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INFERENCE FOR ABSOLUTE LORENZ CURVE AND ABSOLUTE LORENZ ORDERING

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ABSTRACT

Absolute Lorenz curve plays an important role for measuring absolute income inequality. Properties of absolute Lorenz curve are listed. Asymptotically distribution free and consistent tests have been proposed for comparing two absolute Lorenz curves in the whole interval $[p_1, p_2]$ where $0 < p_1 < p_2 < 1$. Absolute Lorenz ordering has been discussed for some distributions.

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Keywords. Absolute Lorenz curve, distribution free, consistent, absolute Lorenz ordering.

1. INTRODUCTION

Inequality can be both relative and absolute. An equiproportional change in all incomes leaves the degree of relative inequality unchanged while an equal increment or decrement to all incomes leaves the degree of absolute inequality unchanged. This concept of absolute and associated indices of inequality has been analyzed by Kolm (1976), Blackorby and Donaldson (1980), Shorrocks (1983), Moyes (1987), Chakraborti *et al.* (1988) and Bishop *et al.* (1989, 1994).

Moyes (1987) introduced the absolute Lorenz curve and showed that the absolute Lorenz curve plays the same role for absolute inequality as the ordinary Lorenz curve for the analysis of relative inequality.

The definition of the absolute Lorenz curve and its properties are listed in Section 2. In Section 3, a testing procedure is suggested for examining the dominance between two absolute Lorenz curves in the whole interval $[p_1, p_2]$ where

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$0 < p_1 < p_2 < 1$. The test has been shown to be asymptotically of size α and a consistent test. Power of the test is obtained using simulation.

Section 4 discusses the concept of absolute Lorenz ordering and illustrates this for exponential, Pareto and uniform distributions.

2. Absolute Lorenz Curve

The absolute Lorenz curve corresponding to the distribution function $F(x)$ with mean μ is defined as

$$\begin{aligned} AL(p) &= \int_0^{F^{-1}(p)} x dF(x) - p\mu \\ &= GL(p) - p\mu = \mu L(p) - p\mu, \end{aligned}$$

where $L(p)$ is Lorenz curve, $F^{-1}(p) = \inf\{y | F(y) \geq p\}$, $0 \leq p \leq 1$ and $GL(p)$ is generalized Lorenz curve (Shorrocks, 1983; Bishop *et al.*, 1989).

Let $I = (1, 2, \dots, n)$ denote the set of n individuals and $\underline{x} = (x_1, x_2, \dots, x_n)^t$ be the vector of ordered incomes received by each individual and $\bar{x}$ be the sample mean.

For $k \in I$, the sample estimate of the absolute Lorenz curve is given as

$$AL\left(\frac{k}{n}\right) = GL\left(\frac{k}{n}\right) - \frac{k}{n}GL(1) = \frac{1}{n}\sum_{i=1}^{k} x_i - \frac{k}{n}\bar{x} = \frac{1}{n}\sum_{i=1}^{k}(x_i - \bar{x}).$$

Geometrically, the absolute Lorenz curve ordinate is the vertical difference between the generalized Lorenz curve ordinate and its equality line, *i.e.*, $p\mu$.

Properties of absolute Lorenz curve are

(i) Absolute Lorenz curve ordinates are always negative,
(ii) Absolute Lorenz curve is bitonic, decreasing when income is less than the mean and increasing otherwise,
(iii) Both the end points are zero, *i.e.*, $AL(0) = 0$ and $AL(1) = 0$,
(iv) When all the income is equally distributed, absolute Lorenz curve coincides with the X-axis, *i.e.*, the line of absolute equality (see Figure 2.1) (Moyes, 1987).

The Table 2.1 displays the expressions for the corresponding absolute Lorenz curve for some distributions.

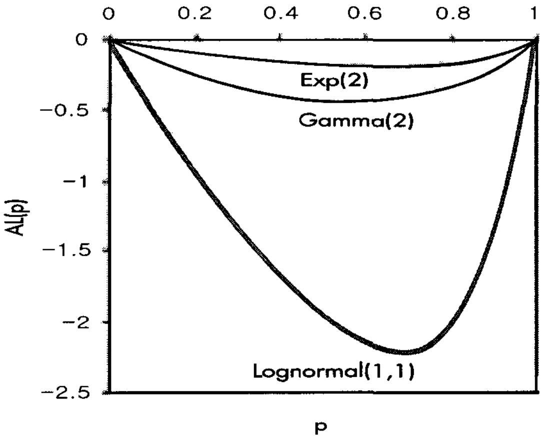


FIGURE 2.1 *Absolute Lorenz curve.*

TABLE 2.1 *Absolute Lorenz curve for some distributions*

Distribution	*c.d.f*	*Absolute Lorenz curve*
Exponential	$1 - e^{-\lambda x}$, $\lambda > 0, x \geq 0$	$(1-p)\log(1-p)/\lambda$
Pareto-I	$1 - (x/x_0)^{-\alpha}$, $\alpha > 1, x \geq x_0 > 0$	$\alpha x_0(\alpha - 1)^{-1}(1-p)$ $\times[1 - (1-p)^{-1/\alpha}]$
Rectangular	$(x-a)/\theta$, $a \leq x \leq a + \theta$, $\theta > 0, a \geq 0$	$\theta p(p-1)/2$

If one wants to compare two income profiles in terms of absolute inequality, one may consider the comparison of two absolute Lorenz curves. We suggest a test of significance for absolute Lorenz dominance in the next section.

3. TESTING FOR ABSOLUTE LORENZ DOMINANCE

In case, absolute Lorenz curve for distribution F dominates the absolute Lorenz curve for distribution G, F is supposed to have less income inequality than G and we say that F dominates G in the absolute Lorenz sense.

Testing for absolute Lorenz dominance can be carried out either at a fixed number of points or using the whole curves. Usually the curves are compared at a fixed number of points using the multivariate set-up. This approach has the limitation that it can not be used to compare two absolute Lorenz curves in the whole interval of interest.

In this paper, we construct an asymptotically distribution free and consistent test for comparing absolute Lorenz curves of two income distributions in the entire region of interest, say $[p_1, p_2]$ where $0 < p_1 < p_2 < 1$.

Let F_1 and F_2 be absolutely continuous income distribution functions and let $AL_1(p)$ and $AL_2(p)$ be the corresponding absolute Lorenz curves. Now we want to test the hypothesis

$$H_0 : AL_1(p) \leq AL_2(p), \quad \text{for some} \quad p \in [p_1, p_2]$$

against the alternative

$$H_1 : AL_1(p) > AL_2(p), \quad \text{for all} \quad p \in [p_1, p_2]$$

with $0 < p_1 < p_2 < 1$.

For testing H_0 *vs.* H_1, we first propose an asymptotic level α and a consistent test for the problem of testing

$$H_0^{'} : AL_1(p) \leq AL_2(p)$$

against

$$H_1^{'} : AL_1(p) > AL_2(p)$$

for any fixed $p \in [p_1, p_2]$ and then reject H_0 in favor of H_1 if $H_0^{'}$ is rejected at level α for all $p \in [p_1, p_2]$.

We define

$$\begin{aligned}
\alpha_p &= F_1^{-1}(p), \\
\beta_p &= F_2^{-1}(p), \\
\lambda_p &= \frac{1}{p} \int_0^{\alpha_p} u dF_1(u), \\
\delta_p &= \frac{1}{p} \int_0^{\beta_p} u dF_2(u), \\
\xi_p^2 &= \frac{1}{p} \int_0^{\alpha_p} u^2 dF_1(u) - \lambda p^2, \\
\eta_p^2 &= \frac{1}{p} \int_0^{\beta_p} u^2 dF_2(u) - \delta p^2.
\end{aligned}$$

Let $F_{1,m}(x)$ and $F_{2,n}(y)$ denote the empirical *cdf*'s based on two random samples of sizes m and n from the distributions with *cdf*'s F_1 and F_2 respectively. Let $X_{(i)}$ and $Y_{(j)}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ be the i^{th} and j^{th} order statistics from the two samples. On the basis of these samples, we have the following consistent estimators

$$\hat{\alpha}_p = X_{([mp]+1)}, \tag{3.1}$$

$$\hat{\beta}_p = Y_{([np]+1)}, \tag{3.2}$$

$$\hat{\lambda}_p = \frac{1}{mp}\sum_{i=1}^{[mp]} X_{(i)}, \tag{3.3}$$

$$\hat{\delta}_p = \frac{1}{np}\sum_{j=1}^{[np]} Y_{(j)}, \tag{3.4}$$

$$\hat{\xi}_p^2 = \frac{1}{mp}\sum_{i=1}^{[mp]} X_{(i)}^2 - (\hat{\lambda}_p)^2, \tag{3.5}$$

$$\hat{\eta}_p^2 = \frac{1}{np}\sum_{j=1}^{[np]} Y_{(j)}^2 - (\hat{\delta}_p)^2, \tag{3.6}$$

where $[x]$ stands for the greatest integer less than or equal to x.

Using (3.3) and (3.4), the consistent estimators of the two absolute Lorenz curves are

$$\widehat{AL}_1(p) = p\hat{\lambda}_p - p\overline{X},$$
$$\widehat{AL}_2(p) = p\hat{\delta}_p - p\overline{Y},$$

where $\overline{X}$ and $\overline{Y}$ denote the respective sample means.

THEOREM 3.1. *(i) The asymptotic distribution of* $\sqrt{m}[\widehat{AL}_1(p) - AL_1(p)]$ *has limiting normal distribution with mean* 0 *and variance* A_{1p}^2 *where* $A_{1p}^2 = \Omega_{1p}^2 + p^2\sigma_{F_1}^2 - 2p^2[\xi_p^2 + (\alpha_p - \lambda_p)(\mu_{F_1} - \lambda_p)]$ *with* $\Omega_{1p}^2 = p[\xi_p^2 + (1-p)(\alpha_p - \lambda_p)^2]$ *for each* $p_1 \le p \le p_2$ *and* $\sigma_{F_1}^2$ *is variance corresponding to* F_1,
(ii) The asymptotic distribution of $\sqrt{n}[\widehat{AL}_2(p) - AL_2(p)]$ *has limiting normal distribution with mean* 0 *and variance* A_{2p}^2 *where* $A_{2p}^2 = \Omega_{2p}^2 + p^2\sigma_{F_2}^2 - 2p^2[\eta_p^2 + (\beta p - \delta_p)(\mu_{F_2} - \delta_p)]$ *with* $\Omega_{2p}^2 = p[\eta_p^2 + (1-p)(\beta_p - \delta_p)^2]$ *for each* $p_1 \le p \le p_2$ *and* $\sigma_{F_2}^2$ *is variance corresponding to* F_2.

PROOF. (i) $\sqrt{m}[\widehat{AL}_1(p) - AL_1(p)] = \sqrt{m}[\widehat{GL}_1(p) - GL_1(p)] - p\sqrt{m}(\overline{X} - \mu_{F_1})$. Since $\sqrt{m}[\widehat{GL}_1(p) - GL_1(p)] \sim N(0, \Omega_{1p}^2)$ asymptotically (Arora and Jain, 2006), $\sqrt{m}(\overline{X} - \mu_{F_1}) \sim N(0, \sigma_{F_1}^2)$ asymptotically and asymptotically, $\mathrm{Cov}(\widehat{GL}_1(p), \overline{X}) = (p/m)[\xi_p^2 + (\alpha_p - \lambda_p)(\mu_{F_1} - \lambda_p)]$ (Beach and Davidson, 1983). Hence $\sqrt{m}[A\hat{L}_1(p) - AL_1(p)] \sim N(0, A_{1p}^2)$ asymptotically.

(ii) The proof follows on similar lines as followed in proof of (i). □

3.1. Testing for H_0' against H_1'

An obvious test statistic for testing H_0' against H_1' is

$$W_{m,n}(p) = \frac{\widehat{AL}_1(p) - \widehat{AL}_2(p)}{\sqrt{\frac{\hat{A}_{1p}^2}{m} + \frac{\hat{A}_{2p}^2}{n}}},$$

where $\hat{A}_{1p}^2$ and $\hat{A}_{2p}^2$ denote the estimates of A_{1p}^2 and A_{2p}^2 obtained by using consistent estimators given in (3.1)–(3.6).

Using results of Theorem 3.1, we can conclude that $W_{m,n}(p)$ follows asymptotically normal distribution. The critical region for testing H_0' against H_1' for any fixed $p \in [p_1, p_2]$ is given by

$$W_{m,n}(p) > Z_\alpha,$$

where Z_α is the $(1-\alpha)^{th}$ percentile of the standard normal distribution.

3.2. Testing for H_0 against H_1

For large sample sizes m and n, we propose a test that rejects H_0 in favor of H_1 if and only if

$$\inf_{p_1 \le p \le p_2} W_{m,n}(p) > Z_\alpha \tag{3.7}$$

or equivalently

$$W_{m,n}(p) > Z_\alpha \text{ for all } p \in [p_1, p_2].$$

REMARK 3.1. The asymptotic normality of $W_{m,n}(p)$ is uniform in p, since the results of Theorem 3.1 hold for all $p \in [p_1, p_2]$.

In the following theorem, we show that the proposed test (3.7) has the prescribed asymptotic size α .

THEOREM 3.2. *Let m, $n \to \infty$ such that $m/(m+n) \to \lambda$ where $0 \le \lambda < 1$, then for F_1 and F_2 in H_0.*

$$\overline{\lim}\, P\left[\inf_{p_1 \le p \le p_2} W_{m,n}(p) > Z_\alpha\right] \le \alpha.$$

PROOF. For proving the above theorem, we prove the following lemma.

LEMMA 3.1. *Let m, $n \to \infty$ such that $m/(m+n) \to \lambda$ where $0 \le \lambda < 1$, then*

$$P[W_{m,n}(p) > Z_\alpha] \longrightarrow \begin{cases} 0, & \text{if } AL_1(p) < AL_2(p), \\ \alpha, & \text{if } AL_1(p) = AL_2(p), \\ 1, & \text{if } AL_1(p) > AL_2(p) \end{cases}$$

for all $p \in [p_1, p_2]$.

PROOF. We can write

$$W_{m,n}(p) = \frac{C_{m,n}(p) + D_{m,n}(p)}{A_{m,n}(p)}, \tag{3.8}$$

where

$$\begin{aligned} C_{m,n}(p) &= \sqrt{m}[\widehat{AL}_1(p) - AL_1(p)] - \frac{\sqrt{m}}{\sqrt{n}}\sqrt{n}[\widehat{AL}_2(p) - AL_2(p)], \\ D_{m,n}(p) &= \sqrt{m}[AL_1(p) - AL_2(p)], \\ A^2_{m,n}(p) &= m\left[\frac{\hat{A}^2_{1p}}{m} + \frac{\hat{A}^2_{2p}}{n}\right]. \end{aligned} \tag{3.9}$$

Using Theorem 2 of Beach and Davidson (1983), it follows that

$$C_{m,n}(p) \xrightarrow{L} N(0, A^2(p)),$$

where $A^2(p) = A^2_{1p} + \lambda/(1-\lambda)A^2_{2p}$. Since $m/(m+n) \to \lambda$ as $m, n \to \infty$ and $\hat{A}^2_{1p}$ and $\hat{A}^2_{2p}$ are consistent estimates,

$$A^2_{m,n}(p) \xrightarrow{a.s} A^2_{1p} + \frac{\lambda}{1-\lambda}A^2_{2p} = A^2(p).$$

Hence using Slutsky's theorem (Serfling, 1980),

$$\frac{C_{m,n}(p)}{A_{m,n}(p)} \xrightarrow{L} N(0,1). \tag{3.10}$$

Using (3.9) for large m and n, we have

$$D_{m,n}(p) = \begin{cases} \infty, & \text{if } AL_1(p) > AL_2(p), \\ 0, & \text{if } AL_1(p) = AL_2(p), \\ -\infty, & \text{if } AL_1(p) < AL_2(p). \end{cases} \tag{3.11}$$

□

The result follows using (3.8), (3.10) and (3.11).

$$\inf_{p_1 \le p \le p_2} W_{m,n}(p) > Z_\alpha \Longrightarrow W_{m,n}(p) > Z_\alpha \text{ for all } p \in [p_1 p_2].$$

Hence,

$$P_{H_0}\left[\inf_{p_1 \le p \le p_2} W_{m,n}(p) > Z_\alpha\right] \le P_{H_0}[W_{m,n}(p) > Z_\alpha]$$

which implies

$$\overline{\lim}\, P_{H_0}\left[\inf_{p_1 \le p \le p_2} W_{m,n}(p) > Z_\alpha\right] \le \overline{\lim}\, P_{H_0}[W_{m,n}(p) > Z_\alpha].$$

The result now follows from (3.8), (3.10) and (3.11). □

The consistency of the test proposed in (3.7) is evident from the following theorem.

THEOREM 3.3. *For $F_1, F_2 \in H_1$,*

$$P\left[\inf_{p_1 \le p \le p_2} W_{m,n}(p) > Z_\alpha\right] \longrightarrow 1 \text{ as } m, n \to \infty.$$

PROOF. The proof follows from Theorem 2.3 in Arora and Jain (2006). □

3.3. Simulation study

We find the power of the test for testing H_0 *vs.* H_1 on the basis of 10,000 replications at 0.05 level of significance.

TABLE 3.1 *Power of the test*

Distributions	m=30, n=30	m=50, n=50	m=100, n=100
Lognormal(0,1), *Lognormal*(1,1)	0.46	0.71	0.85
Exp(2), *Exp*(1)	0.54	0.73	0.89
Gamma(10,1), *Gamma*(10,2)	0.62	0.78	0.91

The above table depicts that power of the test increases with an increase in sample size. Out of the three distributions considered above, it is observed that power is maximum in case of gamma distribution.

3.4. Illustration

For illustrating the testing of H_0' *vs.* H_1', two samples of size $m = n = 100$, are generated respectively from $F_1 \sim Lognormal(0,1)$ and $F_2 \sim Lognormal(0,2)$. The values of $W_{m,n}(p)$, *i.e.*, test statistic (for fixed p) are displayed in the Table 3.2.

TABLE 3.2 *Test statistic for fixed p*

p	$\widehat{AL}_1(p)$	$\widehat{AL}_2(p)$	$\widehat{AL}_1(p) - \widehat{AL}_2(p)$	$W_{m,n}(p)$
0.1	-0.1674880	-0.7581555	0.5906675	2.007840
0.2	-0.3335663	-1.5157551	1.1821888	2.011084
0.3	-0.4955426	-2.2711320	1.7755894	2.016352
0.4	-0.6497269	-3.0198715	2.3701446	2.022554
0.5	-0.7879161	-3.7553737	2.9674566	2.030666
0.6	-0.8987293	-4.4597615	3.5610312	2.042956
0.7	-0.9573909	-5.1042360	4.1468451	2.047853
0.8	-0.9208378	-5.6456279	4.7247901	2.058817
0.9	-0.7380672	-5.8734373	4.8353701	2.086727

Comparison of $W_{m,n}(p)$ with Z_α for $\alpha = 0.05$ shows that absolute Lorenz curve ordinates for F_1 dominate the corresponding ordinates for F_2 at all the selected values of p.

Since for testing H_0 *vs.* H_1,

$$\inf_{0 \le p \le 1} W_{m,n}(p) = 2.007840 > Z_\alpha, \text{ for } \alpha = 0.05,$$

hence the absolute Lorenz curve for F_1 dominates the absolute Lorenz curve for F_2 in the whole interval $[p_1, p_2]$ where $0 < p_1 < p_2 < 1$. This is also apparent from Figure 3.1.

REMARK 3.2. The terms absolute Lorenz dominance and absolute Lorenz ordering are used interchangeably and hence the above testing procedure holds for AL ordering as well.

4. ABSOLUTE LORENZ ORDERING

DEFINITION 4.1. *F_1 is said to have less income inequality than F_2 in the absolute Lorenz sense (i.e., $F_1 \underset{A}{\le} F_2$) iff $AL_1(p)$ lies above $AL_2(p)$, i.e.,*

$$AL_1(p) \ge AL_2(p), \quad 0 \le p \le 1.$$

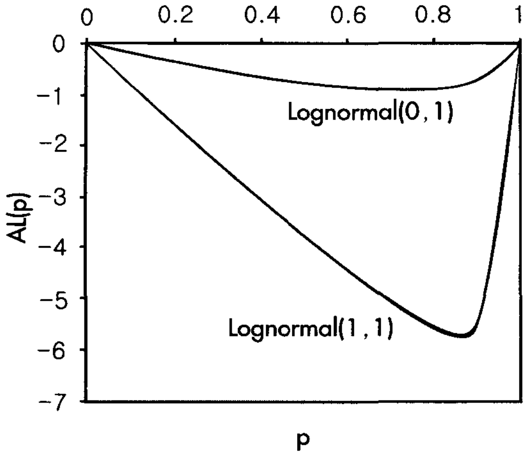


FIGURE 3.1 *Absolute Lorenz curve.*

The shape of the absolute Lorenz curve which depends upon the parameters of the underlying distribution is used to depict the absolute Lorenz ordering. We observe this for some distributions.

(i) *Exponential distribution.* From Table 2.1, the absolute Lorenz curve is given as

$$AL(p) = \frac{(1-p)\log(1-p)}{\lambda} \quad \text{where} \quad p = F(x) = 1 - e^{-\lambda x}, \;\; \lambda > 0, \;\; x \geq 0.$$

It is easy to prove that for $\lambda_1 \leq \lambda_2$, $AL_1(p) \leq AL_2(p) \Longrightarrow F_1 \underset{\text{A}}{\geq} F_2$. This fact is illustrated in the Figure 4.1.

(ii) *Uniform distribution.* From Table 2.1, the absolute Lorenz curve is given as

$$AL(p) = \frac{\theta p(p-1)}{2} \quad \text{where} \quad p = F(x) = \frac{x-a}{\theta}, \;\; a \leq x \leq a + \theta.$$

It can be seen that for $\theta_1 \leq \theta_2, AL_1(p) \geq AL_2(p) \Longrightarrow F_1 \underset{\text{A}}{\leq} F_2$. This is depicted in the Figure 4.1 as well.

(iii) *Pareto distribution.* From Table 2.1, the corresponding absolute Lorenz curve is given as

$$AL(p) = \frac{\alpha x_0}{\alpha - 1}\left[1 - p - (1-p)^{\frac{\alpha-1}{\alpha}}\right],$$

where $p = F(x) = 1 - (x/x_0)^{-\alpha}$, $\alpha > 0$, $x \geq x_0$.

(a) $\alpha_1 \leq \alpha_2$, x_0 fixed $\Longrightarrow AL_1(p) \leq AL_2(p) \Longrightarrow F_1 \underset{\text{A}}{\geq} F_2$.

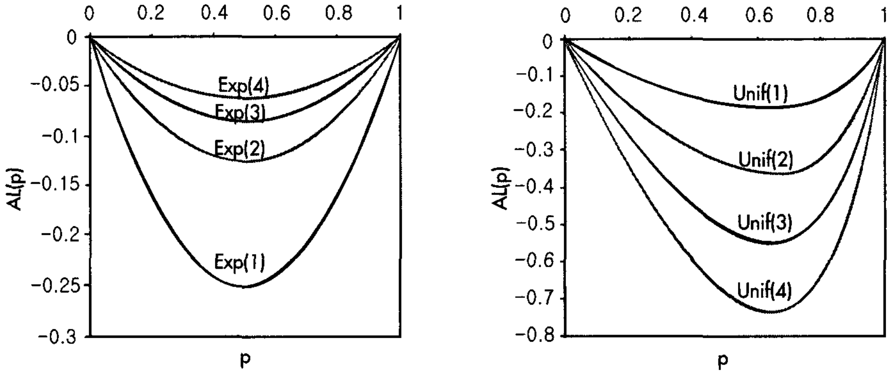
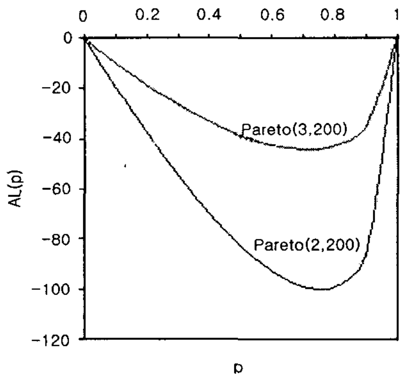
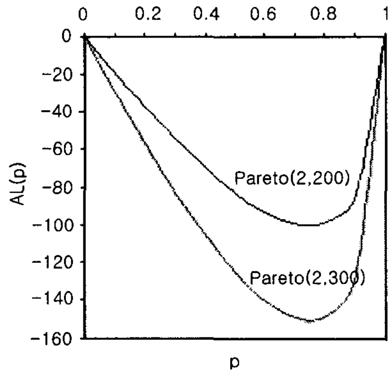


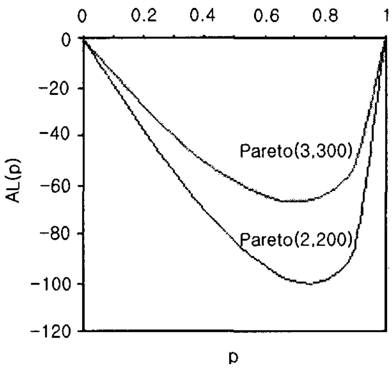
FIGURE 4.1 *Absolute Lorenz curve. Left panel is exponential distribution case and right panel is uniform distribution case.*



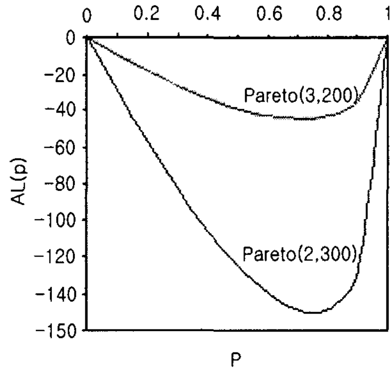
(a) $\alpha_1 \leq \alpha_2$, x_0 *fixed.*



(b) α *fixed and* $x_0^{(1)} \leq x_0^{(2)}$.



(c) $\alpha_1 \leq \alpha_2$ *and* $x_0^{(1)} \leq x_0^{(2)}$.



(d) $\alpha_1 \leq \alpha_2$ *and* $x_0^{(1)} \geq x_0^{(2)}$.

FIGURE 4.2 *Absolute Lorenz curve (Pareto distribution).*

(b) α fixed and $x_0^{(1)} \leq x_0^{(2)} \Longrightarrow AL_1(p) \geq AL_2(p) \Longrightarrow F_1 \underset{A}{\leq} F_2$.

(c) $\alpha_1 \leq \alpha_2$ and $x_0^{(1)} \leq x_0^{(2)} \Longrightarrow AL_1(p) \leq AL_2(p) \Longrightarrow F_1 \underset{A}{\geq} F_2$.

(d) $\alpha_1 \leq \alpha_2$ and $x_0^{(1)} \geq x_0^{(2)} \Longrightarrow AL_1(p) \leq AL_2(p) \Longrightarrow F_1 \underset{A}{\geq} F_2$.

These are depicted in the Figure 4.2 as well, respectively.

Hence it is evident from the above examples that the relation between parameters of two distributions affects the absolute Lorenz dominance.

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