^k x, A^k x\rangle^{\frac{1}{p+1}} ||A^{k-1}x||^{\frac{2p}{p+1}} \le \langle |A|^{2p} A^k x, A^k x\rangle^{\frac{1}{p+1}} ||A^{k-1}x||^{\frac{2p}{p+1}}$$

$$\le \langle |A|^2 A^k x, A^k x\rangle^{\frac{p}{p+1}} ||A^k||^{\frac{2(1-p)}{p+1}} ||A^{k-1}x||^{\frac{2p}{p+1}} = 0.$$

Therefore $asc(A) \le k$ and $asc(A - \lambda) \le k$. Since operators with finite ascent have SVEP [32], A has SVEP at all $\lambda \in \mathbb{C}$. Therefore $f(A)$ has SVEP. Which achieves the proof. □

Before proving the following lemma, we need some notations and definitions.

Let $r(A)$ and $W(A)$ denote the spectral radius and the numerical range of A, respectively. It is well known that $r(A) \le ||A||$ and that $W(A)$ is convex with convex hull $conv\sigma(A) \subseteq \overline{W(A)}$. A is said convexoid if $conv\sigma(A) = \overline{W(A)}$.

Lemma 2.3. *Let A be a (p,k)-quasihyponormal operator and $\lambda \in \mathbb{C}$. If $\sigma(A) = \{\lambda\}$, then $A = \lambda$.*

Proof. We consider two cases:

Case 1 ($\lambda = 0$). Since A is (p,k)-quasihyponormal, A is normaloid [35]. Therefore $A = 0$.

Case 2 ($\lambda \neq 0$). Here A is invertible, and since A is (p,k)-quasihyponormal, A^{-1} is also (p,k)-quasihyponormal ([36], Lemma 3). Therefore A^{-1} is normaloid. On the other hand, $\sigma(A^{-1}) = \{\frac{1}{\lambda}\}$. Hence $||A||||A^{-1}|| = |\lambda||\frac{1}{\lambda}| = 1$. It follows from ([40], Lemma 3) that A is convexoid. Hence $W(A) = \{\lambda\}$ and $A = \lambda$. □

It is shown in [14] that a quasinilpotent algebraically paranormal operator A is nilpotent. By the same way we prove that this result remains hold for a (p,k)-quasihyponormal operator A.

Lemma 2.4. *Let A be a quasinilpotent algebraically (p,k)-quasihyponormal operator. Then A is nilpotent.*

Proof. Assume that $p(A)$ is (p,k)-quasihyponormal for some nonconstant polynomial p. Since $\sigma(p(A)) = p(\sigma(A))$, the operator $p(A) - p(0)$ is quasinilpotent. Thus Lemma 2.3 would imply that

$$cA^m(A - \lambda_1)\cdots(A - \lambda_n) \equiv p(A) - p(0) = 0,$$

where $m \ge 1$. Since $A - \lambda_i$ is invertible for every $\lambda \neq 0$, we must have $A^m = 0$. □

Lemma 2.5. *Let A be an algebraically (p,k)-quasihyponormal operator. Then A is isoloid.*

Proof. Let $\lambda \in iso\sigma(A)$ and let

$$P := \frac{1}{2\pi i}\int_{\partial D}(\mu - A)^{-1}d\mu$$

be the associated Riesz idempotent, where D is a closed disk centered at λ which contains no other points of $\sigma(A)$. We can then represent A as the direct sum

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \text{ where } \sigma(\mathrm{A}_1) = \{\lambda\} \text{ and } \sigma(\mathrm{A}_2) = \sigma(\mathrm{A}) \setminus \{\lambda\}.$$

Since A is algebraically (p,k)-quasihyponormal, $p(A)$ is (p,k)-quasihyponormal for some nonconstant polynomial p. Since $\sigma(A_1) = \lambda$, we must have

$$\sigma(p(A_1)) = p(\sigma(A_1)) = \{p(\lambda)\}.$$

Therefore $p(A_1) - p(\lambda)$ is quasinilpotent. Since $p(A_1)$ is (p,k)-quasihyponormal, it follows from Lemma 2.3 that $p(A_1) - p(\lambda) = 0$. Put $q(z) := p(z) - p(\lambda)$. Then $q(A_1) = 0$, so A_1 is algebraically (p,k)-quasihyponormal . Since $A_1 - \lambda$ is quasinilpotent and algebraically (p,k)-quasihyponormal , it follows from Lemma 2.4 that $A_1 - \lambda$ is nilpotent. Therefore $\lambda \in \pi_0(A_1)$, and hence $\lambda \in \pi_0(A)$. This shows that A is isoloid. □

Theorem 2.6. *Let A be an algebraically (p,k)-quasihyponormal operator. Then generalized Weyl's theorem holds for A.*

Proof. Assume that $\lambda \in \sigma(A) \setminus \sigma_{Bw}(A)$. Then $A - \lambda I$ is B-Weyl and not invertible. We claim that $\lambda \in \partial\sigma(A)$. Assume to the contrary that λ is an interior point of $\sigma(A)$. Then there exists a neigborhood U of λ such that $dim(A - \mu) > 0$ for all $\mu \in U$. It follows from ([21], Theorem 10) that A does not have SVEP. On the other hand, since $p(A)$ is (p,k)-quasihyponormal for nonconstant polynomial p, it follows from Lemma 2.2 that $p(A)$ has SVEP. Hence by ([33], Theorem 3.3.9), A has SVEP, a contradiction. Therefore $\lambda \in \partial\sigma(A)$. Conversely, assume that $\lambda \in E(A)$, then λ is isolated in $\sigma(A)$. From ([31], Theorem 7.1) we have $X = M \oplus N$, where M, N are closed subspaces of X, $U = (A-\lambda I)|_M$ is an invertible operator and $V = (A-\lambda I)|_N$ is a quasinilpotent operator. Since A is algebraically (p,k)-quasihyponormal, V is also algebraically (p,k)-quasihyponormal, from Lemma 2.4 V is nilpotent. Therefore $A - \lambda I$ is Drazin invertible ([44], Proposition 6) and ([39], Corollary 2.2). By ([7], Lemma 4.1) $A - \lambda I$ is a B-Fredholm operator of index 0. □

Theorem 2.7. *Let A be an algebraically paranormal operator. Then generalized Weyl's theorem holds for A.*

Proof. Since a *paranormal* operator has SVEP ([18], Lemma 3.1) and a quasinilpotent algebraically paranormal operator A is nilpotent [14]. Hence the proof can be completed by the same way as the above proof. □

As consequences of the previous theorems, we obtain

Corollary 2.7.

(1) *Every algebraically class A operator satisfies generalized Weyl's theorem. In particular Weyl's theorem holds for an algebarically class A operator.*

(2) *Every algebraically log-hyponormal operator satisfies generalized Weyl's theorem. In particular Weyl's theorem holds for algebraically log-hyponormal operators.*

(3) *Every algebraically p-hyponormal operator satisfies generalized Weyl's theorem. In particular Weyl's theorem holds for algebarically p-hyponormal operators.*

(4) *Every algebraically p-quasihyponormal operator satisfies generalized Weyl's theorem. In particular generalized Weyl's theorem holds for a p-quasihyponormal operators.*

(5) *Every algebraically k-quasihyponormal operator satisfies generalized Weyl's theorem. In particular generalized Weyl's theorem holds for an algebraically k-quasihyponormal operators.*

Before proving the following lemma and theorem we need some notations and definitions

Let

$$\sigma_{BF}(A) = \{\lambda \in \mathbb{C} : \ A - \lambda I \ \text{ is not a B} - \text{Fredholm operator}\}$$

be the B-Fredholm spectrum of T and $\rho_{BF}(A) = \mathbb{C} \setminus \sigma_{BF}(A)$ the B-resolvent set of A

Definition 2.8. Let $A \in B(H)$ we will say that A is of stable index if for each $\lambda, \mu \in \rho_{BF}(A)$, $ind(A - \lambda I)$, $ind(A - \mu I)$ have the same sign index.

Lemma 2.9. *Let $A \in B(H)$ be paranormal or (p,k)-quasihyponormal. Then A is of stable index.*

Proof. We consider the case where A is paranormal. If A is a paranormal operator, then

$$||Ax||^2 \leq ||A^2x||$$

for each unit vecto $x \in H$. This implies that $N(A) = N(A^2)$. Moreover, if A is also a B-Fredholm operator, then there exists an integer n such that $R(A^n)$ is closed and such that

$$A_n : R(A^n) \to R(A^n)$$

is a Fredholm operator. We have

$$ind(A) = ind(A_n) = dimN(A) \cap R(A^n) - dimR(A^n)/R(A^{n+1}) = -dimR(A^n)/R(A^{n+1}).$$

Hence $ind(A) \leq 0$.

Further, if $\lambda \in \rho_{BF}(A)$, then $A - \lambda I$ is a B-Fredholm operator, and $A - \lambda I$ is also paranormal. B y the same way as above, we have $ind(A - \lambda I) \leq 0$. Therefore A is of stable index.

For the case in which A is a (p, k)-quasihyponormal operator. It is konwn [48] that if A is a (p, k)-quasihyponormal operator, then

$$||A^k x|| \leq ||A^{k-1}x|| ||A^{k+1}x||,$$

for every unit vector $x \in H$. The rest of the proof is the same as above. □

Theorem 2.10. *Let A be an algebraically (p, k)-quasihyponormal operator. Then generalized Weyl's theorem holds for $f(A)$ for every function f analytic on a neighborhood of $\sigma(A)$.*

Proof. Assume that A be an algebraically (p, k)-quasihyponormal operator. We prove that $f(\sigma_{Bw}(A)) = \sigma_{Bw}(f(A))$ for every function f analytic on a neighborhood of $\sigma(A)$. Let f be an analytic function on a neighborhood of $\sigma(A)$. Since $\sigma_{Bw}(f(A)) \subseteq f(\sigma_{Bw}(A))$ with no restrection on A, it is sufficient to prove that $f(\sigma_{Bw}(A)) \subseteq \sigma_{Bw}(f(A))$. Assume that $\lambda \notin \sigma_{Bw}(f(A))$. Then $f(A) - \lambda$ is B-Weyl and

$$f(A) - \lambda = c(A - \alpha_1 I)(A - \alpha_2 I) \cdots (A - \alpha_n I)g(A), \tag{2.1}$$

where $c, \alpha_1, \alpha_2, \cdots \alpha_n \in \mathbb{C}$ and $g(A)$ is invertible. Since $f(A) - \lambda I$ is a B-Fredholm operator, from ([5], Theorem 3.4) it follows that for each i, $1 \leq i \leq n$, $A - \alpha_i I$ is a B-Fredholm operator. Moreover, since $ind(f(A) - \lambda I) = 0$ and A is of stable sign index by Lemma 2.6, from ([7], Theorem, 3.2) we have for each i, $1 \leq i \leq n$, $ind(A - \alpha_i I) = 0$. So for each i, $1 \leq i \leq n$, $\alpha_i \notin \sigma_{Bw}(A)$. If $\lambda \in f(\sigma_{Bw}(A))$, there exists $\alpha \in \sigma_{Bw}(A)$ such that $\lambda = f(\alpha)$. Hence

$$0 = f(\alpha) - \lambda = (\alpha - \alpha_1) \cdots (\alpha - \alpha_n)g(\alpha).$$

This implies that $\alpha \in \{\alpha_1, \cdots, \alpha_n\}$. Hence there exists i, $1 \leq i \leq n$, such that $\alpha_i \in \sigma_{Bw}(A)$, contradiction. Hence $\lambda \notin f(\sigma_{Bw}(A))$. It is known ([6], Lemma 2.9) that if A is isoloid then

$$f(\sigma(A) \setminus E(A)) = \sigma(f(A)) \setminus E(f(A))$$

for every analytic function on a neighborhood of $\sigma(A)$. Since A is isoloid by Lemma 2.4 and generalized Weyl's theorem holds for A,

$$\sigma(f(A)) \setminus E(f(A)) = f(\sigma(A)) \setminus E(A)) = f(\sigma_{Bw}(A)) = \sigma_{Bw}(f(A))$$

by ([6], Theorem 2.10). Which achieves the proof. □

Theorem 2.11. *Let A be an algebraically paranormal operator. Then generalized Weyl's theorem holds for $f(A)$ for every function f analytic on a neighborhood of $\sigma(A)$.*

Proof. Since an algebarically paranormal operator is isoloid [14] and it is of stable index by Lemma 2.6, the proof is the same as the proof of the above theorem. □

As consequences of the above theorems, we obtain

Lemma 2.12. *Let $A \in B(H)$. Then the generalized Weyl's theorem holds for $f(A)$ for every analytic function f in a neighborhood of $\sigma(A)$ under either of the following hypothesis*

(1) *A is algebraically class A operator.*

(2) *A is an algebraically log-hyponormal operator.*

(3) *A is an algebraically p-hyponormal operator.*

(4) *A is an algebraically quasihyponormal operator.*

(5) *A is an algebraically p-quasihyponormal operator.*

(6) *A is an algebraically k-quasihyponormal operator.*

Acknowledgement. This research was supported by the K. S. U research center project no. Math/2005/04.

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