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The First Four Terms of Kauffman's Link Polynomial

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Abstract. We give formulas for the first four coefficient polynomials of the Kauffman's link polynomial involving linking numbers and the coefficient polynomials of the Kauffman polynomials of the one- and two-component sublinks. We use mainly the Dubrovnik polynomial, a version of the Kauffman polynomial.

1. Introduction

The Kauffman polynomial [6], [7] of an r-component link L may be written as $F(L;a,z) = \sum_{i\geq 1} F_{i-r}(L;a)z^{i-r}$, where each coefficient $F_{i-r}(L;a)$ is a Laurent polynomial in a, $\mathbf{Z}[a^{\pm 1}]$. We give formulas for the first four coefficient polynomials $F_{i-r}(L;a)$, $1 \leq i \leq 4$, $r \geq 2$, which involve linking numbers and the first i coefficient polynomials of the Kauffman polynomials of the one- and two-component sublinks of L. Although, we will rather use the Dubrovnik polynomial, a version of the Kauffman polynomial, we will also give formulas for the original Kauffman polynomial. In practice, we calculate the difference between the coefficient polynomial of the link L and that of the split union of each component of L.

In a joint work with Miyazawa [5], the author has given similar formulas for the HOMFLY polynomial, the second and third coefficient polynomials $P_{3-r}(L;t)$ and $P_{5-r}(L;t)$ involving linking numbers and the first three coefficient polynomials of the HOMFLY polynomials of the one-, two-, and three-component sublinks of L. They generalize the formula for the first coefficient polynomial $P_{1-r}(L;t)$ by Lickorish and Millett. Also, the first coefficient polynomial of the HOMFLY polynomial agrees with that of the Kauffman polynomial. See Remark 4.2. In this paper, we generalize this to obtain the formulas for the first four coefficient polynomials of the Kauffman polynomial.

In Section 2, we give definitions of the Kauffman and Dubrovnik polynomials. We also give some of their properties. In Section 3, we give some lemmas that we need to prove the theorems. In Section 4, we prove the formulas for the first and

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second coefficient polynomials. In Sections 5 and 6, we prove the formulas for the third and fourth coefficient polynomials, respectively. In Appendix A, we give some errata in [3], since we use some formulas in it.

2. The Kauffman and Dubrovnik polynomials

We give the definitions of the *Kauffman polynomial* $F(L;a,z) \in \mathbf{Z}[a^{\pm 1}, z^{\pm 1}]$ [6], [7] and the *Dubrovnik polynomial* $Y(L;\alpha,\omega) \in \mathbf{Z}[\alpha^{\pm 1}, \omega^{\pm 1}]$, a version of the Kauffman polynomial, which are isotopy invariants of an oriented link L. There are regular isotopy invariants $\Lambda(D) \in \mathbf{Z}[a^{\pm 1}, z^{\pm 1}]$ and $\Lambda^*(D) \in \mathbf{Z}[\alpha^{\pm 1}, \omega^{\pm 1}]$ for an unoriented link diagram D with properties:

$$\Lambda(O) = 1, \tag{2.1}$$

$$\Lambda(D_+) + \Lambda(D_-) = z\left(\Lambda(D_0) + \Lambda(D_\infty)\right), \tag{2.2}$$

$$\Lambda\left(C_+\right) = a\Lambda\left(C\right), \qquad \Lambda\left(C_-\right) = a^{-1}\Lambda\left(C\right); \tag{2.3}$$

$$\Lambda^*(O) = 1, \tag{2.4}$$

$$\Lambda^*(D_+) - \Lambda^*(D_-) = \omega\left(\Lambda^*(D_0) - \Lambda^*(D_\infty)\right), \tag{2.5}$$

$$\Lambda^*\left(C_+\right) = \alpha\Lambda^*\left(C\right), \qquad \Lambda^*\left(C_-\right) = \alpha^{-1}\Lambda^*\left(C\right), \tag{2.6}$$

where O is the diagram of the trivial knot with no crossing, $\{D_+, D_-, D_0, D_\infty\}$ and $\{C_+, C_-, C\}$ are unoriented link diagrams that are identical except near one point where they are as in Fig. 1 and Fig. 2, respectively.

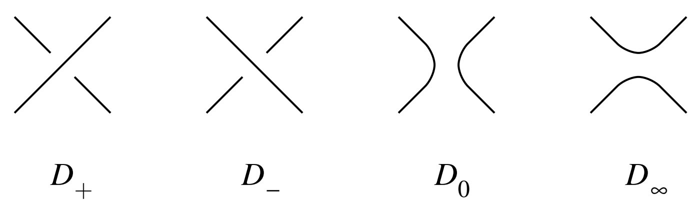

Figure 1:

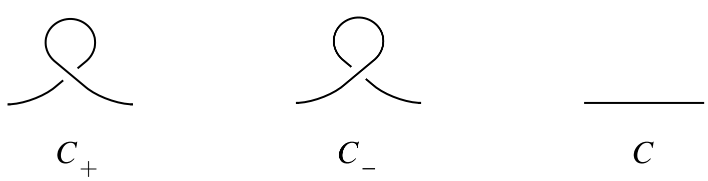

Figure 2:

The Kauffman and Dubrovnik polynomials are defined by

$$F(L;a,z) = a^{-w(D)}\Lambda(D); \tag{2.7}$$
$$Y(L;\alpha,\omega) = \alpha^{-w(D)}\Lambda^*(D), \tag{2.8}$$

where D is a diagram of L and $w(D)$ is its writhe. They are related by the following formula due to Lickorish; see [8, p. 89], [10, p. 177]:

$$Y(L;\alpha,\omega) = (-1)^{r+1}F(L;\sqrt{-1}\alpha, -\sqrt{-1}\omega), \tag{2.9}$$

where r is the number of the components of L. Cf. [3, Eq. (2.2)] and Appendix A.

The Kauffman and Dubrovnik polynomials of an r-component link L are of the form

$$F(L;a,z) = \sum_{i\geq 1} F_{i-r}(L;a)z^{i-r}, \tag{2.10}$$
$$Y(L;\alpha,\omega) = \sum_{i\geq 1} Y_{i-r}(L;\alpha)\omega^{i-r}, \tag{2.11}$$

where $F_{i-r}(L;a) \in \mathbf{Z}[a^{\pm 1}]$, $Y_{i-r}(L;\alpha) \in \mathbf{Z}[\alpha^{\pm 1}]$ and the powers of a and α which appear in them are all even or odd, depending on whether $i-r$ is even or odd; see [3, Proposition 2.1], [14, Proposition 3(ii)]. The proof of these formulas are analogous to that of [11, Proposition 22].

We call $F_{i-r}(L;a)$ and $Y_{i-r}(L;\alpha)$ the *coefficient polynomials* of the Kauffman polynomial $F(L;a,z)$ and the Dubrovnik polynomial $Y(L;\alpha,\omega)$, respectively.

Substituting (2.10) and (2.11) for (2.9), we obtain

$$Y_{i-r}(L;\alpha) = (-1)^{i+1}\sqrt{-1}^{\,i-r}F_{i-r}(L;\sqrt{-1}\alpha) \tag{2.12}$$

for $i \geq 1$.

For an r-component link $L = K_1 \cup K_2 \cup \cdots \cup K_r$, we denote by $\dot{L}$ the split union of the r knots $K_1, K_2, \cdots, K_r$; $\dot{L} = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_r$. Then

$$F(\dot{L};a,z) = \big((a+a^{-1})z^{-1}-1\big)^{r-1}\prod_{j=1}^{r}F(K_j;a,z); \tag{2.13}$$
$$Y(\dot{L};\alpha,\omega) = \big((\alpha-\alpha^{-1})\omega^{-1}+1\big)^{r-1}\prod_{j=1}^{r}Y(K_j;\alpha,\omega); \tag{2.14}$$

cf. [10, Proposition 16.2(iv)]. We put

$$\varPhi_n(L;a) = (-a^2)^{\lambda}F_n(L;a) - F_n(\dot{L};a); \tag{2.15}$$
$$\varUpsilon_n(L;\alpha) = \alpha^{2\lambda}Y_n(L;\alpha) - Y_n(\dot{L};\alpha), \tag{2.16}$$

where λ is the total linking number of L.

In the following, we will mainly consider the Dubrovnik polynomial. However, we will also give equivalent formulas in terms of the Kauffman polynomial.

For an r-component link L, from the observation of Lickorish [14, Proposition 3(i)], we have $F(L;\sqrt{-1},z)=(-1)^{r-1}$ and $Y(L;1,\omega)=1$. Moreover, we have the following; cf. [3, Proposition 2.2].

Proposition 2.1. *Let L be an r-component link.*

(i) *If $1\le i<r$, then the coefficient polynomial $F_{i-r}(L)$ (resp. $Y_{i-r}(L)$) is divisible by $(a+a^{-1})^{r-i}$ (resp. $(\alpha-\alpha^{-1})^{r-i}$).*

(ii) $F_0(L;\sqrt{-1})=(-1)^{r-1}, \quad Y_0(L;1)=1.$

(iii) *If $r<i$, then the coefficient polynomial $F_{i-r}(L)$ (resp. $Y_{i-r}(L)$) is divisible by $a+a^{-1}$ (resp. $\alpha-\alpha^{-1}$).*

Also, we have:

Proposition 2.2. *Suppose that L is an r-component link and $1\le i\le r-1$. Then it holds that*

$$\left[\left(a+a^{-1}\right)^{i-r}F_{i-r}(L;a)\right]_{a=\sqrt{-1}} = (-1)^{i+1}\binom{r-1}{r-i}; \tag{2.17}$$

$$\left[\left(\alpha-\alpha^{-1}\right)^{i-r}Y_{i-r}(L;\alpha)\right]_{\alpha=1} = \binom{r-1}{r-i}. \tag{2.18}$$

Proof. Let $m=r-i$. Then by [3, Corollary 2.1(i)] (see Remark 2.3 below) and [3, Lemma 2.1], we have:

$$\begin{aligned}\left[\frac{F_m(L;a)}{\left(a+a^{-1}\right)^m}\right]_{a=\sqrt{-1}} &= \frac{F_m^{(m)}(L;\sqrt{-1})}{\left[(d^m/da^m)\left(a+a^{-1}\right)^m\right]_{a=\sqrt{-1}}}\\ &= \frac{(-1)^{m+r-1}m!2^m\binom{r-1}{m}}{m!2^m},\end{aligned} \tag{2.19}$$

which gives (2.17). By using (2.12), (2.17) implies (2.18). □

Remark 2.3. The equations in [3, Corollary 2.1(i)] and in [3, Page 417, Line −2] contain errata. The correct forms are given in [4, Appendix].

3. Lemmas

In this section, we prepare some lemmas that we need to prove the theorems in Sects 4-6.

Lemma 3.1. *Let L be the split union of an $(r-1)$-component link L' and a knot*

K; $L = L' \sqcup K$. Then it holds the following.

$$\Upsilon_{m-r}(L) = \begin{cases} (\alpha-\alpha^{-1})\sum_{i=0}^{m-1}\Upsilon_{(m-i)-(r-1)}(L')Y_i(K) \\ \quad +\sum_{j=0}^{m-2}\Upsilon_{(m-j-1)-(r-1)}(L')Y_j(K) & \text{if } m \geq 2; \\ (\alpha-\alpha^{-1})\Upsilon_{2-r}(L')Y_0(K) & \text{if } m = 1. \end{cases} \tag{3.1}$$

Proof. Since

$$\begin{aligned} Y(L) &= \big((\alpha-\alpha^{-1})\omega^{-1}+1\big)\,Y(L')Y(K) \\ &= \big((\alpha-\alpha^{-1})\omega^{-1}+1\big)\left(\sum_{i\geq 0} Y_{i-r+2}(L')\omega^{i-r+2}\right)\left(\sum_{j\geq 0} Y_j(K)\omega^j\right), \end{aligned} \tag{3.2}$$

we have

$$\begin{aligned} & Y_{m-r}(L) \\ = {} & (\alpha-\alpha^{-1})\sum_{i=0}^{m-1} Y_{(m-i)-(r-1)}(L')Y_i(K) + \sum_{j=0}^{m-2} Y_{(m-j-1)-(r-1)}(L')Y_j(K). \end{aligned} \tag{3.3}$$

Similarly, we have

$$\begin{aligned} & Y_{m-r}(\dot{L}) \\ = {} & (\alpha-\alpha^{-1})\sum_{i=0}^{m-1} Y_{(m-i)-(r-1)}(\dot{L}')Y_i(K) + \sum_{j=0}^{m-2} Y_{(m-j-1)-(r-1)}(\dot{L}')Y_j(K). \end{aligned} \tag{3.4}$$

Let λ be the total linking number of L, which is equal to that of L'. Then we have

$$\begin{aligned} & \Upsilon_{m-r}(L) \\ = {} & a^{2\lambda}Y_{m-r}(L) - Y_{m-r}(\dot{L}) \\ = {} & (\alpha-\alpha^{-1})\sum_{i=0}^{m-1} \Upsilon_{(m-i)-(r-1)}(L')Y_i(K) + \sum_{j=0}^{m-2} \Upsilon_{(m-j-1)-(r-1)}(L')Y_j(K), \end{aligned} \tag{3.5}$$

completing the proof. □

Let L_+, L_-, L_0, L_∞ be four oriented links that are identical except near one point where they are as in Fig. 3, where the dotted curves show how the arcs are connected, meaning that the crossings of L_+ and L_- shown in Fig. 3 are between different components. We call $(L_+, L_-, L_0, L_\infty)$ a *mixed skein quadruple.*

Suppose further that L_+ and L_- are r-component links, $r \geq 2$, that differ at a crossing c between the $(r-1)$th and rth components. Then L_0 and L_∞ are

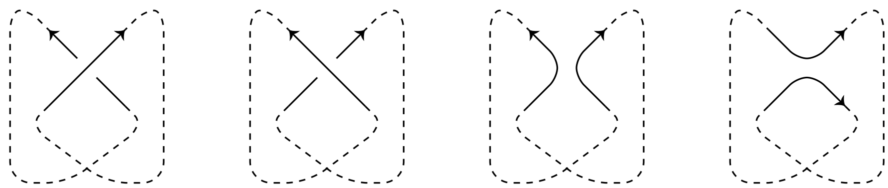

Figure 3:

$(r-1)$-component links. We denote:

$$L_+ = K_+^1 \cup K_+^2 \cup \cdots \cup K_+^{r-1} \cup K_+^r, \quad (3.6)$$
$$L_- = K_-^1 \cup K_-^2 \cup \cdots \cup K_-^{r-1} \cup K_-^r,$$
$$L_0 = K_0^1 \cup K_0^2 \cup \cdots \cup K_0^{r-1},$$
$$L_\infty = K_\infty^1 \cup K_\infty^2 \cup \cdots \cup K_\infty^{r-1},$$

where K_+^r and K_-^{r-1} cross under K_+^{r-1} and K_-^r at the crossing c, respectively. For each component,

- $K_+^i = K_-^i = K_0^i = K_\infty^i$ for $1 \le i \le r-2$, which we simply denote by K^i;
- $K_+^i = K_-^i$ for $1 \le i \le r$.

We denote by L_δ^{ij} the two-component sublink $K_\delta^i \cup K_\delta^j$ of L_δ, $\delta = +, -, 0, \infty$. Then

- $(L_+^{r-1,r}, L_-^{r-1,r}, K_0^{r-1}, K_\infty^{r-1})$ is a mixed skein quadruple;
- $L_+^{ij} = L_-^{ij} = L_0^{ij} = L_\infty^{ij}$ if $1 \le i < j \le r-2$;
- $L_+^{i,r-1} = L_-^{i,r-1}$ and $L_+^{ir} = L_-^{ir}$ if $1 \le i \le r-2$.

We denote by λ_δ^{ij} the linking number of K_δ^i and K_δ^j, $\delta = +, -, 0, \infty$; $\lambda_\delta^{ij} = \mathrm{lk}(K_\delta^i, K_\delta^j)$, and by λ_δ the total linking number of L_δ, $\delta = +, -, 0, \infty$; $\lambda_\delta = \sum_{i<j} \lambda_\delta^{ij}$. Then we have:

$$\lambda_+ = \lambda_0 + \lambda_+^{r-1,r}, \quad (3.7)$$
$$\lambda_- = \lambda_0 + \lambda_-^{r-1,r} = \lambda_+ - 1, \quad (3.8)$$
$$\lambda_\infty = \lambda_0 - 2\sum_{i=1}^{r-2} \lambda_+^{ir}. \quad (3.9)$$

Let μ be the linking number of K_+^r with the remainder of L_+;

$$\mu = \mathrm{lk}(K_+^r, L_+ - K_+^r) = \sum_{i=1}^{r-1} \lambda_+^{ir}. \quad (3.10)$$

Then we have:

$$\alpha Y(L_+) - \alpha^{-1} Y(L_-) = \omega\left(Y(L_0) - \alpha^{-4\mu+2} Y(L_\infty)\right); \tag{3.11}$$

cf. [9, Sect. 3].

Putting

$$\nu = 2\lambda_+ - 1 = 2\lambda_- + 1, \tag{3.12}$$

we have:

Lemma 3.2.

$$\Upsilon_n(L_+) - \Upsilon_n(L_-) = \alpha^\nu \left(Y_{n-1}(L_0) - \alpha^{-4\mu+2} Y_{n-1}(L_\infty)\right). \tag{3.13}$$

Proof. Since $\dot{L}_+$ and $\dot{L}_-$ are isotopic, we have

$$\begin{aligned} \Upsilon_n(L_+) - \Upsilon_n(L_-) &= \alpha^{2\lambda_+} Y_n(L_+) - \alpha^{2\lambda_-} Y_n(L_-) \\ &= \alpha^\nu \left(\alpha Y_n(L_+) - \alpha^{-1} Y_n(L_-)\right), \end{aligned} \tag{3.14}$$

which implies (3.13) by (3.11). □

For an r-component link $L = K^1 \cup K^2 \cup \cdots \cup K^r$, put

$$f_n(L) = \sum_{i<j} \left(\Upsilon_n(L^{ij}) \prod_{k \neq i,j} Y_0(K^k) \right); \tag{3.15}$$

$$g(L) = \sum_{i<j,\ k\neq i,j} \left(\Upsilon_1(L^{ij}) Y_1(K^k) \prod_{l \neq i,j,k} Y_0(K^l) \right), \tag{3.16}$$

where $L^{ij} = K^i \cup K^j$. Then we have:

Lemma 3.3. *Let* $\mu' = \mathrm{lk}(K_+^{r-1}, K_+^r)$. *Then*

$$\begin{aligned} &f_n(L_+) - f_n(L_-) \\ &= \alpha^{2\mu'-1} \left(Y_{n-1}(K_0^{r-1}) - \alpha^{-4\mu'+2} Y_{n-1}(K_\infty^{r-1}) \right) \prod_{k=1}^{r-2} Y_0(K^k); \end{aligned} \tag{3.17}$$

$$\begin{aligned} &g(L_+) - g(L_-) \\ &= \alpha^{2\mu'-1} \left(Y_0(K_0^{r-1}) - \alpha^{-4\mu'+2} Y_0(K_\infty^{r-1}) \right) \sum_{k=1}^{r-2} \left(Y_1(K^k) \prod_{l \neq k, r-1, r} Y_0(K^l) \right). \end{aligned} \tag{3.18}$$

Proof. If $(i,j) \neq (r-1,r)$, then $L_+^{ij} = L_-^{ij}$, and so we denote it by L^{ij}. Then for $\delta = +, -$,

$$f_n(L_\delta) = \sum_{\substack{i<j \\ (i,j)\neq(r-1,r)}} \left(\Upsilon_n(L^{ij}) \prod_{k\neq i,j} Y_0(K^k) \right) + \Upsilon_n(L_\delta^{r-1,r}) \prod_{k=1}^{r-2} Y_0(K^k), \tag{3.19}$$

and so

$$f_n(L_+) - f_n(L_-) = \left(\Upsilon_n(L_+^{r-1,r}) - \Upsilon_n(L_-^{r-1,r}) \right) \prod_{k=1}^{r-2} Y_0(K^k), \tag{3.20}$$

which implies (3.17) by Lemma 3.2.

Next, for $\delta = +, -$,

$$\begin{aligned} g(L_\delta) &= \sum_{\substack{i<j,\ k\neq i,j \\ (i,j)\neq(r-1,r)}} \left(\Upsilon_1(L^{ij}) Y_1(K^k) \prod_{l\neq i,j,k} Y_0(K^l) \right) \\ &\quad + \Upsilon_1(L_\delta^{r-1,r}) \sum_{k=1}^{r-2} \left(Y_1(K^k) \prod_{l\neq k,r-1,r} Y_0(K^l) \right), \end{aligned} \tag{3.21}$$

and so

$$\begin{aligned} & g(L_+) - g(L_-) \\ = & \left(\Upsilon_1(L_+^{r-1,r}) - \Upsilon_1(L_-^{r-1,r}) \right) \sum_{k=1}^{r-2} \left(Y_1(K^k) \prod_{l\neq k,r-1,r} Y_0(K^l) \right), \end{aligned} \tag{3.22}$$

which implies (3.18) by Lemma 3.2. □

4. The first and second terms

In this section, we prove the following theorem giving formulas for the first and second coefficient polynomials.

Theorem 4.1. *For an r-component link L, $r \geq 2$,* it holds that

$$\Upsilon_{1-r}(L) = 0; \tag{4.1}$$

$$\Upsilon_{2-r}(L) = 0. \tag{4.2}$$

Proof. Let $(L_+, L_-, L_0, L_\infty)$ be a mixed skein quadruple with L_+ an r-component link. Suppose that $n = 1-r$ or $2-r$. Then by Lemma 3.2,

$$\Upsilon_n(L_+) = \Upsilon_n(L_-), \tag{4.3}$$

since $Y_{n-1}(L_0) = Y_{n-1}(L_\infty) = 0$ by (2.11). Therefore, putting $L = K^1 \cup K^2 \cup \cdots \cup K^r$, we have $\Upsilon_n(L) = \Upsilon_n(L' \sqcup K^r)$ with $L' = L - K^r$, and thus we obtain $\Upsilon_n(L) = \Upsilon_n(\dot{L}) = 0$ by (2.16), completing the proof. □

By using (2.14) and (2.11), (4.1) and (4.2) imply the following equivalent formulas:

$$Y_{1-r}(L) = \alpha^{-2\lambda}(\alpha - \alpha^{-1})^{r-1} \prod_{i=1}^{r} Y_0(K^i); \tag{4.4}$$

$$\begin{aligned} Y_{2-r}(L) &= \alpha^{-2\lambda}(\alpha - \alpha^{-1})^{r-1} \sum_{i=1}^{r} \left(Y_1(K^i) \prod_{j \neq i} Y_0(K^j) \right) \\ &\quad + (r-1)\alpha^{-2\lambda}(\alpha - \alpha^{-1})^{r-2} \prod_{i=1}^{r} Y_0(K^i). \end{aligned} \tag{4.5}$$

Equations (4.4) and (4.5) immediately imply (2.18) with $i = 1, 2$.

Remark 4.2. Since the first terms of the HOMFLY and Kauffman polynomials agree (see (4.10) below), we obtain (4.4) from a formula (4.9) below by Lickorish and Millett as follows:

The *HOMFLY polynomial* $P(L; t, z) \in \mathbf{Z}[t^{\pm 1}, z^{\pm 1}]$ [1], [13] is an invariant of the isotopy type of an oriented link L, which is defined, as in [2], by the following formulas:

$$P(U; t, z) = 1, \tag{4.6}$$

$$t^{-1}P(L_+; t, z) - tP(L_-; t, z) = zP(L_0; t, z). \tag{4.7}$$

where U is the unknot and L_+, L_-, L_0 are three links that are identical except near one point where they are as in Fig. 4. By [11, Proposition 22], the HOMFLY polynomial of an r-component link L is of the form

$$P(L; t, z) = \sum_{i \geq 0} P_{2i-r+1}(L; t) z^{2i-r+1}, \tag{4.8}$$

where each polynomial $P_{2i-r+1}(L; t) \in \mathbf{Z}[t^{\pm 1}]$ is called the *coefficient polynomial* of the HOMFLY polynomial $P(L; t, z)$; the powers of t which appear in it are all even or odd, depending on whether $2i - r + 1$ is even or odd. Also, for an r-component link $L = K^1 \cup K^2 \cup \cdots \cup K^r$ with total linking number λ, we have:

$$P_{1-r}(L; t) = t^{2\lambda}(t^{-1} - t)^{r-1} \prod_{i=1}^{r} P_0(K^i; t); \tag{4.9}$$

cf. [5, (2.2)]. Furthermore, it holds that:

$$P_{1-r}(L; t) = Y_{1-r}(L; t^{-1}) = \sqrt{-1}^{\,1-r} F_{1-r}(L; \sqrt{-1}t^{-1}). \tag{4.10}$$

In particular, for a knot K,

$$P_0(K;t) = Y_0(K;t^{-1}) = F_0(K;\sqrt{-1}t^{-1}); \tag{4.11}$$

see [10, Proposition 16.9]. By using (4.10) and (4.11), we obtain (4.4) from (4.9).

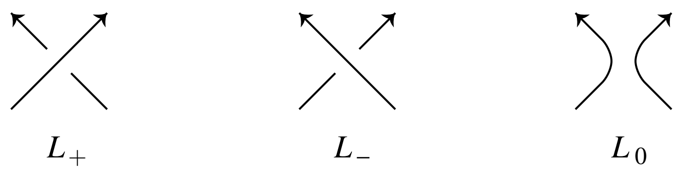

Figure 4:

Remark 4.3. Equation (4.4) and the first equalities of (4.10) and (4.11) are given in (2.4) in [3]. However, it contains errata. See Appendix for the correct form.

Theorem 4.1 is equivalent to the following:

Theorem 4.1(F). *For an r-component link L, $r \geq 2$, it holds that*

$$\varPhi_{1-r}(L) = 0; \tag{4.12}$$

$$\varPhi_{2-r}(L) = 0. \tag{4.13}$$

Equivalently,

$$F_{1-r}(L) = (-a^{-2})^{\lambda}(a+a^{-1})^{r-1}\prod_{i=1}^{r} F_0(K^i); \tag{4.14}$$

$$\begin{aligned} F_{2-r}(L) &= (-a^{-2})^{\lambda}(a+a^{-1})^{r-1}\sum_{i=1}^{r}\left(F_1(K_i)\prod_{j\neq i}F_0(K_j)\right) \\ &\quad - (r-1)(-a^{-2})^{\lambda}(a+a^{-1})^{r-2}\prod_{i=1}^{r}F_0(K_i). \end{aligned} \tag{4.15}$$

5. The third term

In this section, we prove the following theorem giving a formula for the third coefficient polynomial.

Theorem 5.1. *For an r-component link $L = K^1 \cup K^2 \cup \cdots \cup K^r$, $r \geq 2$, it holds that*

$$\varUpsilon_{3-r}(L) = (\alpha - \alpha^{-1})^{r-2}\sum_{i<j}\left(\varUpsilon_1(L^{ij})\prod_{k\neq i,j}Y_0(K^k)\right), \tag{5.1}$$

where $L^{ij} = K^i \cup K^j$, a two-component sublink of L.

Proof. We prove (5.1) by induction on (r, s), where r is the number of the components of L, and s is the number of the crossings in a regular projection of L at which the rth component K^r crosses under the other components $K^1, K^2, \ldots, K^{r-1}$. We order (r, s) lexicographically. For $r = 2$, (5.1) is trivial. So the proof of (5.1) is divided into two steps:

Step 1. Assuming (5.1) for all links with less than r components, $r \geq 2$, we prove for $(r, 0)$, that is, for a split link $L = L' \sqcup K^r$, where $L' = K^1 \cup K^2 \cup \cdots \cup K^{r-1}$.

Step 2. Assuming (5.1) for all links with less than r components and r-component links but with smaller s, $r \geq 2$, we prove for (r, s).

Step 1. By Lemma 3.1, we have

$$\Upsilon_{3-r}(L) = (\alpha - \alpha^{-1})\Upsilon_{4-r}(L')Y_0(K^r) \tag{5.2}$$

since $\Upsilon_{2-r}(L') = \Upsilon_{3-r}(L') = 0$ by Theorem 4.1. By the inductive hypothesis,

$$\Upsilon_{4-r}(L') = (\alpha - \alpha^{-1})^{r-3} \sum_{i<j<r} \left(\Upsilon_1(L^{ij}) \prod_{k \neq i,j} \Upsilon_0(K^k) \right). \tag{5.3}$$

Also, since each L^{ir}, $1 \leq i \leq r-1$, is a split link, $\Upsilon_1(L^{ir}) = 0$, and thus we obtain (5.1) for this case.

Step 2. Let $(L_+, L_-, L_0, L_\infty)$ be a mixed skein quadruple as given in Section 3. Supposing that Theorem 5.1 is true for L_- (resp. L_+), we wish to verify Theorem 5.1 for L_+ (resp. L_-).

By Lemma 3.2, we have

$$\Upsilon_{3-r}(L_+) - \Upsilon_{3-r}(L_-) = \alpha^\nu \left(Y_{2-r}(L_0) - \alpha^{-4\mu+2} Y_{2-r}(L_\infty) \right). \tag{5.4}$$

Since L_0 and L_∞ have $r-1$ components, Theorem 4.1 yields $Y_{2-r}(L_0) = \alpha^{-2\lambda_0} Y_{2-r}(\dot{L}_0)$ and $Y_{2-r}(L_\infty) = \alpha^{-2\lambda_\infty} Y_{2-r}(\dot{L}_\infty)$, and so, by using (3.7)-(3.10) and (3.12), (5.4) becomes

$$\Upsilon_{3-r}(L_+) - \Upsilon_{3-r}(L_-) = \alpha^{2\mu'-1} Y_{2-r}(\dot{L}_0) - \alpha^{-2\mu'+1} Y_{2-r}(\dot{L}_\infty), \tag{5.5}$$

where $\mu' = \lambda_+^{r-1,r}$.

From (2.14), we have:

$$\begin{aligned} &Y_{2-r}(\dot{L}_0) \\ = \; &(\alpha - \alpha^{-1})^{r-2} \prod_{j=1}^{r-1} Y_0(K_0^j) = (\alpha - \alpha^{-1})^{r-2} Y_0(K_0^{r-1}) \prod_{j=1}^{r-2} Y_0(K^j); \end{aligned} \tag{5.6}$$

$$\begin{aligned} &Y_{2-r}(\dot{L}_\infty) \\ = \; &(\alpha - \alpha^{-1})^{r-2} \prod_{j=1}^{r} Y_0(K_\infty^j) = (\alpha - \alpha^{-1})^{r-2} Y_0(K_\infty^{r-1}) \prod_{j=1}^{r-2} Y_0(K^j). \end{aligned} \tag{5.7}$$

Substituting (5.6) and (5.7) for (5.5), we obtain:

$$\begin{aligned}(5.8)\qquad &\Upsilon_{3-r}(L_+) - \Upsilon_{3-r}(L_-) \\ &= \alpha^{2\mu'-1}(\alpha-\alpha^{-1})^{r-2}\left(Y_0(K_0^{r-1}) - \alpha^{-4\mu'+2}Y_0(K_\infty^{r-1})\right)\prod_{j=1}^{r-2}Y_0(K^j),\end{aligned}$$

and so, by (3.17) in Lemma 3.3, we obtain:

$$(5.9)\qquad \Upsilon_{3-r}(L_+) - \Upsilon_{3-r}(L_-) = (\alpha-\alpha^{-1})^{r-2}\left(f_1(L_+) - f_1(L_-)\right),$$

which completes the proof since $\Upsilon_{3-r}(L_\delta) = (\alpha-\alpha^{-1})^{r-2}f_1(L_\delta)$ means that Theorem 5.1 is true for L_δ, $\delta = +, -$. □

Theorem 5.1 is equivalent to the following:

Theorem 5.1(F). *For an r-component link $L = K^1 \cup K^2 \cup \cdots \cup K^r$, $r \geq 2$, it holds that*

$$(5.10)\qquad \Phi_{3-r}(L) = (a+a^{-1})^{r-2}\sum_{i<j}\left(\Phi_1(L^{ij})\prod_{k\neq i,j}F_0(K^k)\right).$$

6. The fourth term

In this section, we prove the following theorem giving a formula for the fourth coefficient polynomial.

Theorem 6.1. *For an r-component link $L = K^1 \cup K^2 \cup \cdots \cup K^r$, $r \geq 2$, it holds that*

$$\begin{aligned}(6.1)\quad \Upsilon_{4-r}(L) \quad = \quad &(\alpha-\alpha^{-1})^{r-2}\sum_{i<j}\left(\Upsilon_2(L^{ij})\prod_{k\neq i,j}Y_0(K^k)\right) \\ &+ (\alpha-\alpha^{-1})^{r-2}\sum_{\substack{i<j\\ k\neq i,j}}\left(\Upsilon_1(L^{ij})Y_1(K^k)\prod_{l\neq i,j,k}Y_0(K^l)\right) \\ &+ (r-2)(\alpha-\alpha^{-1})^{r-3}\sum_{i<j}\left(\Upsilon_1(L^{ij})\prod_{k\neq i,j}Y_0(K^k)\right).\end{aligned}$$

Proof. We prove (6.1) by similar induction to the proof of Theorem 5.1. For $r = 2$, (6.1) is trivial. So the proof of (6.1) is divided into two steps:

Step 1. Assuming (6.1) for all links with less than r components, $r \geq 2$, we prove for a split link $L = L' \sqcup K^r$, where $L' = K^1 \cup K^2 \cup \cdots \cup K^{r-1}$.

Step 2. Let $(L_+, L_-, L_0, L_\infty)$ be a mixed skein quadruple as given in Section 3. Assuming that Theorem 6.1 is true for L_- (resp. L_+), we show Theorem 6.1 for L_+ (resp. L_-).

Step 1. By Lemma 3.1, we have:

$$\Upsilon_{4-r}(L) = (\alpha - \alpha^{-1}) \sum_{i=0}^{3} \Upsilon_{5-i-r}(L') Y_i(K^r) + \sum_{i=0}^{2} \Upsilon_{4-j-r}(L') Y_j(K^r). \tag{6.2}$$

By Theorem 4.1, we have:

$$\Upsilon_{2-r}(L') = \Upsilon_{3-r}(L') = 0. \tag{6.3}$$

By Theorem 5.1, we have:

$$\Upsilon_{4-r}(L') = (\alpha - \alpha^{-1})^{r-3} \sum_{i<j<r} \left(\Upsilon_1(L^{ij}) \prod_{k \neq i,j,r} Y_0(K^k) \right). \tag{6.4}$$

By the inductive hypothesis, we have:

$$\begin{aligned} \Upsilon_{5-r}(L') \quad = \quad & (\alpha - \alpha^{-1})^{r-3} \sum_{i<j<r} \left(\Upsilon_2(L^{ij}) \prod_{k \neq i,j,r} Y_0(K^k) \right) \\ & + (\alpha - \alpha^{-1})^{r-3} \sum_{\substack{i<j<r \\ k \neq i,j,r}} \left(\Upsilon_1(L^{ij}) Y_1(K^k) \prod_{l \neq i,j,k,r} Y_0(K^l) \right) \\ & + (r-3)(\alpha - \alpha^{-1})^{r-4} \sum_{i<j<r} \left(\Upsilon_1(L^{ij}) \prod_{k \neq i,j,r} Y_0(K^k) \right). \end{aligned} \tag{6.5}$$

Substituting (6.3)-(6.5) for (6.2), we obtain:

$$\begin{aligned} & \Upsilon_{4-r}(L) \\ = \quad & (\alpha - \alpha^{-1})^{r-2} \sum_{i<j<r} \left(\Upsilon_2(L^{ij}) \prod_{k \neq i,j,r} Y_0(K^k) \right) Y_0(K^r) \\ & + (\alpha - \alpha^{-1})^{r-2} \sum_{\substack{i<j<r \\ k \neq i,j,r}} \left(\Upsilon_1(L^{ij}) Y_1(K^k) \prod_{l \neq i,j,k,r} Y_0(K^l) \right) Y_0(K^r) \\ & + (r-3)(\alpha - \alpha^{-1})^{r-3} \sum_{i<j<r} \left(\Upsilon_1(L^{ij}) \prod_{k \neq i,j,r} Y_0(K^k) \right) Y_0(K^r) \end{aligned} \tag{6.6}$$

$$+ (\alpha - \alpha^{-1})^{r-2} \sum_{i<j<r} \left(\Upsilon_1(L^{ij}) \prod_{k \neq i,j,r} Y_0(K^k) \right) Y_1(K^r)$$

$$+ (\alpha - \alpha^{-1})^{r-3} \sum_{i<j<r} \left(\Upsilon_1(L^{ij}) \prod_{k \neq i,j,r} Y_0(K^k) \right) Y_0(K^r).$$

This implies (6.1) since $\Upsilon_2(L^{ir}) = \Upsilon_1(L^{ir}) = 0$, $i < r$.

Step 2. By Lemma 3.2, we have:

$$\Upsilon_{4-r}(L_+) - \Upsilon_{4-r}(L_-) = \alpha^{\nu} \left(Y_{3-r}(L_0) - \alpha^{-4\mu+2} Y_{3-r}(L_\infty) \right). \tag{6.7}$$

Since L_0 and L_∞ have $r-1$ components, Theorem 4.1 yields $Y_{3-r}(L_0) = \alpha^{-2\lambda_0} Y_{3-r}(\dot{L}_0)$ and $Y_{3-r}(L_\infty) = \alpha^{-2\lambda_\infty} Y_{3-r}(\dot{L}_\infty)$, and so, using (3.7)-(3.10) and (3.12), (6.7) becomes

$$\Upsilon_{4-r}(L_+) - \Upsilon_{4-r}(L_-) = \alpha^{2\mu'-1} Y_{3-r}(\dot{L}_0) - \alpha^{-2\mu'+1} Y_{3-r}(\dot{L}_\infty), \tag{6.8}$$

where $\mu' = \lambda_+^{r-1,r}$. From (2.14), for $\delta = 0$, ∞, we have:

$$\begin{aligned}
(6.9) \quad & Y_{3-r}(\dot{L}_\delta) \\
= \quad & (\alpha - \alpha^{-1})^{r-2} \sum_{i=1}^{r-1} \left(Y_1(K_\delta^i) \prod_{j \neq i} Y_0(K_\delta^j) \right) + (r-2)(\alpha - \alpha^{-1})^{r-3} \prod_{k=1}^{r-1} Y_0(K_\delta^k) \\
= \quad & (\alpha - \alpha^{-1})^{r-2} Y_0(K_\delta^{r-1}) \sum_{i=1}^{r-2} \left(Y_1(K^i) \prod_{j \neq i, r-1} Y_0(K^j) \right) \\
& + (\alpha - \alpha^{-1})^{r-2} Y_1(K_\delta^{r-1}) \prod_{j=1}^{r-2} Y_0(K^j) \\
& + (r-2)(\alpha - \alpha^{-1})^{r-3} Y_0(K_\delta^{r-1}) \prod_{k=1}^{r-2} Y_0(K^k).
\end{aligned}$$

Thus we have:

$$\begin{aligned}
(6.10) \quad & \Upsilon_{4-r}(L_+) - \Upsilon_{4-r}(L_-) \\
= \quad & (\alpha - \alpha^{-1})^{r-2} \alpha^{2\mu'-1} \left(Y_0(K_0^{r-1}) - \alpha^{-4\mu'+2} Y_0(K_\infty^{r-1}) \right) \\
& \sum_{i=1}^{r-2} \left(Y_1(K^i) \prod_{j \neq i, r-1, r} Y_0(K^j) \right) \\
& + (\alpha - \alpha^{-1})^{r-2} \alpha^{2\mu'-1} \left(Y_1(K_0^{r-1}) - \alpha^{-4\mu'+2} Y_1(K_\infty^{r-1}) \right) \prod_{j=1}^{r-2} Y_0(K^j)
\end{aligned}$$

$$+ (r-2)(\alpha-\alpha^{-1})^{r-3}\alpha^{2\mu'-1}\left(Y_0(K_0^{r-1}) - \alpha^{-4\mu'+2}Y_0(K_\infty^{r-1})\right)\prod_{k=1}^{r-2} Y_0(K^k),$$

and so by Lemma 3.3, this becomes

$$\begin{aligned}(6.11)\qquad & \Upsilon_{4-r}(L_+) - \Upsilon_{4-r}(L_-) \\ = \; & (\alpha-\alpha^{-1})^{r-2}\left(g(L_+) - g(L_-)\right) + (\alpha-\alpha^{-1})^{r-2}\left(f_2(L_+) - f_2(L_-)\right) \\ & + (r-2)(\alpha-\alpha^{-1})^{r-3}\left(f_1(L_+) - f_1(L_-)\right),\end{aligned}$$

which completes the proof since $\Upsilon_{4-r}(L_\delta) = (\alpha-\alpha^{-1})^{r-2}f_2(L_\delta)+(\alpha-\alpha^{-1})^{r-2}g(L_\delta)+(r-2)(\alpha-\alpha^{-1})^{r-3}f_1(L_\delta)$ means that Theorem 6.1 is true for L_δ, $\delta = +, -$. □

Theorem 6.1 is equivalent to the following:

Theorem 6.1(F). *For an r-component link $L = K^1 \cup K^2 \cup \cdots \cup K^r$, $r \geq 2$, it holds that*

$$\begin{aligned}(6.12)\quad \Phi_{4-r}(L) \;=\; & (a+a^{-1})^{r-2}\sum_{i<j}\left(\Phi_2(L^{ij})\prod_{k\neq i,j} F_0(K^k)\right) \\ & + (a+a^{-1})^{r-2}\sum_{\substack{i<j \\ k\neq i,j}}\left(\Phi_1(L^{ij})F_1(K^k)\prod_{l\neq i,j,k} F_0(K^l)\right) \\ & - (r-2)(a+a^{-1})^{r-3}\sum_{i<j}\left(\Phi_1(L^{ij})\prod_{k\neq i,j} F_0(K^k)\right).\end{aligned}$$

7. Final remarks

From the formulas of [5], we may ask if $P_{2i-r-1}(L;t)$, the ith coefficient polynomial of the HOMFLY polynomial of an r-component link L, with $i < r$ is obtained from linking numbers and the HOMFLY polynomials of the j-component sublinks of L with $1 \leq j \leq i$; the case $i = 1, 2, 3$ is true by the theorems in [5]. The simplest case is: if L is a Brunnian link, then is it true that $P_{2i-r-1}(L;t) = 0$ for $1 < i < r$? See [5, Question 4.3]. Przytycki and Taniyama [12] solved this question affirmatively in a more general context and further proved a similar result for the Kauffman polynomial.

For the Kauffman polynomial, from the formulas given in this paper, we may ask if $F_{i-r}(L;a)$ (or $Y_{i-r}(L;\alpha)$), the ith coefficient polynomial of the Kauffman polynomial of an r-component link L, with $i < r$ is obtained from linking numbers and the Kauffman polynomials of the j-component sublinks of L with $1 \leq j \leq (i+1)/2$. The case where $1 \leq i \leq 4$ is true by the theorems in this paper.

Errata in [3]

In this paper, we use several formulas given in [3]. However, some of them contain errata. In [4, Appendix], some errata are corrected, but regrettably not all. Here we correct remaining errata.

- Equation (2.2) in [3] contains an erratum. The correct form is

$$Y(L; a, z) = (-1)^{c+1} F(L; \sqrt{-1}a, -\sqrt{-1}z).$$

- Equation (2.4) in [3] contains errata. The correct form is

$$Y_{1-c}(L; a) = P_{1-c}(L; a^{-1}) = a^{-2\lambda}(a - a^{-1})^{c-1} \prod_{j=1}^{c} Y_0(K_j; a).$$

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