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Asymptotic Results for a Class of Fourth Order Quasilinear Difference Equations

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Abstract. In this paper, the authors first classify all nonoscillatory solutions of equation

$$(1) \qquad \Delta^2 \;\; |\Delta^2 y_n|^{\alpha-1}\Delta^2 y_n \;\; + q_n|y_{\sigma(n)}|^{\beta-1}y_{\sigma(n)} = 0, \quad n \in \mathbb{N}$$

into six disjoint classes according to their asymptotic behavior, and then they obtain necessary and sufficient conditions for the existence of solutions in these classes. Examples are inserted to illustrate the results.

1. Introduction

Consider the fourth order quasilinear difference equation

$$(1) \qquad \Delta^2\left(|\Delta^2 y_n|^{\alpha-1}\Delta^2 y_n\right) + q_n|y_{\sigma(n)}|^{\beta-1}y_{\sigma(n)} = 0, \quad n \in \mathbb{N},$$

where $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$, α and β are positive constants ≥ 1; $\{q_n\}$ is a nonnegative real sequence and $\{\sigma(n)\}$ is a positive sequence of integers such that $\sigma(n) \to \infty$ as $n \to \infty$.

By a solution of equation (1), we mean a real sequence $\{y_n\}$ such that $\{y_n\}$ defined for all $n \in \mathbb{N}$ and satisfies equation (1) for all sufficiently large n. A nontrivial solution $\{y_n\}$ of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

Determining asymptotic behavior of difference equations has received a great deal of attention in the last few years. See, for example, the monographs by Agarwal [1], Agarwal and Wong [2], Kocic and Ladas [6] and the references cited therein. Compared to second order difference equations, the study of higher order equations,

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and in particular fourth order equation, has received considerably less attention even though such equations appear in the problem related to bending of beams (see, for example [3], [8], [9] and the references cited therein). Further an important special case of fourth order difference equations is the discrete version of the Schrödinger equation. This motivated our interest in studying the asymptotic behavior of solutions of equation (1).

In [5], [7], [10], [11], [12], the authors considered equation of type (1) and classified all nonoscillatory solutions of such equations into two types and obtained criteria for the existence of solution in these types. In this paper, we first classify all nonoscillatory solutions of equation (1) into six disjoint classes according to their asymptotic behavior, and then obtain necessary and sufficient conditions for the existence of solutions in these classes. Hence the results obtained in this paper gives a more detailed information on the asymptotic behavior of nonoscillatory solutions of equation (1). Examples are inserted to illustrate the results.

2. Nonoscillation Theorems

In this section, we study in detail the structure of the set of all possible nonoscillatory solutions of equation (1). It suffices to restrict our attention to eventually positive solutions of equation (1), since if $\{y_n\}$ is a solution of equation (1), then so is $\{-y_n\}$. Let $\{y_n\}$ be one such solution of equation (1). Then $\{y_n\}$ satisfies (see [5]) either

(I) $\quad \Delta y_n > 0, \Delta^2 y_n > 0, \quad \Delta\left(\left|\Delta^2 y_n\right|^{\alpha-1}\Delta^2 y_n\right) > 0$ for all large n

or

(II) $\quad \Delta y_n > 0, \Delta^2 y_n < 0, \quad \Delta\left(\left|\Delta^2 y_n\right|^{\alpha-1}\Delta^2 y_n\right) > 0$ for all large n.

From the above, it is easy to see that $\{y_n\}$, $\{\Delta y_n\}$, $\left\{\Delta^2 y_n\right\}$ and $\Delta\left(\left|\Delta^2 y_n\right|^{\alpha-1}\Delta^2 y_n\right)$ are tend to finite or infinite as $n \to \infty$.

Let $\lim\limits_{n\to\infty} \Delta^i y_n = \eta_i, i = 0,1,2$ and $\lim\limits_{n\to\infty} \Delta\left(\left|\Delta^2 y_n\right|^{\alpha-1}\Delta^2 y_n\right) = \eta_3$. It is clear from equation (1) that η_3 is a finite nonnegative number.

If $\{y_n\}$ satisfies Case (I), then the values $\eta_i, \;\; i = 0,1,2,3$ falls into one of the following three cases:

(c_1) $\eta_0 = \eta_1 = \eta_2 = \infty, \eta_3 \in (0,\infty)$;

(c_2) $\eta_0 = \eta_1 = \eta_2 = \infty, \eta_3 = 0$;

(c_3) $\eta_0 = \eta_1 = \infty, \eta_2 \in (0,\infty), \eta_3 = 0$.

If $\{y_n\}$ satisfies Case (II), then the values $\eta_i, i = 0,1,2,3$ falls into one of the following three cases:

(c_4) $\eta_0 = \infty, \eta_1 \in (0,\infty), \eta_2 = \eta_3 = 0$;

(c_5) $\eta_0 = \infty, \eta_1 = \eta_2 = \eta_3 = 0$;

(c_6) $\eta_0 \in (0,\infty), \eta_1 = \eta_2 = \eta_3 = 0$.

Equivalent expressions for these six classes of positive solutions of equation (1) are as follows:

(c_1) $\lim_{n\to\infty} \frac{y_n}{n^{2+\frac{1}{\alpha}}} = \text{constant} > 0$;

(c_2) $\lim_{n\to\infty} \frac{y_n}{n^{2+\frac{1}{\alpha}}} = 0$, $\lim_{n\to\infty} \frac{y_n}{n^2} = \infty$;

(c_3) $\lim_{n\to\infty} \frac{y_n}{n^2} = \text{constant} > 0$;

(c_4) $\lim_{n\to\infty} \frac{y_n}{n} = \text{constant} > 0$;

(c_5) $\lim_{n\to\infty} \frac{y_n}{n} = 0$; $\lim_{n\to\infty} y_n = \infty$;

(c_6) $\lim_{n\to\infty} y_n = \text{constant} > 0$.

Next we obtain necessary and sufficient conditions for the existence of positive solutions of the four types $(c_1), (c_3), (c_4)$ and (c_6) of equation (1).

Theorem 1. *The equation* (1) *has a positive solution of type* (c_1) *if and only if*

$$\sum_{n=n_0}^{\infty} (\sigma(n))^{\left(2+\frac{1}{\alpha}\right)\beta} q_n < \infty. \tag{2}$$

Proof. Assume equation (1) has a positive solution $\{y_n\}$ of type (c_1). Then summing (1) from n to ∞, we have

$$\Delta\left(\left|\Delta^2 y_n\right|^{\alpha-1} \Delta^2 y_n\right) = \eta_3 + \sum_{s=n}^{\infty} q_s y_{\sigma(s)}^{\beta}, \tag{3}$$

for $n \geq n_0$. If n_0 is sufficiently large, (3) implies that

$$\sum_{n=n_0}^{\infty} q_n y_{\sigma(n)}^{\beta} < \infty.$$

This together with the asymptotic relation $\lim_{n\to\infty} \frac{y_n}{n^{2+\frac{1}{\alpha}}} = \text{constant} > 0$ shows that condition (2) is satisfied.

Suppose now that (2) holds. Let $k > 0$ be any given constant. Choose an integer $N > n_0 > 0$ large enough so that

$$\left(\frac{\alpha^2}{(\alpha+1)(2\alpha+1)}\right)^{\beta} \sum_{n=N}^{\infty} (\sigma(n))^{\left(2+\frac{1}{\alpha}\right)^{\beta}} q_n \leq \frac{(2k)^{\alpha} - k^{\alpha}}{(2k)^{\beta}}.$$

Let $N_0 = \min\left\{N, \inf_{n\geq N} \sigma(n)\right\}$, and define

$$G(n, N) = \begin{cases} \sum_{s=N}^{n-1} (n-s-1)(s-N)^{\frac{1}{\alpha}}, & n > N, \\ 0, & N_0 \leq n \leq N. \end{cases}$$

Consider the Banach space $\mathcal{B}$ of all real sequences $y = \{y_n\}$, $n \geqslant N_0$ with the sup-norm $\|y\| = \sup_{n \geqslant N_0} |y_n|$. Let $\mathcal{S}$ be the subset of $\mathcal{B}$ defined by

$$\mathcal{S} = \{y \in \mathcal{B} : kG(n, N) \leq y_n \leq 2kG(n, N), \quad n \geqslant N_0\}.$$

Clearly, $\mathcal{S}$ is a nonempty, bounded, closed and convex subset of $\mathcal{B}$.We define the partial ordering on $\mathcal{B}$ in the usual sense, that is $x \leq y$ means $x_n \leq y_n$ for all $n \geq N$. Then, for every subset A of $\mathcal{S}$, both $\sup A$ and $\inf A$ exists in $\mathcal{S}$.

Define the operator $\mathcal{T} : \mathcal{S} \to \mathcal{B}$ by

$$(4) \qquad (\mathcal{T}y)_n = \begin{cases} \sum\limits_{s=N}^{n-1} (n-s-1) \left[\sum\limits_{t=N}^{s-1} \left(k^\alpha + \sum\limits_{j=t}^{\infty} q_j y_{\sigma(j)}^{\beta} \right) \right]^{\frac{1}{\alpha}}, & n \geq N+1, \\ 0, & N_0 \leqslant n \leqslant N. \end{cases}$$

If $y_n \in \mathcal{S}$, then for $n \geq N$

$$(\mathcal{T}y)_n \geqslant k \sum_{s=N}^{n-1} (n-s-1)(s-N)^{\frac{1}{\alpha}} = kG(n, N)$$

and

$$\begin{aligned} (\mathcal{T}y)_n &\leqslant \sum_{s=N}^{n-1} (n-s-1) \left[\sum_{t=N}^{s-1} \left(k^\alpha + \sum_{j=t}^{\infty} q_j \left(2kG(\sigma(j), N)\right)^\beta \right) \right]^{\frac{1}{\alpha}} \\ &\leqslant \sum_{s=N}^{n-1} (n-s-1) \left[\sum_{t=N}^{s-1} \left(k^\alpha + \left(\frac{2k\alpha^2}{(\alpha+1)(2\alpha+1)} \right)^\beta \sum_{j=t}^{\infty} q_j \sigma(j)^{\left(2+\frac{1}{\alpha}\right)\beta} \right) \right]^{\frac{1}{\alpha}} \\ &\leqslant 2k \sum_{s=N}^{n-1} (n-s-1)(s-N)^{\frac{1}{\alpha}} = 2kG(n, N) \end{aligned}$$

and hence $\mathcal{T}y \in \mathcal{S}$. Thus, T maps S into itself. Clearly $\mathcal{T}$ is increasing. Therefore, by Knaster-Tarski fixed point theorem, we conclude that there exists $y \in \mathcal{S}$ such that $\mathcal{T}y = y$. That is, $\{y_n\}$ is a positive solution of equation (1). From (4), we see that $\lim_{n\to\infty} \Delta\left(\left(\Delta^2 y_n\right)^\alpha\right) = k^\alpha > 0$ and hence $\{y_n\}$ is a desired solution of type (c_1). This completes the proof. □

Theorem 2. *Equation* (1) *has a positive solution of type* (c_3) *if and only if*

$$(5) \qquad \sum_{n=n_0}^{\infty} n\left(\sigma(n)\right)^{2\beta} q_n < \infty.$$

Proof. Assume equation (1) has a positive solution $\{y_n\}$ of type (c_3). Summing equation (3) from n to ∞ and then rearranging, we have

$$\sum_{n=N}^{\infty} (n-N+1)\, q_n y^{\beta}_{\sigma(n)} < \infty,$$

where N is chosen sufficiently large. Since $\lim\limits_{n\to\infty} \frac{y_n}{n^2} =$ constant > 0, we see that condition (5) holds.

Now, assume that condition (5) holds. Let $k > 0$ be an arbitrary fixed constant and choose an integer $N > n_0 > 0$ so large that

$$\sum_{n=N}^{\infty} n\,(\sigma(n))^{2\beta}\, q_n \leqslant \frac{[(2k)^{\alpha} - k^{\alpha}]}{k^{\beta}}.$$

Let $N_0 = \min\left\{N, \inf\limits_{n\geq N} \sigma(n)\right\}$ and consider the Banach space $\mathcal{B}$ of all real sequences $y = \{y_n\}$, $n \geqslant N_0$ with the sup-norm $\|y\| = \sup\limits_{n\geqslant N_0} \left\{\frac{|y_n|}{n^3}\right\}$. Let $\mathcal{S}$ be the subset of $\mathcal{B}$ defined by

$$\mathcal{S} = \left\{y \in \mathcal{B} : \frac{k}{2}(n-N)_+^2 \leqslant y_n \leqslant k(n-N)_+^2, n \geqslant N_0\right\},$$

where $(n-N)_+^2 = (n-N)^2$ if $n > N$ and $(n-N)_+^2 = 0$ if $N_0 \leq n \leq N$.

Clearly, $\mathcal{S}$ is a closed ,bounded,and convex subset of $\mathcal{B}$.We define an operator $\mathcal{T} : \mathcal{S} \to \mathcal{B}$ as follows:

$$(6)\quad (\mathcal{T}y)_n = \begin{cases} \sum\limits_{s=N}^{n-1} (n-s-1)\left[(2k)^{\alpha} - \sum\limits_{t=s}^{\infty} (t-s+1)\, q_t y^{\beta}_{\sigma(t)}\right]^{\frac{1}{\alpha}}, & n \geqslant N+1, \\ 0, & N_0 \leqslant n \leqslant N. \end{cases}$$

We first show that $\mathcal{TS} \subset \mathcal{S}$. Indeed, if $y \in \mathcal{S}$ it is clear from (6) that $(\mathcal{T}y)_n \leq k(n-N)^2$. Furthermore for $n > N$, we have

$$\begin{aligned} (\mathcal{T}y)_n &\geqslant \sum_{s=N}^{n-1} (n-s-1)\left[(2k)^{\alpha} - k^{\beta}\sum_{t=s}^{\infty} (t-s+1)\, q_t\, (\sigma(t)-N)^{\beta}\right]^{\frac{1}{\alpha}} \\ &\geqslant \sum_{s=N}^{\infty} (n-s-1)\left[(2k)^{\alpha} - k^{\beta}\frac{[(2k)^{\alpha} - k^{\alpha}]}{k^{\beta}}\right]^{\frac{1}{\alpha}} \\ &\geqslant \frac{k}{2}(n-N)^2. \end{aligned}$$

Thus, T maps S into itself. Next, we prove $\mathcal{T}$ is continuous. Let $y^i = \{y^i_n\}$ be a sequence in $\mathcal{S}$ such that $\lim\limits_{i\to\infty} ||y^i - y|| = 0$. Because $\mathcal{S}$ is closed, $y \in \mathcal{S}$. Then from (6), one can easily prove that $\lim\limits_{i\to\infty} ||\left(\mathcal{T}y^i\right)_n - (\mathcal{T}y)_n|| = 0$ and so $\mathcal{T}$ is continuous.

Finally, in order to apply Schauder fixed point theorem, we need to show that $\mathcal{TS}$ is relatively compact. In view of a result of Cheng and Patula [4] it suffices to show that $\mathcal{TS}$ is uniformly Cauchy. To see this, we have to show that, given $\varepsilon > 0$, there is an integer $N_1 > N$, such that $m > n > N_1$,

$$\left| \frac{(Ty)_m}{m^3} - \frac{(Ty)_n}{n^3} \right| < \varepsilon$$

for any $y \in \mathcal{S}$. Indeed , by (6) we have

$$\left| \frac{(Ty)_m}{m^3} - \frac{(Ty)_n}{n^3} \right| \leqslant \frac{2k}{n-N} \to 0.$$

Therefore, by Schauder fixed point theorem, $\mathcal{T}$ has a fixed element y in $\mathcal{S}$, that is,

$$y_n = \sum_{s=N}^{n-1} (n-s-1) \left[(2k)^\alpha - \sum_{t=s}^{\infty} (t-s+1)\, q_t y_{\sigma(t)}^{\beta} \right]^{\frac{1}{\alpha}}, \quad n \geqslant N. \tag{7}$$

From (7), we have see that $\{y_n\}$ is a positive solution of equation (1) for all large n with the property that $\lim_{n\to\infty} \Delta^2 y_n = 2k > 0$. Thus, $\{y_n\}$ is a type (c_3) solution of equation (1). This completes the proof. □

Theorem 3. *Equation* (1) *has a positive solution of type* (c_4) *if and only if*

$$\sum_{n=n_0}^{\infty} \left[\sum_{s=n}^{\infty} (s-n+1)\, q_s\, (\sigma(s))^{\beta} \right]^{\frac{1}{\alpha}} < \infty. \tag{8}$$

Proof. To prove the necessary part of the theorem it suffices to observe that a positive solution $\{y_n\}$ of type (c_4) satisfies $\lim_{n\to\infty} \frac{y_n}{n} =$ constant > 0 and

$$\sum_{n=N}^{\infty} \left[\sum_{s=n}^{\infty} (s-n+1)\, q_s y_{\sigma(s)}^{\beta} \right]^{\frac{1}{\alpha}} < \infty.$$

To prove the sufficient part, assume (8) holds, and for any fixed constant $k > 0$ choose an integer $N > n_0 > 0$ such that

$$\sum_{n=N}^{\infty} \left[\sum_{s=n}^{\infty} (s-n+1)\, (\sigma(s))^{\beta}\, q_s \right]^{\frac{1}{\alpha}} < 2^{-\frac{\beta}{\alpha}} k^{1-\frac{\beta}{\alpha}}.$$

Let $N_0 = \min\left\{ N, \inf_{n\geq N} \sigma(n) \right\}$, and consider the Banach space $\mathcal{B}$ of all real sequences $y = \{y_n\}, n \geqslant N_0$ with the sup-norm $\|y\| = \sup_{n\geqslant N_0} \left\{ \frac{|y_n|}{n} \right\}$. Consider $\mathcal{S} \subset \mathcal{B}$ and a mapping $\mathcal{T} : \mathcal{S} \to \mathcal{B}$ defined by

$$\mathcal{S} = \{ y \in \mathcal{B} : kn \leqslant y_n \leqslant 2kn, \quad n \geqslant N_0 \}$$

and

$$(Ty)_n = \begin{cases} kn + \sum_{s=N}^{n-1} \sum_{t=s}^{\infty} \left[\sum_{j=t}^{\infty} (j-t+1) q_j y_{\sigma(j)}^{\beta} \right]^{\frac{1}{\alpha}}, & n \geqslant N+1, \\ kn, & N_0 \leqslant n \leqslant N. \end{cases}$$

Clearly, $\mathcal{S}$ is a nonempty, bounded,closed and convex subset of $\mathcal{B}$. We define the partial ordering on $\mathcal{B}$ as in Theorem 1. Then, for every subset A of $\mathcal{S}$, both $\sup A$ and $\inf A$ exists in $\mathcal{S}$. It can be easily verified that $\mathcal{TS} \subset \mathcal{S}$ and $\mathcal{T}$ is an increasing mapping. Therefore, by Knaster-Tarski fixed point theorem, $\mathcal{T}$ has a fixed point $y \in \mathcal{S}$, which gives rise to a positive type (c_4) solution of equation (1), since it satisfies

$$y_n = kn + \sum_{s=N}^{n-1} \sum_{t=s}^{\infty} \left[\sum_{j=t}^{\infty} (j-t+1) q_j y_{\sigma(j)}^{\beta} \right]^{\frac{1}{\alpha}}, \quad n \geqslant N. \tag{9}$$

From (9), we have $\lim_{n\to\infty} \Delta y_n = k$. This completes the proof. □

Theorem 4. *Equation* (1) *has a positive solution of type* (c_6) *if and only if*

$$\sum_{n=n_0}^{\infty} n \left[\sum_{s=n}^{\infty} (s-n+1)\, q_s \right]^{\frac{1}{\alpha}} < \infty. \tag{10}$$

Proof. Let $\{y_n\}$ be a type (c_6) solution of equation (1). Summing first, equation (3) with $\eta_3 = 0$ three times yield

$$y_n = \eta_0 - \sum_{s=n}^{\infty} (s-n+1) \left[\sum_{t=s}^{\infty} (t-s+1)\, q_t y_{\sigma(t)}^{\beta} \right]^{\frac{1}{\alpha}}$$

which implies

$$\sum_{n=N}^{\infty} n \left[\sum_{s=n}^{\infty} (t-n+1)\, q_s y_{\sigma(s)}^{\beta} \right]^{\frac{1}{\alpha}} < \infty.$$

Since $\lim_{n\to\infty} y_n =$ constant > 0, (10) follows immediately from the last inequality.

Suppose now that (10) holds. Let $k > 0$ be any fixed constant and choose an integer $N > n_0 > 0$ so large that

$$\sum_{n=N}^{\infty} n \left[\sum_{s=n}^{\infty} (s-n+1)\, q_s \right]^{\frac{1}{\alpha}} \leqslant \frac{1}{2} k^{1-\frac{\beta}{\alpha}}.$$

Let $N_0 = \min\{N, \inf_{n\geq N} \sigma(n)\}$, and consider the Banach space $\mathcal{B}$ of all real sequences $y = \{y_n\}$, $n \geqslant N_0$ with the sup-norm $\|y\| = \sup_{n\geqslant N_0} |y_n|$. Consider the set

$\mathcal{S} \subset \mathcal{B}$ and the mapping $\mathcal{T} : \mathcal{S} \to \mathcal{B}$ defined by

$$\mathcal{S} = \left\{ y \in \mathcal{B} : \frac{k}{2} \leqslant y_n \leqslant k, n \geqslant N_0 \right\}$$

and

$$(\mathcal{T}y)_n = \begin{cases} k - \sum\limits_{s=n}^{\infty} (s-n+1) \left[\sum\limits_{t=s}^{\infty} (t-s+1)\, q_t y_{\sigma(t)}^{\beta} \right]^{\frac{1}{\alpha}} & n \geqslant N+1, \\ (\mathcal{T}y)_N , & N_0 \leqslant t \leqslant N. \end{cases}$$

Then it can be easily verified that $\mathcal{T}$ has a fixed point y in $\mathcal{S}$. This fixed point gives rise to a required positive type (c_6) solution of equation (1), since it satisfies

$$y_n = k - \sum_{s=n}^{\infty} (s-n+1) \left[\sum_{t=s}^{\infty} (t-s+1)\, q_t y_{\sigma(t)}^{\beta} \right]^{\frac{1}{\alpha}}, \quad n \geqslant N. \tag{11}$$

From (11), we have $\lim\limits_{n\to\infty} y_n = k$.This completes the proof. □

Next we discuss about the existence of positive solutions of types (c_2) and (c_5) of equation (1). We are content with the sufficient conditions for the existence of positive solutions with "intermediate" growth.

Theorem 5. *Equation* (1) *has a positive solution of type* (c_2) *if*

$$\sum_{n=n_0}^{\infty} (\sigma(n))^{\left(2+\frac{1}{\alpha}\right)\beta} q_n < \infty \tag{12}$$

and

$$\sum_{n=n_0}^{\infty} n\, (\sigma(n))^{2\beta} q_n = \infty \tag{13}$$

are hold.

Proof. Choose an integer $N \geq n_0 > 0$ large enough so that $N_0 = \min\{N, \inf\limits_{n\geq N} \sigma(n)\}$, and

$$\sum_{n=N}^{\infty} (\sigma(n))^{\left(2+\frac{1}{\alpha}\right)\beta} q_n < \frac{1}{2^{\alpha+1}} \left(\frac{(\alpha+1)(2\alpha+1)}{\alpha^2} \right)^{\alpha}. \tag{14}$$

Consider the Banach space $\mathcal{B}$ of all real sequences $y = \{y_n\}$, $n \geqslant N_0$ with the sup-norm $\|y\| = \sup\limits_{n\geqslant N_0} \left\{ \frac{|y_n|}{n^2} \right\}$. Consider the set $\mathcal{S} \subset \mathcal{B}$ and the mapping $\mathcal{T} : \mathcal{S} \to \mathcal{B}$ defined by

$$\mathcal{S} = \left\{ y \in \mathcal{B} : \frac{1}{2^{1+\frac{1}{\alpha}}} (n-N)_+^2 \leqslant y_n \leqslant n^{2+\frac{1}{\alpha}}, \quad n \geqslant N_0 \right\} \tag{15}$$

and

$$(16) \qquad (\mathcal{T}y)_n = \begin{cases} \sum_{s=N}^{n-1} (n-s+1) \left[\frac{1}{2} + \sum_{t=N}^{s-1} \sum_{j=t}^{\infty} q_j y_{\sigma(j)}^{\beta} \right]^{\frac{1}{\alpha}} & n > N, \\ 0, & N_0 \leqslant n \leqslant N. \end{cases}$$

If $y \in S$, then, using the inequality $(A+B)^{\frac{1}{\alpha}} \leqslant (2A)^{\frac{1}{\alpha}} + (2B)^{\frac{1}{\alpha}}$, $A \geqslant 0$, $B \geqslant 0$, and (16) we have for $n \geq N$

$$(\mathcal{T}y)_n \geq \frac{1}{2^{1+\frac{1}{\alpha}}} (n-N)^2$$

and

$$\begin{aligned} (\mathcal{T}y)_n &\leqslant \sum_{s=N}^{n-1} (n-s+1) \left[1 + \left\{ 2 \sum_{j=N}^{\infty} q_j \left(\sigma(j)\right)^{\left(2+\frac{1}{\alpha}\right)\beta} \right\}^{\frac{1}{\alpha}} (s-N)^{\frac{1}{\alpha}} \right] \\ &\leqslant \frac{1}{2} \sum_{s=N}^{n-1} (n-s+1) + \frac{(\alpha+1)(2\alpha+1)}{2\alpha^2} \sum_{s=N}^{n-1} (n-s+1)(s-N)^{\frac{1}{\alpha}} \\ &\leqslant \frac{1}{2} (n-N)^2 + \frac{1}{2} (n-N)^{2+\frac{1}{\alpha}} \leqslant n^{2+\frac{1}{\alpha}}. \end{aligned}$$

This implies that $\mathcal{T}$ maps $\mathcal{S}$ into itself.

Since it is easy to verify that all conditions of Knaster-Tarski fixed point theorem are satisfied, there exists an element $y \in \mathcal{S}$ such that $\mathcal{T}y = y$, which satisfies the equation

$$(17) \qquad y_n = \sum_{s=N}^{n-1} (n-s+1) \left[\frac{1}{2} + \sum_{t=N}^{s-1} \sum_{j=t}^{\infty} q_j y_{\sigma(j)}^{\beta} \right]^{\frac{1}{\alpha}}, \quad n \geqslant N.$$

From (17), we have

$$\begin{aligned} \Delta^2 y_n &= \left[\frac{1}{2} + \sum_{t=N}^{n-1} \sum_{j=t}^{\infty} q_j y_{\sigma(j)}^{\beta} \right]^{\frac{1}{\alpha}} \\ &\geqslant \left[\frac{1}{2} + \sum_{t=N}^{n-1} \sum_{j=t}^{\infty} q_j \frac{(\sigma(j) - N)_+^{2\beta}}{2^{\left(1+\frac{1}{\alpha}\right)\beta}} \right]^{\frac{1}{\alpha}} \\ &\geqslant \left[\frac{1}{2} + \frac{1}{2^{\left(1+\frac{1}{\alpha}\right)\beta}} \sum_{t=N}^{n-1} (t-N+1) q_t (\sigma(t) - N)^{2\beta} \right]^{\frac{1}{\alpha}}. \end{aligned}$$

Now (13) implies that $\{y_n\}$ satisfies $\lim_{n\to\infty} \Delta^2 y_n = \infty$. This completes the proof. □

Theorem 6. *Equation* (1) *has a positive solution of type* (c_5) *if*

$$\sum_{n=n_0}^{\infty}\left(\sum_{s=n}^{\infty}(s-n+1)\left(\sigma\left(s\right)\right)^{\beta} q_s\right)^{\frac{1}{\alpha}} < \infty \tag{18}$$

and

$$\sum_{n=n_0}^{\infty} n\left(\sum_{s=n}^{\infty}(s-n+1)q_s\right)^{\frac{1}{\alpha}} = \infty \tag{19}$$

are hold.

Proof. Let $k > 0$ be any fixed constant and choose an integer $N > n_0 > 0$ large enough so that

$$N_0 = \min\{N, \inf_{n\geq N} \sigma\left(n\right)\} \geq 1$$

and

$$\sum_{n=N}^{\infty}\left(\sum_{s=n}^{\infty}(s-n+1)\left(\sigma\left(s\right)\right)^{\beta} q_s\right)^{\frac{1}{\alpha}} < 2^{-\frac{\beta}{\alpha}} k^{1-\frac{\beta}{\alpha}}.$$

Consider the Banach space $\mathcal{B}$ of all real sequences $y = \{y_n\}$, with the sup-norm $\|y\| = \sup_{n\geqslant N_0} |y_n|$. Define the set $\mathcal{S} \subset \mathcal{B}$ and the mapping $\mathcal{T} : \mathcal{S} \to \mathcal{B}$ by

$$\mathcal{S} = \{y \in \mathcal{B} : k \leqslant y_n \leqslant 2kt,\ n \geqslant N_0\}$$

and

$$(Ty)_n = \begin{cases} k + \sum\limits_{s=N}^{n-1}\sum\limits_{t=s}^{\infty}\left(\sum\limits_{j=t}^{\infty}\left(j-t+1\right)q_j y_{\sigma(j)}^{\beta}\right)^{\frac{1}{\alpha}} & n \geqslant N+1, \\ k, & N_0 \leqslant n \leqslant N. \end{cases}$$

Then the Knaster-Taraski fixed point theorem can be applied to the existence of a fixed point $y \in \mathcal{S}$ of $\mathcal{T}$. This $y = \{y_n\}$ gives a solution of equation (1) for all $n \geq N$. Further

$$\begin{aligned} y_n &\geqslant k + \sum_{s=N}^{n-1}\left(s-N+1\right)\left[\sum_{t=s}^{\infty}\left(t-s+1\right)q_t y_{\sigma(t)}^{\beta}\right]^{\frac{1}{\alpha}} \\ &\geqslant k + k^{\frac{\beta}{\alpha}}\sum_{s=N}^{n-1}\left(s-N+1\right)\left[\sum_{t=s}^{\infty}\left(t-s+1\right)q_t\right]^{\frac{1}{\alpha}}, \quad n \geqslant N. \end{aligned}$$

Therefore from (19) we see that $\{y_n\}$ satisfies $\lim_{n\to\infty} y_n = \infty$. Hence $\{y_n\}$ is a positive solution of type (c_5). □

We conclude this paper with the following example.

Example 1. Consider the difference equation

$$\Delta^2\left(|\Delta^2 y_n|^{\alpha-1}\Delta^2 y_n\right) + n^{-\lambda}|y_{\sigma(n)}|^{\beta-1}y_{\sigma(n)} = 0 \qquad (E_1)$$

where α, β are positive constants, λ is a varying parameter and $\sigma(n) = n^\gamma$, γ a positive integer. It is easy to check that

(i) equation (E_1) has a type (c_1) solution if and only if $\lambda > 1 + \left(2 + \frac{1}{\alpha}\right)\beta\gamma$;

(ii) equation (E_1) has a type (c_3) solution if and only if $\lambda > 2 + 2\beta\gamma$;

(iii) equation (E_1) has a type (c_4) solution if and only if $\lambda > 2 + \alpha + \beta\gamma$;

(iv) equation (E_1) has a type (c_6) solution if and only if $\lambda > 2 + 2\alpha$.

Further, it follows that equation (E_1) has solutions of all types (c_1), (c_3), (c_4) and (c_6) if either

$$\alpha \le \beta\gamma \ \text{ and } \lambda > 1 + \left(2 + \frac{1}{\alpha}\right)\beta\gamma$$

or

$$\alpha > \beta\gamma \ \text{ and } \lambda > 2 + 2\alpha$$

holds. It is easy to see that for equation (E_1) the existence of solutions of types (c_2) and (c_4) may be realized only when $\alpha > \beta\gamma$. The conclusions of Theorem 5 and 6 are

(v) equation (E_1) has a type (c_2) solution if

$$\alpha > \beta\gamma \text{ and } 1 + \left(2 + \frac{1}{\alpha}\right)\beta\gamma < \lambda \leqslant 2 + 2\beta\gamma; \tag{20}$$

(vi) equation (E_1) has a type (c_5) solution if

$$\alpha > \beta\gamma \text{ and } 2 + \alpha + \beta\gamma < \lambda \leqslant 2 + 2\alpha. \tag{21}$$

Note that if (20) holds, then equation (E_1) has no solutions of types (c_3), (c_4) and (c_6), and that if (21) holds, then equation (E_1) has no solution of type (c_6).

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