KYUNGPOOK Math. J. 46(2006), 463-476

Structure Eigenvectors of the Ricci Tensor in a Real Hypersurface of a Complex Projective Space

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Abstract. It is known that there are no real hypersurfaces with parallel Ricci tensor in a nonflat complex space form ([6], [9]). In this paper we investigate real hypersurfaces in a complex projective space $P_n\mathbb{C}$ using some conditions of the Ricci tensor S which are weaker than $\nabla S = 0$. We characterize Hopf hypersurfaces of $P_n\mathbb{C}$.

0. Introduction

An n-dimensional complex space form $M_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature c.

As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}_n$ or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0$, $c = 0$ or $c < 0$.

Let M be a real hypersurface of $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure J and the Kaehlerian metric of $M_n(c)$ (for details see section 1). The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. A real hypersurface is said to be a *Hopf hypersurface* if the structure vector field ξ of M is principal.

Tagaki ([16], [17]) classified all homogeneous real hypersurfaces of $P_n\mathbb{C}$ as six model spaces which are said to be A_1, A_2, B, C, D and E, and Cecil-Ryan ([2]) and Kimura ([11]) proved that they are realized as the tubes of constant radius over the Kaehlerian submanifolds. Namely, he proved the following:

Theorem T ([16]). *Let M be a homogeneous real hypersurface of $P_n\mathbb{C}$. Then M*

Received March 10, 2005, and, in revised form, February 22, 2006.
2000 Mathematics Subject Classification: 53C40, 53C15.
Key words and phrases: real hypersurface, structure vector field, Ricci tensor, Hopf hypersurface.

is a tube of radius r over one of the following Kaehlerian submanifolds:

(A_1) *a hyperplane* $P_{n-1}\mathbb{C}$, *where* $0 < r < \frac{\pi}{2}$,

(A_2) *a totally geodesic* $P_k\mathbb{C}$ $(1 \leq k \leq n-2)$, *where* $0 < r < \frac{\pi}{2}$,

(B) *a complex quadric* Q_{n-1}, *where* $0 < r < \frac{\pi}{4}$,

(C) $P_1\mathbb{C} \times P_{(n-1)/2}\mathbb{C}$, *where* $0 < r < \frac{\pi}{4}$ *and* $n(\geq 5)$ *is odd,*

(D) *a complex Grassmann* $G_{2,5}\mathbb{C}$, *where* $0 < r < \frac{\pi}{4}$ *and* $n = 9$,

(E) *a Hermitian symmetric space* $SO(10)/U(5)$, *where* $0 < r < \frac{\pi}{4}$ *and* $n = 15$.

Also Berndt([1]) classified all Hopf real hypersurfaces of $H_n\mathbb{C}$ with constant principal curvatures as follows:

Theorem B ([1]). *Let M be a real hypersurface of $H_n\mathbb{C}$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following*:

(A_0) *a self-tube, that is, a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a hyperplane* $H_{n-1}\mathbb{C}$,

(A_2) *a tube over a totally geodesic* $H_k\mathbb{C}(1 \leq k \leq k-2)$,

(B) *a tube over a totally real hyperbolic space* $H_n\mathbb{R}$.

We denote by ∇ and S be the Levi-Civita connection and the Ricci tensor of M. There are many studies about Ricci tensors of real hypersurfaces ([3], [4], [5], [6], [7], [8], [9], [12], [15], etc.). Very important fact is that there are no real hypersurfaces with parallel Ricci tensor in $M_n(c)$, $n \geq 2$, $c \neq 0$ ([6], [9], [10]). So it is natural to investigate real hypersurfaces M by using some conditions about derivative of S which are weaker than $\nabla S = 0$. For each Hopf hypersurface M in a nonflat complex space form, the structure vector field ξ is an eigenvector of the Ricci tensor S of M, and the scalar $g(\nabla_\xi \xi, \nabla_\xi \xi)$ vanishes identically on M. So it is natural to consider a problem that if $S\xi = g(S\xi,\xi)\xi$ holds or $g(\nabla_\xi \xi, \nabla_\xi \xi) = \text{const.}$, is M a Hopf hypersurface? Nagai and one of the present authors ([8]) proved the following which gives a partial answer to this problem:

Theorem KN ([8]). *Let M be a real hypersurface in a complex projective space $P_n\mathbb{C}$. Then the following are equivalent*:

(1) *M is a Hopf hypersurface in the ambient space $P_n\mathbb{C}$.*

(2) *The structure vector ξ is an eigenvector with constant eigenvalue of the Ricci tensor S of M and $\nabla_{\phi\nabla_\xi\xi}S = 0$ holds.*

The purpose of this paper is to establish the following:

Theorem. *Let M be a real hypersurface of $P_n\mathbb{C}$. The the following are equivalent*:

(1) *M is a Hopf hypersurface in $P_n\mathbb{C}$.*

(2) *$S\xi = g(S\xi, \xi)\,\xi$ and $\nabla_{\phi\nabla_\xi\xi}S = 0$ hold, and $g(\nabla_\xi\xi, \nabla_\xi\xi)$ is constant on M.*

1. Preliminaries

Let M be a real hypersurface immersed in a complex space form $(M_n(c), G)$ with almost complex structure J and the Kaehler metric G of constant holomorphic sectional curvature c, and let N be a unit normal vector field on M. The Riemannian connection $\tilde{\nabla}$ in $M_n(c)$ and ∇ in M are related by the following formulas for any vector fields X and Y on M:

$$\tilde{\nabla}_Y X = \nabla_Y X + g(AY, X)\,N, \quad \tilde{\nabla}_X N = -AX,$$

where g denotes the Riemannian metric on M induced from that G of $M_n(c)$ and A is the shape operator in the direction of N in $M_n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)\,N, \quad JN = -\xi.$$

Then we may see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, namely, we have

$$\phi^2 X = -X + \eta(X)\,\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y),$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi),$$

for any vector fields X and Y on M.

From the fact $\tilde{\nabla} J = 0$ and above equations we verify that

$$(1.1) \qquad (\nabla_X \phi)\,Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX.$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations respectively:

$$(1.2) \quad \begin{aligned} R(X, Y)\,Z \;=\; & \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z\} + g(AY, Z)\,AX - g(AX, Z)AY, \end{aligned}$$

$$(1.3) \qquad (\nabla_X A)\,Y - (\nabla_Y A)\,X = \frac{c}{4}\{\eta(X)\,\phi Y - \eta(Y)\,\phi X - 2g(\phi X, Y)\,\xi\}$$

for any vector fields X, Y and Z on M, where R denotes Riemann-Christoffel curvature tensor of M. We shall denote the Ricci tensor of type (1.1) by S. Then it follows from (1.2) that

$$(1.4) \qquad SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\,\xi\} + hAX - A^2 X,$$

where $h=$ trace A. Further, using (1.1), we obtain

$$(\nabla_X S)\,Y = -\frac{3}{4}c\{g(\phi AX,Y)\xi+\eta\,(Y)\,\phi AX\}+(Xh)\,AY + (hI-A)\,(\nabla_X A)\,Y-(\nabla_X A)\,AY, \tag{1.5}$$

where I is the identity map.

In what follows, to write our formulas in convention forms, we denote by $\alpha=\eta\,(A\xi)\,,\ \beta=\eta\left(A^2\xi\right)$, and for a function f we denote by ∇f the gradient vector field of f.

We put $U=\nabla_\xi\xi$, then U is orthogonal to the structure vector field ξ. Thus it is, using (1.1), seen that

$$\phi U=-A\xi+\alpha\xi, \tag{1.6}$$

which enables us to obtain $g\,(U,U)=\beta-\alpha^2$. Thus we easily see that ξ is a principal curvature vector, that is, $A\xi=\alpha\xi$ if and only if $\beta-\alpha^2=0$.

Now differentiating (1.6) covariantly along M and using (1.1), we find

$$\eta\,(X)\,g(AU+\nabla\alpha,Y)+g(\phi X,\nabla_Y U) = g((\nabla_Y A)\,X,\xi)-g\,(A\phi AX,Y)+\alpha g\,(A\phi X,Y)\,, \tag{1.7}$$

which shows that

$$(\nabla_\xi A)\,\xi=2AU+\nabla\alpha, \tag{1.8}$$

because of (1.3). From (1.7) we also have

$$\nabla_\xi U=3\phi AU+\alpha A\xi-\beta\xi+\phi\nabla\alpha, \tag{1.9}$$

where we have used (1.1).

If $\beta-\alpha^2\neq 0$, then we can put

$$A\xi=\alpha\xi+\mu W, \tag{1.10}$$

where W is a unit vector field orthogonal to ξ. Then by (1.1) we see that $U=\mu\phi W$ and hence $g\,(U,U)=\mu^2$, and W is also orthogonal to U. Thus, we see, making use of (1.1) and (1.10), that

$$\mu g\,(\nabla_X W,\xi) = g\,(AU,X)\,, \tag{1.11}$$

$$g\,(\nabla_X\xi,U) = \mu g\,(AW,X)\,. \tag{1.12}$$

2. Structure eigenvectors of the Ricci tensor

Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. Now, suppose that the structure vector ξ is an eigenvector of the Ricci tensor, that is,

$$S\xi = g(S\xi, \xi)\,\xi. \tag{2.1}$$

We then have by (1.4)

$$A^2\xi = hA\xi + (\beta - h\alpha)\,\xi. \tag{2.2}$$

In the following we assume that $\mu \neq 0$ on M, that is, ξ is not a principal curvature vector field and we put $\Omega = \{p \in M \mid \mu(p) \neq 0\}$. Then Ω is an open subset of M, and from now on we discuss our arguments on Ω unless otherwise stated.

From (1.10) and (2.2) we verify that

$$AW = \mu\xi + (h - \alpha)\,W \tag{2.3}$$

and hence

$$A^2W = hAW + (\beta - h\alpha)\,W \tag{2.4}$$

because $\mu \neq 0$.

Differentiating (2.3) covariantly along Ω, we find

$$(\nabla_X A)\,W + A\nabla_X W = (X\mu)\,\xi + \mu\nabla_X\xi + X(h - \alpha)\,W + (h - \alpha)\,\nabla_X W. \tag{2.5}$$

If we take an inner product with W in this, then we obtain

$$g((\nabla_X A)\,W, W) = -2g(AU, X) + Xh - X\alpha \tag{2.6}$$

because W is a unit vector field orthogonal to ξ. We also have by applying ξ to (2.5)

$$\mu g((\nabla_X A)\,W, \xi) = (h - 2\alpha)\,g(AU, X) + \mu(X\mu), \tag{2.7}$$

where we have used (1.11).

Putting $X = \xi$ in (2.5) and making use of (1.3) and (2.7), we find

$$\begin{aligned}(h - 2\alpha)\,AU - \frac{c}{4}U + \mu\nabla\mu + \mu\{A\nabla_\xi W - (h-\alpha)\nabla_\xi W\} \\ = \mu(\xi\mu)\,\xi + \mu^2 U + \mu(\xi h - \xi\alpha)\,W.\end{aligned} \tag{2.8}$$

On the other hand, it is, using $\phi U = -\mu W$, seen that

$$g(AU, X)\,\xi - \phi\nabla_X U = (X\mu)\,W + \mu\nabla_X W.$$

Replacing X by ξ in this and taking account of (1.6) and (1.9), we have

$$\mu\nabla_\xi W = 3AU - \alpha U - (\xi\alpha)\,\xi - (\xi\mu)\,W, \tag{2.9}$$

which shows that

$$W\alpha = \xi\mu. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8), we get

$$\begin{aligned} &3A^2U - 2hAU + A\nabla\alpha + \frac{1}{2}\nabla\beta - h\nabla\alpha + \Big(\alpha h - \beta - \frac{c}{4}\Big)U \\ = \;& 2\mu\,(W\alpha)\,\xi + \mu\,(\xi h)\,W - (h-2\alpha)\,(\xi\alpha)\,\xi, \end{aligned} \tag{2.11}$$

where we have used $\mu^2 = \beta - \alpha^2$.

Differentiating (2.2) covariantly and making use of (1.1), we find

$$\begin{aligned} &(\nabla_X A)\,A\xi + A\,(\nabla_X A)\,\xi + A^2\phi AX - hA\phi AX \\ = \;& (Xh)A\xi + h(\nabla_X A)\xi + X(\beta - h\alpha)\xi + (\beta - h\alpha)\phi AX, \end{aligned} \tag{2.12}$$

which together with (1.3) yields

$$\begin{aligned} &\frac{c}{4}\{u(Y)\eta(X) - u(X)\eta\,(Y)\} + \frac{c}{2}(h-\alpha)g\,(\phi Y, X) - g(A^2\phi AX, Y) \\ &+g(A^2\phi AY, X) + 2hg\,(\phi AX, AY) - (\beta - h\alpha)\{g(\phi AY, X) - g(\phi AX, Y)\} \\ = \;& g(AY, (\nabla_X A)\,\xi) - g(AX, (\nabla_Y A)\,\xi) + (Yh)g(A\xi, X) - (Xh)g(A\xi, Y) \\ &+Y(\beta - h\alpha)\eta\,(X) - X(\beta - h\alpha)\eta\,(Y)\,, \end{aligned} \tag{2.13}$$

where we have defined a 1-form u by $u(X) = g(U, X)$ for any vector field X.

Putting $X = \mu W$ in (2.12) and taking account of (1.3), (1.8), (2.3), (2.4) and (2.7), we obtain

$$\begin{aligned} &(3\alpha - 2h)\,A^2U + 2(h^2 + \beta - 2\alpha h + \frac{c}{4})AU + (h-\alpha)(\beta - \alpha h - \frac{c}{2})U \\ = \;& \mu A\nabla\mu + (\alpha h - \beta)\nabla\alpha - \frac{1}{2}\,(h-\alpha)\,\nabla\beta + \mu^2\nabla h - \mu\,(Wh)\,A\xi - \mu W(\beta - \alpha h)\xi. \end{aligned} \tag{2.14}$$

Because of (1.10), we have from (2.12)

$$\begin{aligned} &A(\nabla_X A)\xi + (\alpha - h)(\nabla_X A)\xi + \mu(\nabla_X A)W \\ = \;& (Xh)A\xi + X(\beta - h\alpha)\xi + (\beta - h\alpha)\phi AX + hA\phi AX - A^2\phi AX. \end{aligned}$$

Therefore, replacing X by $\alpha\xi + \mu W$ in this and using (1.1), (1.3), (1.8), (1.10), (2.6) and (2.7), we find

$$\begin{aligned} &2hA^2U + 2(\alpha h - \beta - h^2 - \frac{c}{4})AU + (h^2\alpha - h\beta + \frac{c}{2}h - \frac{3}{4}c\alpha)U \\ = \;& g\,(A\xi, \nabla h)\,A\xi - \frac{1}{2}A\nabla\beta + \frac{1}{2}(h-2\alpha)\nabla\beta + \beta\nabla\alpha \\ &-\mu^2\nabla h + g\,(A\xi, \nabla\,(\beta - \alpha h))\,\xi. \end{aligned} \tag{2.15}$$

In the following we assume that $\nabla_{\phi U} S = 0$ and hence $\nabla_W S = 0$ since we assume that $\mu \neq 0$. Then, by replacing X by W, we have from (1.5)

$$\begin{aligned} & -\frac{3}{4}c(h-\alpha)\left(u(Y)\xi + \eta\left(Y\right)U\right) + \mu\left(Wh\right)AY + \mu h\left(\nabla_W A\right)Y \\ = \quad & \mu A\left(\nabla_W A\right)Y + \mu\left(\nabla_W A\right)AY, \end{aligned}$$

where we have used (1.1) and (2.3). Putting $Y = W$ in this and making use of (1.3), (2.6) and (2.7), we find

$$(Wh)AW = hAU - \frac{c}{2}U - 2A^2U + \frac{1}{2}\nabla\beta - \alpha\nabla h + A\nabla h - A\nabla\alpha \tag{2.16}$$

because $\mu \neq 0$.

Differentiating (2.1) covariantly and using $\nabla_W S = 0$, we find

$$S\nabla_W\xi = W(\alpha h - \beta)\xi + \left\{\frac{c}{2}(n-1) + h\alpha - \beta\right\}\nabla_W\xi, \tag{2.17}$$

which implies

$$W\left(\beta - h\alpha\right) = 0. \tag{2.18}$$

By the way we see, using (1.1) and (2.3), that $\mu\nabla_W\xi = (h-\alpha)\,U$, it follows from (1.4) and (2.17) that

$$(h-\alpha)\left\{A^2U - hAU - \left(\beta - h\alpha + \frac{3}{4}c\right)U\right\} = 0. \tag{2.19}$$

3. Real hypersurfaces with $g\left(\nabla_\xi\xi, \nabla_\xi\xi\right) =$ const.

We continue now, our arguments under the same hypotheses $S\xi = g(S\xi,\xi)\xi$ and $\nabla_{\phi U} S = 0$ as in section 2. Further, suppose that $g\left(U,U\right) =$ const., that is, $\nabla\mu = 0$. Then we have

$$\nabla\beta = 2\alpha\nabla\alpha, \tag{3.1}$$

which together with (2.10) gives

$$W\alpha = 0. \tag{3.2}$$

Using these facts, (2.14) and (2.16) turn out respectively to

$$\begin{aligned} & (3\alpha - 2h)\,A^2U + 2(h^2 + \beta - 2\alpha h + \frac{c}{4})AU + (h-\alpha)\left(\beta - h\alpha - \frac{c}{2}\right)U \\ = \quad & \mu^2(\nabla h - \nabla\alpha) - \mu^2\left(Wh\right)W, \end{aligned} \tag{3.3}$$

$$(3.4)\qquad 2A^2U - hAU + \frac{c}{2}U + (Wh)\,AW = A\nabla h - A\nabla\alpha - \alpha(\nabla h - \nabla\alpha).$$

We notice here that $h - \alpha \neq 0$ on Ω. In fact, if not, then we have $h = \alpha$. So (3.3) and (3.4) are reduced respectively to

$$(3.5)\qquad \alpha A^2U + 2\left(\beta - \alpha^2 + \frac{c}{4}\right)AU = 0, \quad 2A^2U = \alpha AU - \frac{c}{2}U$$

because of (3.2), which enables us to obtain

$$\alpha AU = 2\left(\alpha^2 - \beta - \frac{c}{4}\right)U$$

on this set. However, we verify that $\alpha \neq 0$ on this subset by virtue of (2.11) and (3.4) with $h = \alpha$, it follows that $AU = \nu U$, where a function ν given by $\alpha\nu = 2\left(\alpha^2 - \beta - \frac{c}{4}\right)$ is defined. From this and (3.5) we see that ν is constant by virtue of $\mu =$ constant and hence $\nabla\alpha = 0$. Further, we have

$$\nu^2 + \beta - \alpha^2 + \frac{c}{2} = 0.$$

Since α is constant, (2.11) implies

$$3\nu^2 - 2\alpha\nu + \alpha^2 - \beta - \frac{c}{4} = 0,$$

which will produce a contradiction. Hence $h - \alpha \neq 0$ on Ω is proved. Thus (2.19) becomes

$$(3.6)\qquad A^2U = hAU + \left(\beta - h\alpha + \frac{3}{4}c\right)U.$$

Now, we are going to prove $Wh = 0$ on Ω. From (2.18), (3.1) and (3.2) we verify that $\alpha\,(Wh) = 0$. Suppose that $Wh \neq 0$. Then we have $\alpha = 0$. So (2.11) implies that

$$3A^2U = 2hAU + \left(\beta + \frac{c}{4}\right)U$$

because U is orthogonal to ξ, or using (3.6)

$$hAU + 2(\beta + c)U = 0.$$

Applying this by A and making use of (3.6), we find

$$\left\{h^2 + 2(\beta + c)\right\}AU + h\left(\beta + \frac{3}{4}c\right)U = 0.$$

Combining the last two equations, it follows that

$$h^2\left(\beta + \frac{5}{4}c\right) + 4(\beta + c)^2 = 0,$$

which shows that $Wh = 0$ by virtue of (3.1), a contradiction. Thus, $Wh = 0$ on Ω is proved.

Using (3.6) and the fact that $Wh = 0$, (3.3) and (3.4) turn out respectively to

$$\mu^2(\nabla h - \nabla\alpha) = (2\beta - h\alpha + \frac{c}{2})AU + \left\{(\beta - h\alpha)(2\alpha - h) + \frac{c}{4}(11\alpha - 8h)\right\}U, \tag{3.7}$$

$$A\nabla h - A\nabla\alpha = \alpha(\nabla h - \nabla\alpha) + hAU + 2(\beta - h\alpha + c)U. \tag{3.8}$$

Applying to the both sides of (3.7) by A and making use of (3.6) and (3.8), we find

$$(3\alpha - 2h)AU = (\alpha^2 - h\alpha - \frac{c}{2})U. \tag{3.9}$$

Let Ω_0 be a set of points in Ω such that $\|AU - \lambda U\|_p \neq 0$ at $p \in \Omega$ and suppose that Ω_0 is nonvoid. If $\alpha^2 - h\alpha - \frac{c}{2} \neq 0$, then from (3.9) we get $3\alpha - 2h \neq 0$ and hence Ω_0 is empty. Thus, it is, using (3.9), seen that

$$\alpha^2 - h\alpha - \frac{c}{2} = 0, \quad 3\alpha = 2h,$$

which shows that $\alpha^2 + c = 0$. Hence α is nonzero constant. So does h on Ω_0. Therefore (3.8) is reduced to

$$3\alpha AU + (4\beta - 10\alpha^2)U = 0, \tag{3.10}$$

which together with (3.6) implies that

$$\left(8\beta - 11\alpha^2\right)AU + \alpha\left(6\beta - \frac{27}{2}\alpha^2\right)U = 0. \tag{3.11}$$

On the other hand, by using the fact that $\alpha^2 + c = 0$ and $3\alpha = 2h$ we have from (3.7)

$$4\left(\beta - \alpha^2\right)AU + \alpha\left(\beta - 4\alpha^2\right)U = 0.$$

Combining (3.10) and (3.11) to this, we see that $\beta - \alpha^2 = 0$ and hence $\Omega_0 = \emptyset$. Thus we have from (3.9)

$$AU = \lambda U, \tag{3.12}$$

where the function λ given by

$$(3\alpha - 2h)\lambda = \alpha^2 - h\alpha - \frac{c}{2}, \tag{3.13}$$

is defined.

Because of (3.12), it follows, making use of (3.6) and (3.7), that

$$\lambda^2 = h\lambda + \beta - h\alpha + \frac{3}{4}c, \tag{3.14}$$

$$\mu^2(\nabla h - \nabla\alpha) = \left\{(2\beta - h\alpha + \frac{c}{2})\lambda + (\beta - h\alpha)(2\alpha - h) + \frac{c}{4}(11\alpha - 8h)\right\}U. \tag{3.15}$$

Since $\nabla\mu = 0$ by assumption, we find from the last equation

$$(Xf)\,u(Y) - (Yf)\,u(X) + f du(X,Y) = 0 \tag{3.16}$$

for any vector fields X and Y, where we have put

$$f = (2\beta - h\alpha + \frac{c}{2})\lambda + (\beta - h\alpha)(2\alpha - h) + \frac{c}{4}(11\alpha - 8h), \tag{3.17}$$

and the exterior derivation du of u is given by

$$du(Y,X) = \frac{1}{2}\{Yu(X) - Xu(Y) - u([Y,X])\}.$$

Putting $X = \xi$ in (3.16), we find

$$(\xi f)\,u(Y) + f du(\xi, Y) = 0. \tag{3.18}$$

$du(\xi, U)$ being vanish identically on Ω, it follows that

$$\xi f = 0$$

because we have (1.9), (3.2), (3.12) and $\nabla\mu = 0$. Therefore (3.18) becomes

$$f du(\xi, X) = 0. \tag{3.19}$$

for any vector field X.

Finally we have from (2.15)

$$\begin{aligned} &\left\{2\lambda\left(\alpha h - \beta - \frac{c}{4}\right) + h\beta - \alpha h^2 + 2ch - \frac{3}{4}c\alpha\right\}U \\ = \;& \alpha(\xi h)A\xi + \alpha\{(2\alpha - h)(\xi\alpha) - \alpha(\xi h)\}\xi \\ & -\alpha A\nabla\alpha + (\beta + \alpha h - 2\alpha^2)\nabla\alpha - (\beta - \alpha^2)\nabla h, \end{aligned}$$

where we have used (3.1), (3.2), (3.6), (3.12) and the fact that $Wh = 0$. If we take an inner product ξ with this and make use of (3.2), then we obtain

$$\xi\alpha = \xi h. \tag{3.20}$$

Thus, above equation can be written as

$$\begin{aligned} &\left\{2\lambda\left(\alpha h - \beta - \frac{c}{4}\right) + h\beta - \alpha h^2 + 2ch - \frac{3}{4}c\alpha\right\}U \\ = \;& \alpha(\xi\alpha)\{(2\alpha - h)\xi + \mu W\} - \alpha A\nabla\alpha + (\beta + \alpha h - 2\alpha^2)\nabla\alpha - (\beta - \alpha^2)\nabla h. \end{aligned} \tag{3.21}$$

4. Real hypersurfaces satisfying $du(\xi, X) = 0$

We will continue, our arguments under the same hypotheses $S\xi = g(S\xi, \xi)\xi, \nabla_{\phi U} S = 0$ and $g(U, U) =$ const. as in section 3.

Now, suppose that $du(\xi, X) = 0$ for any vector field X on Ω. Then we have $g(\nabla_\xi U, X) + g(\nabla_X \xi, U) = 0$, or using (1.1), (1.9), (1.12) and (3.12), $\phi\nabla\alpha = \mu(h - 3\lambda)W$ and hence

$$\nabla\alpha - (\xi\alpha)\xi = (h - 3\lambda)U. \tag{4.1}$$

Differentiating (3.14) covariantly and using (3.1), we find

$$(2\lambda - h)(X\lambda) = (\lambda - \alpha)(Xh) + (2\alpha - h)(X\alpha).$$

Putting $X = \xi$ in this and taking account of (3.20), we get

$$(2\lambda - h)\xi\lambda = (\lambda + \alpha - h)\xi\alpha. \tag{4.2}$$

In the same way, we also have from (3.13)

$$(3\alpha - 2h)\xi\lambda = (-\lambda + \alpha - h)\xi\alpha, \tag{4.3}$$

which together with (4.2) yields

$$(-3\alpha + h - 2\lambda)\xi\lambda = 2\lambda(\xi\alpha), \quad (3\alpha - 3h + 2\lambda)\xi\lambda = 2(\alpha - h)\xi\alpha.$$

Let $\Omega_1 = \{p \in \Omega | \{(\xi\lambda)^2 + (\xi\alpha)^2\}(p) \neq 0\}$. Assume that $\Omega_1 \subset \Omega$ and $\Omega_1 \neq \emptyset$. Then we have by above equations

$$h^2 - 4\alpha h + 3\alpha^2 + 2\lambda^2 + \lambda(\alpha - h) = 0 \tag{4.4}$$

on Ω_1. Differentiating this covariantly and using (3.20), we find

$$(4\lambda + \alpha - h)\xi\lambda = 2(h - \alpha)\xi\alpha.$$

From this and (4.3), we verify that

$$5h^2 - 12h\alpha + 7\alpha^2 - 4\lambda^2 + 3\lambda(\alpha - h) = 0. \tag{4.5}$$

Using Sylvester's elimination method to (4.4) and (4.5), we deduce that

$$\left(204\alpha^2 - 121c\right)(\alpha^2 + c) = 0 \tag{4.6}$$

on Ω_1. (We use a computer to calculate this.) It is contradictory for $c > 0$ or $c < 0$. Thus $\Omega_1 = \emptyset$ and hence $\xi\lambda = \xi\alpha = 0$ on Ω. Thus (4.1) is reduced to

$$\nabla\alpha = (h - 3\lambda)U. \tag{4.7}$$

on Ω_2, where $\Omega_2 = \{p \in \Omega | f(p) \neq 0\} \neq \emptyset$ since we have (3.19).

We notice here that α does not vanish on Ω_2. In fact, if not, we have $h = 3\lambda$ because of (4.7). By (3.13) we see that $h\lambda = \frac{c}{4}$ and hence $3\lambda^2 = \frac{c}{4}$. So we have $c > 0$ on Ω_2. We also have from (3.14), $2\lambda^2 + \beta + \frac{3}{4}c = 0$, a contradiction by virtue of $c > 0$.

Using (4.7), the equations (3.15) and (3.21) turn out respectively to

$$\mu^2\nabla h = \{(\beta-\alpha^2)(h-3\lambda) + (2\beta - h\alpha + \frac{c}{2})\lambda + (\beta - h\alpha)(2\alpha - h) + \frac{c}{4}(11\alpha - 8h)\}U. \tag{4.8}$$

$$\mu^2\nabla h = \{(h-3\lambda)(\beta + \alpha h - 2\alpha^2 - \alpha\lambda) + (\beta - h\alpha)(2\lambda - h) + \frac{c}{2}\lambda - 2ch + \frac{3}{4}c\alpha\}U. \tag{4.9}$$

Comparing the last two equations, we obtain

$$h^2 + h\alpha - 2\beta - 5h\lambda + 3\alpha\lambda + 3\lambda^2 - 2c = 0$$

because $\alpha \neq 0$ on Ω_2, which together with (3.13) and (3.14) gives

$$\beta - h\alpha + (h-\alpha)^2 - \frac{c}{4} = 0. \tag{4.10}$$

From this and (3.14) we have

$$\lambda^2 = h\lambda - (h-\alpha)^2 + c,$$

which connected with (3.13) implies that

$$4h^4 - 22\alpha h^3 + (43\alpha^2 - 5c)h^2 + \left(\frac{29}{2}c - 35\alpha^2\right)\alpha h + 10\alpha^4 - 10c\alpha^2 + \frac{c^2}{4} = 0. \tag{4.11}$$

Differentiating (4.10) covariantly and making use of (3.1), we find on Ω_2

$$(3\alpha - 2h)\nabla h + (3h - 4\alpha)\nabla\alpha = 0.$$

Similarly we also have from (4.11)

$$\left\{16h^3 - 66\alpha h^2 + (86\alpha^2 - 10c)h - 35\alpha^3 + \frac{29}{2}c\alpha\right\}\nabla h + \left\{-22h^3 + 86\alpha h^2 - 10\alpha^2 h + \frac{29}{2}ch + 40\alpha^3 - 20c\alpha\right\}\nabla\alpha = 0.$$

Since $(\nabla\alpha)^2 + (\nabla h)^2 \neq 0$ on Ω_2 with the aid of (3.15) and (3.17), it follows, using the last two equations, that

$$2h^4 - 12\alpha h^3 + \left(27\alpha^2 - \frac{c}{2}\right)h^2 - 27\alpha^3 h + 10\alpha^4 + c\alpha^2 = 0. \tag{4.12}$$

Using the same method as that used to derive (4.6) from (4.4) and (4.5), we can deduce from (4.11) and (4.2) the following: (We use a computer to calculate this.)

$$\left(\alpha^2 + c\right)\left(80\alpha^4 - 260c\alpha^2 + c^2\right) = 0$$

on Ω_2. It is contradictory for $c > 0$ or $c < 0$. Therefore $\Omega_2 = \emptyset$ and consequently $f = 0$ on Ω because of (3.19).

5. Proof of Theorems

First of all, we prove

Lemma 5.1. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If it satisfies $S\xi = g(S\xi, \xi)\xi, \nabla_{\phi U} S = 0$ and $g(U, U) = const.$, then we have*

$$g(U, U) + 9\lambda^2 + \frac{9}{4}c = 0. \tag{5.1}$$

Proof. As is already shown in section 4, we have $f = 0$ on Ω and hence $\nabla h = \nabla\alpha$ because of (3.15). Thus (3.8) becomes

$$h\lambda + 2(\beta - h\alpha + c) = 0, \tag{5.2}$$

which together with (3.14) implies that

$$\lambda^2 + \beta - h\alpha + \frac{5}{4}c = 0, \quad \lambda^2 = \frac{1}{2}h\lambda - \frac{c}{4}. \tag{5.3}$$

Since $\nabla h = \nabla\alpha$, we see, using (5.3), that $\nabla\alpha = 0$ on Ω. Thus, (3.21) implies

$$(h - 2\lambda)(\beta - h\alpha) - \frac{c}{2}\lambda + 2ch - \frac{3}{4}c\alpha = 0, \tag{5.4}$$

which connected to (5.2) and (5.3) yields $h = \alpha - 2\lambda, \alpha\lambda = 4\lambda^2 + \frac{c}{2}$. Substituting these into (5.2), we verify that $\beta - \alpha^2 + 9\lambda^2 + \frac{9}{4}c = 0$. This completes the proof.□

According to Lemma 5.1, we see that $\Omega = \emptyset$ if $c > 0$. Thus, we have

Theorem 5.1. *Let M be a real hypersurface of a complex projective space $P_n\mathbb{C}$. The the following are equivalent*:

1. *M is a Hopf hypersurface in the ambient space $P_n\mathbb{C}$.*
2. *The structure vector ξ is an eigenvector of the Ricci tensor S of M and satisfies $g(\nabla_\xi\xi, \nabla_\xi\xi)$ is constant on M and $\nabla_{\phi\nabla_\xi\xi}S = 0$ holds.*

Remark. For a real hypersurface of a nonflat complex space form, Theorem 5.1 is valid provided that $\|\nabla_\xi\xi\|^2 + \frac{9}{4}c \geq 0$.

Acknowledgement. The authors would like to express their gratitude to the referee for valuable comments.

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