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Intuitionistic Fuzzy Subgroups and Level Subgroups

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Abstract

We introduce the concept of level subgroups of an intuitionistic fuzzy subgroup and study some of it's properties. These level subgroups in turn play an important role in the characterization of all intuitionistic fuzzy subgroup of a prime cyclic group.

Key Words : Intuitionistic fuzzy set, intuitionistic fuzzy subgroup, level subgroup.

1. Introduction

The notion of fuzzy sets was introduced by Zadeh in [20]. Since its inception, the theory of fuzzy sets has developed in many directions and is finding applications in a wide variety of fields. In particular, several researchers [3, 7-9, 18, 19] have applied the notion of fuzzy sets to group theory.

In 1986, Atanassov [1] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. Recently, Çoker and his colleagues [5,6,10], Hur and his colleagues [13], and Lee and his colleague[17] applied the notion of intuitionistic fuzzy sets to topology. In particular, Hur and his colleagues [15] applied one to topological group. In 1989, Biswas [4] introduced the concept of intuitionistic fuzzy subgroups and investigated some of it's properties. Moreover, Hur and his colleagues [11,12,14,16] redefined the concept of intuitionistic fuzzy subgroupoids, subgroups and rings, and studied some of their properties. In particular, they gave a characterization of all intuitionistic fuzzy subgroups of a prime cyclic group in tems of the complex mapping [11, Proposition 2.14].

In this paper, we obtain a similar characterization of all intuitionistic fuzzy subgroups of finite cyclic groups. For this, we study some properties of level subgroupss of an intuitionistic fuzzy subgroup in the first part of the paper. These level subgroups in turn play an important role in the above characterization.

2. Preliminaries

We will list some concepts and results needed in the later sections.

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For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I.

Definition 1.1[1,5]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) on X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2[1]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.
(3) $A^c = (\nu_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

Definition 1.3[5]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(a) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.
(b) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4[14]. Let A be an IFS in a set X and let $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$. Then the set

$A^{(\lambda,\mu)} = \{x \in X : \mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \mu\}$ is called a (λ,μ)-*level subset* of A.

Result 1.A[12, Proposition 2.2]. Let A be an IFS in a set X and let $(\lambda_1,\mu_1),(\lambda_2,\mu_2) \in$ ImA. If $\lambda_1 \leq \lambda_2$ and $\mu_1 \geq \mu_2$, then $A^{(\lambda_2,\mu_2)} \subset A^{(\lambda_1,\mu_1)}$.

Definition 1.5[11]. Let G be a group and let $A \in IFS(G)$. Then A is called an *intuitionistic fuzzy subgroup* (in short, IFG) of G if it satisfies the following conditions:

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and
$\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ for each $x, y \in G$.

(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and
$\nu_A(x^{-1}) \leq \nu_A(x)$ for each $x \in G$.

We will denote the set of all IFGs of G as IFG(G).

Result 1.B[11, Proposition 2.6]. Let A be an IFG of a group G. Then $A(x^{-1}) = A(x)$ and $\mu_A(x) \leq \mu_A(e), \nu_A(x) \geq \nu_A(e)$ for each $x \in G$, where e is the identity element of G.

Result 1.C[11, Proposition 2.17 and Proposition 2.18]. Let A be an IFS of a group G. Then $A \in$ IFG(G) if and only if for each $(\lambda,\mu) \in I \times I$ with $(\lambda,\mu) \leq A(e)$, i.e., $\lambda \leq \mu_A(e)$ and $\mu \geq \nu_A(e)$, $A^{(\lambda,\mu)}$ is a subgroup of G.

Result 1.D[11, Proposition 2.14]. Let G_p be the cyclic group of prime order p. Then $A \in$ IFG(G_p) if and only if $A(x) = A(y) \leq A(e)$, i.e., $\mu_A(x) = \mu_A(y) \leq \mu_A(e)$ and $\nu_A(x) = \nu_A(y) \geq \nu_A(e)$ for any $x, y \in G_p$ such that $x \neq e$ and $y \neq e$.

3. Level subgroups

From Result 1.C, we define the following concept.

Definition 2.1. Let G be a group and let $A \in$ IFG(G) and let $(\lambda,\mu) \in I \times I$ with $(\lambda,\mu) \leq A(e)$. $A^{(\lambda,\mu)}$ is called a (λ,μ)-*level subgroup* of A.

Let G be a finite group. Then the number of subgroups of G is finite. However, the number of level subgroups of an IFG A appears to be infinite. Indeed, since every level subgroup is a subgroup of G, not all these level subgroups are distinct.

Example 2.1 Let G be the Klein four-group:

$$G = \{a,\, b,\, a^2 = b^2 = (ab)^2 = e\}.$$

Then the elements of G are e, a, b and ab. Moreover, it is clear that the number of subgroups of G is finite. We define a complex mapping $A = (\mu_A, \nu_A) : G \to I \times I$ as follows;

$\mu_A(e) = (\lambda_0,\mu_0)$, $\mu_A(a) = (\lambda_1,\mu_1)$,
$\mu_A(b) = (\lambda_2,\mu_2)$ and $\mu_A(ab) = (\lambda_3,\mu_3)$,

where $(\lambda_i,\mu_i) \in I \times I$ ($i = 0,1,2$ and 3), $\lambda_0 \geq \lambda_i$, $\mu_0 \leq \mu_i$ ($i = 1,2,3$) and $\lambda_3 \geq \lambda_1 \wedge \lambda_2$, $\mu_3 \leq \mu_1 \vee \mu_2$. Then we can easily see that $A \in$ IFG(G). Consider the family $\mathcal{P} = \{A^{(\lambda,\mu)} : (\lambda,\mu) \in I \times I$ with $(\lambda,\mu) \leq A(e)\}$. Then, by Result 1.C, $\mathcal{P}$ is a family of level subgroups of G. Furthermore, $\mathcal{P}$ is infinite. But we can see that all members of $\mathcal{P}$ are not distinct.

Proposition 2.2. Let G be a group and let $A \in$ IFG(G). Two level subgroups $A^{(t_1,s_1)}$ and $A^{(t_2,s_2)}$ (with $(t_1,s_1) < (t_2,s_2)$, i.e., $t_1 < t_2$ and $s_1 > s_2$) of A are equal if and only if there is no $x \in G$ such that $t_1 < \mu_A(x) < t_2$ and $s_1 > \nu_A(x) > s_2$.

Proof.($\Rightarrow$): Suppose $A^{(t_1,s_1)} = A^{(t_2,s_2)}$. Assume that there exists an $x \in G$ such that $t_1 < \mu_A(x) < t_2$ and $s_1 > \nu_A(x) > s_2$. Then $x \in A^{(t_1,s_1)}$ and $x \notin A^{(t_2,s_2)}$. Thus, by Result 1.A, $A^{(t_2,s_2)} \subsetneq A^{(t_1,s_1)}$. This contradicts the hypothesis.

($\Leftarrow$): Suppose the necessary condition holds. Since $t_1 < t_2$ and $s_1 > s_2$, by Result 1.A, $A^{(t_2,s_2)} \subset A^{(t_1,s_1)}$. Let $x \in A^{(t_1,s_1)}$. Then $t_1 \leq \mu_A(x)$ and $s_1 \geq \nu_A(x)$. By the hypothesis, $t_2 \leq \mu_A(x)$ and $s_2 \geq \nu_A(x)$. Thus $x \in A^{(t_2,s_2)}$. So $A^{(t_1,s_1)} \subset A^{(t_2,s_2)}$. Hence $A^{(t_1,s_1)} = A^{(t_2,s_2)}$. ■

Corollary 2.2. Let G be a finite group of order n and let $A \in$ IFG(G). Let ImA $= \{(t_i,s_i)) : A(x) = (t_i,s_i)$ for some $x \in G\}$. Then $\{A^{(t_i,s_i)}\}$ is the set of the only level subgroups of A.

Proof. By Result 1.C, $A^{(t_i,s_i)}$ is a subgroup of G. Let $(\lambda,\mu) \in I \times I$ such that $(\lambda,\mu) \notin$ ImA and $\lambda + \mu \leq 1$.

Case(i) : Suppose $t_i < \lambda < t_j$ and $s_i > \mu > s_j$, where $(t_i,s_i),(t_j,s_j) \in$ ImA. Then, by Proposition 2.2, $A^{(t_i,s_i)} = A^{(t_j,s_j)} = A^{(\lambda,\mu)}$.

Case(ii) : Suppose $\lambda < t_r$ and $\mu > s_r$, where (t_r,s_r) is the least element in ImA. Then, by Proposition 2.2, $A^{(t_r,s_r)} = G = A^{(\lambda,\mu)}$.

Case(iii) : Suppose $t_0 < \lambda$ and $s_0 > \mu$, where (t_0,s_0) is the greast element of ImA. Then, by Proposition 2.2, $A^{(\lambda,\mu)} = A^{(t_i,s_i)} = \{e\}$.

Hence, in any cases, for each $(\lambda,\mu) \in I \times I$ with $\lambda + \mu \leq 1$, the (λ,μ)-level subgroup is one of $\{A^{(t_i,s_i)}\}$, where $(t_i,s_i) \in$ ImA. ■

Proposition 2.3. Any subgroup H of a group G can be realized as a level subgroup of some IFG of G.

Proof. We define a complex mappings $A = (\mu_A,\nu_A) : G \to I \times I$ as follows: for each $x \in G$,

$$A(x) = (t,s) \text{ if } x \in H$$

and

$$A(x) = (0,1) \text{ if } x \notin H,$$

where $(t,s) \in I \times I$ such that $t + s \leq 1$. Then clearly $A = (\mu_A,\nu_A) \in$ IFS(G). Let $x, y \in G$.

Case (i): Suppose $x, y \in H$. Then $xy \in H$. Thus $\mu_A(xy) = \mu_A(x) = \mu_A(y) = t$ and $\nu_A(xy) = \nu_A(x) =$

$\nu_A(y) = s$. So $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$. Since $x \in H$, $x^{-1} \in H$. Thus $\mu_A(x^{-1}) = t$ and $\nu_A(x^{-1}) = s$. So $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$.

Case (ii): Suppose $x \in H$ and $y \notin H$. Then $xy \notin H$. Thus $\mu_A(x) = t$, $\mu_A(y) = \mu_A(xy) = 0$ and $\nu_A(x) = s$, $\nu_A(y) = \nu_A(xy) = 1$. So $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$. Also, we have $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$.

Case (iii): Suppose $x \notin H$ and $y \in H$. Then, we have the same ones as results of case (ii).

Case (iv): Suppose $x \notin H$ and $y \notin H$. Then xy may or may not belong to H. In any case, we have $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ and $\mu_A(x^{-1}) \geq \mu_A(x)$, $\nu_A(x^{-1}) \leq \nu_A(x)$. Hence, in all cases, $A \in \text{IFG}(G)$. In fact, $H = A^{(t,s)}$. This completes the proof. ■

The following result is the generalization of Proposition 2.3.

Proposition 2.4. Let G be a group and let the following be any chain of subgroups

$$G_0 \subset G_1 \subset \cdots \subset G_r = G.$$

Then there exists an intuitionistic fuzzy subgroup of G whose level subgroups are precisely the members of this chain.

Proof. Consider the following set of real numbers:

$$t_0 > t_1 > \cdots > t_r \text{ and } s_0 < s_1 < \cdots < s_r,$$

where $(t_i, s_i) \in I \times I$ and $t_i + s_i \leq 1$ for each i. We define a complex mapping $A = (\mu_A, \nu_A) : G \to I \times I$ as follows:

$$A(G_0) = (t_0, s_0) \text{ and } A(\hat{G}_i) = (t_i, s_i),$$

where $\hat{G}_i = G_i \setminus G_{i-1}$ for $i = 1, 2, \cdots, r$. Then it is clear that $A \in \text{IFS}(G)$ from the definition of A. Let $x, y \in G$.

Case (i) : Suppose $x, y \in \hat{G}_i$. Then $A(x) = (t_i, s_i) = A(y)$. Since G_i is a subgroup, $xy \in G_i$. Thus either $xy \in G_i$ or $xy \in G_{i-1}$. In any case, $\mu_A(xy) \geq t_i = \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq s_i = \nu_A(x) \vee \nu_A(y)$. On the other hand, $x^{-1} \in G_i$. Thus $\mu_A(x^{-1}) \geq t_i = \mu_A(x)$ and $\nu_A(x^{-1}) \leq s_i = \nu_A(x)$.

Case (ii) : Suppose $x \in \hat{G}_i$, $y \in \hat{G}_j$ and $i > j$. Then $A(x) = (t_i, s_i)$ and $A(y) = (t_j, s_j)$. Since $G_j \subset G_i$ and G_i is a subgroup, $xy \in G_i$. Thus

$$\mu_A(xy) \geq t_i = \mu_A(x) \wedge \mu_A(y)$$

and

$$\nu_A(xy) \leq s_i = \nu_A(x) \vee \nu_A(y).$$

On the other hand, $x^{-1} \in C_i$. Thus

$$\mu_A(x^{-1}) \geq t_i = \mu_A(x)$$

and

$$\nu_A(x^{-1}) \leq s_i = \nu_A(x).$$

So, in either case, we can see that $A \in \text{IFG}(G)$.

Now, from the definition of A, ImA$= \{(t_0, s_0), \cdots, (t_r, s_r)\}$. Thus the level subgroups of A are given by the chain of subgroups

$$A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_r,s_r)} = G.$$

We claim that $A^{(t_i,s_i)} = G_i, 0 < i \leq r$. By the definitions of A and $A^{(t_i,s_i)}$, it is clear that $G_i \subset A^{(t_i,s_i)}$. Let $x \in A^{(t_i,s_i)}$. Then $\mu_A(x) \geq t_i$ and $\nu_A(x) \leq s_i$. Thus $x \notin G_j$ for $j > i$. So $A(x) \in \{(t_1, s_1), \cdots, (t_i, s_i)\}$, i.e., $x \in G_k$ for some $k \leq i$. Since $G_k \subset G_i$, $x \in G_i$. Thus $A^{(t_i,s_i)} \subset G_i$. Hence $A^{(t_i,s_i)} = G_i, 0 \leq i \leq r$. This completes the proof. ■

As a consequence of Proposition 2.4, the level subgroups of an IFG A form a chain. Since $\mu_A(x) \leq \nu_A(e)$ and $\nu_A(x) \geq \nu_A(e)$ for each $x \in G$, $A^{(t_0,s_0)}$ is the smallest level subgroup of A, where $A(e) = (t_0, s_0)$. Thus we have the chain

$$(e) = A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_r,s_r)} = G, \quad (2.1)$$

where $t_0 > t_1 > \cdots > t_r$ and $s_0 < s_1 < \cdots < s_r$. We denote this chain (2.1) of level subgroups by $C(A)$. In general, as all the subgroups of G do not form a chain, it follows that not all subgroups of G are level subgroups of a given intutionistic fuzzy subgroup. So it is an interesting problem to find an IFG A of G which accommodates as many subgroups of G as possible in $C(A)$.

Proposition 2.5. Let G be a finite group such that $G = G_{p_1} \times G_{p_2} \times \cdots \times G_{p_r}$, where the G_{p_i} are prime cyclic groups of orders p_i. Then there exists an $A \in \text{IFG}(G)$ such that $C(A)$ is a maximal chain of length $r+1$.

Proof. We prove by induction on r. Suppose $r = 1$. Then $G = C_{p_1}$. Then, by Result 1.D, there exists an $A \in \text{IFG}(G)$ such that $A(e) = (t_0, s_0)$, $A(x) = (t_1, s_1)$ for each $e \neq x \in G$ and $t_2 < t_1$ and $s_2 > s_1$. Thus $A^{(t_0,s_0)} = (e)$ and $A^{(t_1,s_1)} = G$. So $A^{(t_0,s_0)} \subset A^{(t_1,s_1)}$ is the maximal chain and of length 2. Hence the theorem is true for $r = 1$. Now let $r > 1$ and suppose the theorem is true for the integers $\leq r - 1$. Let $H = G_{p_1} \times G_{p_2} \times \cdots \times G_{p_{r-1}}$. Then $G = H \times G_{p_r}$. Define the complex mapping $A = (\mu_A, \nu_A) : G \longrightarrow I \times I$ by $A(e) = (t_0, s_0)$, $A(\widehat{G_{p_1}}) = (t_1, s_1)$, $A(\widehat{G_{p_1} \times G_{p_2}}) = (t_2, s_2)$, $\cdots$, $A(\widehat{H \times G_{p_r}}) = (t_r, s_r)$, where $t_0 > t_1 > t_2 > \cdots > t_r$, $s_0 < s_1 < s_2 < \cdots < s_r$, $t_i + s_i \leq 1$ and $\widehat{G_{p_1}} = G_{p_1} \setminus (e)$, $\widehat{G_{p_1} \times G_{p_2}} = G_{p_1} \times G_{p_2} \setminus G_{p_1}$, and so on. Then it is clear that $A \in \text{IFS}(G)$ from the definition of A. We will show that $A \in \text{IFG}(G)$. Let $x, y \in G$.

Case (i): Suppose $x, y \in H$. Then $xy \in H$. By the induction,

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y), \quad \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$$

and

$$\mu_A(x^{-1}) \geq \mu_A(x), \quad \nu_A(x^{-1}) \leq \nu_A(x).$$

Case (ii): Suppose $x \in H$ and $y \in G \setminus H$. Then $xy \notin H$. Thus $A(xy) = (t_r, s_r), \mu_A(x) \geq t_{r-1}, \nu_A(x) \leq s_{r-1}$ and $A(y) = (t_r, s_r)$. So

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y), \quad \nu_A(xy) \leq_A (x) \vee_A(y)$$

and

$$\mu_A(x^{-1}) \geq \mu_A(x), \quad \nu_A(x^{-1}) \leq \nu_A(x).$$

Case (iii): Suppose $x \in G \setminus H$ and $y \in H$. Then, we have the same ones as the results of case(ii).

Case (iv): Suppose $x \notin H$ and $y \notin H$. Then also we can easily see that

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y), \quad \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$$

and

$$\mu_A(x^{-1}) \geq \mu_A(x), \quad \nu_A(x^{-1}) \leq \nu_A(x).$$

So, in either cases, $A \in \text{IFG}(G)$. Moreover, $A^{(t_0,s_0)} = (e)$, $A^{(t_1,s_1)} = G_{r_1}, A^{(t_2,s_2)} = G_{r_1} \times G_{r_2}, \cdots, A^{(t_r,s_r)} = H \times G_{r_r}$. Hence $A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_r,s_r)}$ is $C(A)$ which is maximal and of length $r+1$. This completes the proof. ■

Remark 2.6. In the same way, we can find an IFG A with the maximal $C(A)$ in the following cases :

(i) G is a cyclic p-group.
(ii) G is the direct product of cyclic p-group.
(iii) G is a finite abelian group.

We can easily check these cases by adopting the same technique as proof in Proposition 2.6.

In the following example, we show that two intuitionistic fuzzy subgroups of a group may have an identical family of level subgroups but the intuitionistic fuzzy subgroups may not be equal.

Example 2.7. Consider the Klein four-group G given in Example 2.1. Let $(t_i, s_i) \in I \times I$ such that $t_0 > t_1 > t_2, s_0 < s_1 < s_2$ and $t_i + s_i \leq 1$, where $i = 0, 1, 2$. We define a complex mapping $A : G \to I \times I$ as follows :

$A(e) = (t_0, s_0), A(a) = (t_1, s_1)$,
$A(b) = (t_2, s_2), A(ab) = (t_2, s_2)$.

Then clearly $A \in \text{IFG}(G)$ and ImA$= \{(t_0, s_0), (t_1, s_1), (t_2, s_2)\}$. Moreover, the level subgroups of A are

$$A^{(t_0,s_0)} = \{e\}, A^{(t_1,s_1)} = \{e, a\}, A^{(t_2,s_2)} = G.$$

Now let $(\lambda_i, \mu_i) \in I \times I$ such that

$\lambda_0 > \lambda_1 > \lambda_2, \mu_0 < \mu_1 < \mu_2, \lambda_i + \mu_i \leq 1$,
for $i = 0, 1, 2$

and

$\{(t_0, s_0), (t_1, s_1), (t_2, s_2)\} \cap$
$\{(\lambda_0, \mu_0), (\lambda_1, \mu_1), (\lambda_2, \mu_2)\} = \phi$.

We define a complex mapping $B : G \to I \times I$ a follows :

$B(e) = (\lambda_0, \mu_0), B(a) = (\lambda_1, \mu_1)$,
$B(b) = (\lambda_2, \mu_2), B(ab) = (\lambda_2, \mu_2)$.

Then clearly $B \in \text{IFG}(G)$. Moreover, the level subgroups of B are

$$B^{(\lambda_0,\mu_0)} = \{e\}, B^{(\lambda_1,\mu_1)} = \{e, a\}, B^{(\lambda_2,\mu_2)} = G.$$

Hence A and B have the same family of level subgroups but $A \neq B$.

The following is the immediate result of Definition 1.4.

Lemma 2.8. Let G be a finite group and let $A \in \text{IFG}(G)$. If $(t_i, s_i), (t_j, s_j) \in$ ImA such that $A^{(t_i,s_i)} = A^{(t_j,s_j)}$, then $(t_i, s_i) = (t_j, s_j)$.

Proposition 2.9 Let G be a finite group and let $A, B \in \text{IFG}(G)$ with identical family of level subgroups. If ImA$= \{(t_0, s_0), \cdots, (t_r, s_r)\}$ and ImB$= \{(\lambda_0, \mu_0), \cdots, (\lambda_k, \mu_k)\}$, where $t_0 > \cdots > t_r$, $s_0 < \cdots < s_r$ and $\lambda_0 > \cdots > \lambda_k$, $\mu_0 < \cdots < \mu_k$, then we have

(i) $r = k$.
(ii) $A^{(t_i,s_i)} = B^{(\lambda_i,\mu_i)}, 0 \leq i \leq r$.
(iii) if $x \in G$ such that $A(x) = (t_i, s_i)$, then $B(x) = (\lambda_i, \mu_i), 0 \leq i \leq r$.

Proof. (i) By corollary 2.2, the only level subgroups of A and B are the two families $\{A^{(t_i,s_i)}\}$ and $\{B^{(\lambda_i,\mu_i)}\}$. Hence, by hypothesis, $r = k$.

(ii) By (i) and corollary 2.2, there exist two chains of level subgroups :

$$A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_r,s_r)} = G$$

and

$$B^{(\lambda_0,\mu_0)} \subset B^{(\lambda_1,\mu_1)} \subset \cdots \subset B^{(\lambda_k,\mu_k)} = G.$$

From this, it follows clearly that

Suppose $(t_i, s_i), (t_j, s_j) \in$ ImA such that $t_i > t_j$ and $s_i > s_j$. Then

$$A^{(t_i,s_i)} \subset A^{(t_j,s_j)}. \tag{2.2}$$

Suppose $(\lambda_i, \mu_i), (\lambda_j, \mu_j) \in$ ImB such that $\lambda_i > \lambda_j$ and $\mu_i > \mu_j$. Then

$$B^{(\lambda_i,\mu_i)} \subset B^{(\lambda_j,\mu_j)}. \tag{2.3}$$

Since $\{A^{(t_i,s_i)}\} = \{B^{(\lambda_i,\mu_i)}\}$, it is clear that $A^{(t_0,s_0)} = B^{(\lambda_0,\mu_0)}$. Now by hypothesis, $A^{(t_1,s_1)} = B^{(\lambda_j,\mu_j)}$ for some $j > 0$. Assume that $A^{(t_1,s_1)} = B^{(\lambda_j,\mu_j)}$ for some $j > 1$. Again, we have that $B^{(\lambda_1,\mu_1)} = A^{(t_i,s_i)}$ for some $t_i > t_1$ and $s_i < s_1$. It is clear that $(t_i, s_i) = (t_1, s_1)$. Thus, by (2.2),

$$A^{(t_i,s_i)} = B^{(\lambda_1,\mu_1)} \subset B^{(\lambda_j,\mu_j)}. \tag{2.4}$$

Also, by (2.3),

$$B^{(\lambda_j,\mu_j)} = A^{(t_1,s_1)} \subset A^{(t_i,s_i)}. \tag{2.5}$$

However, (2.4) and (2.5) contradict one another as the inclusions are both proper inclusions. So, we must have that

$$A^{(t_1,s_1)} = B^{(\lambda_1,\mu_1)}.$$

The rest of the proof follows by induction on i by using arguments exactly on the same lines as above. Hence $A^{(t_i,s_i)} = B^{(\lambda_j,\mu_j)}, 0 \le i \le r$.

(iii) Let $x \in G$ such that $A(x) = (t_i, s_i)$ and $B(x) = (\lambda_j, \mu_j)$. Then, by (ii), $A^{(t_i,s_i)} = B^{(\lambda_i,\mu_i)}$. Since $x \in A^{(t_i,s_i)}, x \in B^{(\lambda_i,\mu_i)}$. Thus $B(x) = (\lambda_j, \mu_j)$, where $\lambda_j \ge \lambda_i$ and $\mu_j \le \mu_i$. By (2.3), $B^{(\lambda_j,\mu_j)} \subset B^{(\lambda_i,\mu_i)}$. By (ii), $B^{(\lambda_j,\mu_j)} = A^{(t_j,s_j)}$. Since $x \in B^{(\lambda_j,\mu_j)}, x \in A^{(t_j,s_j)}$. Thus $A(x) = (t_i, s_i)$, where $t_i \ge t_j$ and $s_i \le s_j$. By (2.2), $A^{(t_i,s_i)} \subset A^{(t_j,s_j)}$. On the other hood, by (ii), $A^{(t_i,s_i)} = B^{(\lambda_i,\mu_i)}$ and $A^{(t_j,s_j)} = B^{(\lambda_j,\mu_j)}$. Consequently, we have that $B^{(\lambda_i,\mu_i)} \subset B^{(\lambda_j,\mu_j)}$. So $B^{(\lambda_i,\mu_i)} = B^{(\lambda_j,\mu_j)}$. Hence, by Lemma 2.8, $(\lambda_i, \mu_i) = (\lambda_j, \mu_j)$. This completes the proof. ■

Theorem 2.10. Let A, B be two IFGs of a finite group G such that the families of level subgroups of A and B are identical. Then $A = B$ if and only if ImA = ImB.

Proof. $(\Rightarrow)$: It is obvious.

$(\Leftarrow)$: Suppose ImA = ImB. Let ImA $= \{(t_0, s_0), \cdots, (t_r, s_r)\}$ and let ImB $= \{(\lambda_0, \mu_0), \cdots, (\lambda_r, \mu_r)\}$ such that $t_0 > \cdots > t_r, s_0 < \cdots < s_r$ and $\lambda_0 < \cdots < \lambda_r, \mu_0 > \cdots > \mu_r$. Since $(\lambda_0, \mu_0) \in$ ImB and ImA = ImB, $(\lambda_0, \mu_0) = (t_{k_0}, s_{k_0})$ for some k_0. Suppose $(t_{k_0}, s_{k_0}) \ne (t_0, s_0)$. Then $t_{k_0} < t_1$ and $s_{k_0} > s_1$. Since $(\lambda_1, \mu_1) \in$ ImA, $(\lambda_1, \mu_1) = (t_{k_1}, s_{k_1})$ for some k_1. Thus we have $\lambda_0 > \lambda_1$ and $\mu_0 < \mu_1$ implies that $t_{k_0} > t_{k_1}$ and $s_{k_0} < s_{k_1}$.
By proceeding in this way, we have

$$t_{k_0} > t_{k_1} > \cdots > t_{k_r} \quad \text{and} \quad s_{k_0} < s_{k_1} < \cdots < s_{k_r},$$

where $(\lambda_0, \mu_0) = (t_{k_0}, s_{k_0})$, $t_{k_0} > t_0$ and $s_{k_0} < s_0$. They contradict the fact that ImA = ImB. So we must have that $(\lambda_0, \mu_0) = (t_0, s_0)$. Arguing in this manner, we obtain that

$$(\lambda_i, \mu_i) = (t_i, s_i), \quad 0 \le i \le r.$$

Now let $g_0, \cdots, g_r$ be distinct elements of G such that $A(g_i) = (t_i, s_i), 0 \le i \le r$. Then, by Proposition 2.9, $B(g_i) = (\lambda_i, \mu_i), 0 \le i \le r$. Since $(\lambda_i, \mu_i) = (t_i, s_i)$, $A(x) = B(x)$ for each $x \in G$. Hence $A = B$. This completes the proof. ■

The following result is easy to prove.

Lemma 2.11. Let G be a finite group. We define a relation $\sim$ on IFG(G) as follows : for any $A, B \in$ IFG(G), $A \sim B$ if and only of they have an identical family of level subgroups. Then $\sim$ is an equivalence relation on IFG(G).

We note that by Example 2.5, two elements A and B of IFG(G) may be such that $A \sim B$ but A and B need not be equal.

For each $A \in$ IFG(G), let $[A]$ denote the equivalence class of A. If G is finite, then the number of possible distinct level subgroups of G is finite since each level subgroup is a subgroup of G in the usual sense. By Proposition 2.3, since any subgroup of a group G can be realized as the level subgroup of some intuitionistic fuzzy subgroup of G, it follows that the number of possible chains of level subgroups is also finite. As each equivalence class is characterized completely by its chain of level subgroups, we have the following result.

Corollary 2.11. Let G be a finite group and let $\sim$ be the equivalence relation on IFG(G) defined by Lemma 2.11. Then IFG$(G)/\sim$ is finite.

Theorem 2.12. Let G be a finite group and let LG$(G) = \{A^{(\lambda,\mu)} : A^{(\lambda,\mu)}$ is a level subgroup of G and $A \in$ IFG(G)$\}$. Let SG(G) denote the set of all subgroups of G. Then there is a one-to-one correspondence between SG(G) and LG$(G)/\sim$, where $\sim$ denote a suitable equivalence relation on LG(G).

Proof. Let $\sim$ be the equivalence relation on IFG(G) defined by Lemma 2.11. Then IFG$(G)/\sim$ is an partition of G. Thus

$$\text{IFG(G)} = [S_1] \cup [S_2] \cup \cdots [S_k],$$

where $[S_i], 1 \le i \le k$, are all distinct equivalence classes. Let us denote

$$\{A_i^{(t_j,s_j)} : 0 \le j \le \lambda_i \text{ and } t_j + s_j \le 1\}$$

to be the chain of level subgroups associated with the equivalence class $[S_i]$. Then LG$(G)/\sim$ is a finite set give by

$$\text{LG}(G)/\sim = \{[A_i^{(t_j,s_j)}] : 0 \le j \le \lambda_i \text{ and } 1 \le i \le k\},$$

where $[A_i^{(t_j,s_j)}]$ denotes the equivalence class containing $A_i^{(t_j,s_j)}$.

From Proposition 2.3, it follows that each subgroup of G is of the form $A_i^{(t_j,s_j)}$. We define a mapping f : LG$(G)/\sim \to$ SG(G) as follows :

$$f([A_i^{(t_j,s_j)}]) = A_i^{(t_j,s_j)}.$$

Then we can easily show that f is bijective. This completes the proof. ■

4. Characterization of intuitionistic fuzzy subgroups of finite cyclic groups.

Proposition 3.1. Let G be a cyclic p-group of order p^n, where p is a prime. Let $A \in IFG(G)$, let $x, y \in G$ and let $O(x)$ denote the order of x.

(1) If $O(x) > O(y)$, then $A(y) \ge A(x)$,
i.e., $\mu_A(y) \ge \mu_A(x)$ and $\nu_A(y) \le \nu_A(x)$.

(2) If $O(x) = O(y)$, then $A(x) = A(y)$.

Proof. We prove by induction on n. Suppose $n = 1$. Then $O(G) = p$. Thus the theorem is true by Result 1.B.

Let $n > 1$ and suppose the theorem is true for all integers $\leq n-1$. Let H be a subgroup of order p^{n-1} and let $x, y \in G$.

Case (i): Suppose $x, y \in H$. Then, by the induction, the results follow.

Case (ii): Suppose $x \notin H$ and $y \in H$. Then $O(x) = p^n$ and $O(y) = p^r$, where $r \leq n-1$. Thus x is a generator of G. So there exists an integer l such that $y = x^l$. Hence

$$\mu_A(y) = \mu_A(x) \wedge \cdots \wedge \mu_A(x)(l \text{ times}) \geq \mu_A(x)$$

and

$$\nu_A(y) = \nu_A(x) \vee \cdots \vee \nu_A(x)(l \text{ times}) \leq \nu_A(x).$$

Case (iii) : Suppose $x \in H$ and $y \notin H$. Then, we have the same ones as the result of case(ii).

Case (iv) : Suppose $x \notin H$ and $y \notin H$. Then $O(x) = O(y) = p^n$. Thus x and y are generators of G . So there exist integers l and m such that $y = x^l$ and $x = y^m$. Hence

$$\mu_A(x) \geq \mu_A(y) \wedge \cdots \wedge \mu_A(y)(m\text{times}) \quad \geq \mu_A(y),$$

$$\nu_A(x) \leq \nu_A(y) \vee \cdots \vee \nu_A(y)(m\text{times}) \quad \leq \nu_A(y)$$

and

$$\mu_A(y) = \mu_A(x) \wedge \cdots \wedge \mu_A(x)(l\text{times}) \quad \geq \mu_A(x),$$

$$\nu_A(y) = \nu_A(x) \vee \cdots \vee \nu_A(x)(l\text{times}) \quad \leq \nu_A(x).$$

Therefore $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$, i.e., $A(x) = A(y)$. This completes the proof. ■

Proposition 3.1 is not true in general as shown in the following examples.

Example 3.2. Consider the Kleins 4 - group :

$$G = \{a, b : a^2 = b^2 = (ab)^2 = e\}.$$

We define a complex mapping $A = (\mu_A, \nu_A) : G \to I \times I$ as follows:

$$A(e) = (t_0, s_0),\ A(a) = (t_1, s_1),\ A(b) = (t_2, s_2) = A(ab),$$

where $(t_0, s_0) > (t_1, s_1) > (t_2, s_2)$ and $t_i + s_i \leq 1$ for $i = 0, 1, 2$.
Then clearly $A \in IFG(G)$. But, even though $O(a) = O(b)$, $A(a) \neq A(b)$.

For a cyclic group it can be seen that $O(a) = O(b)$ implies $A(a) = A(b)$. But $O(a) \neq O(b)$ may also imply $A(a) = A(b)$.

Example 3.3. Let $G = (a)$ be a cyclic group of order 6. We define a complex mapping $A = (\mu_A, \nu_A) : G \to I \times I$ as follows:
$A(e) = (t_0, s_0)$, $A(a) = A(a^3) = A(a^5) = (t_1, s_1)$,
$A(a^2) = A(a^4) = (t_2, s_2)$,
where $(t_0, s_0) > (t_1, s_1) > (t_2, s_2)$ and $t_i + s_i \leq 1$ for $i = 0, 1, 2$. Then clearly $A \in IFG(G)$ and $O(a^3) \neq O(a)$. But $A(a) = A(a^3)$.

Now we give the characterization of all IFGs of a finite cyclic group in the following. In fact, the following result is the spacial case of Proposition 2.4.

Theorem 3.4. Let G be a finite cyclic group and let $A \in \mathrm{IFS}(G)$. Then A is an IFG of G if and only if there exists a maximal chain of subgroups $(e) = G_0 \subset G_1 \subset \cdots \subset G_r = G$ such that for any (t_0, s_0), $(t_1, s_1), \cdots, (t_r, s_r) \in I_m(A)$ with $t_0 > t_1 > \cdots > t_r$ and $s_0 < s_1 < \cdots < s_r$, $A(e) = (t_0, s_0)$, $A(\hat{G_1}) = (t_1, s_1), \cdots, A(\hat{G_r}) = (t_r, s_r)$, where $\hat{G_i} = G_i \setminus G_{i-1}$ for $i = 1, 2, \cdots, r$.

Proof. ($\Leftarrow$) : Suppose the necessary condition holds. We define a complex mapping $A = (\mu_A, \nu_A) : G \to I \times I$ by $A(e) = (t_0, s_0)$, $A(\hat{G_1}) = (t_1, s_1), \cdots, A(\hat{G_r}) = (t_r, s_r)$. Then clearly $A \in \mathrm{IFS}(G)$ from the definition of A. Let $x, y \in G$.

Case (i) : $x, y \in G_i$ but not in G_{i-1}. Then $A(x) = A(y) = (t_i, s_i)$ and $xy \in G_i$ or G_{i-1}. Thus $\mu_A(xy) \geq t_i = \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq s_i = \nu_A(x) \vee \nu_A(y)$. Moreover, $\mu_A(x^{-1}) \geq t_i = \mu_A(x)$ and $\nu_A(x^{-1}) \leq s_i = \nu_A(x)$.

Case (ii) : Suppose $x \in G_i$ but not in G_{i-1} and $y \in G_j$ but not G_{j-1}, where $i > j$. Then $A(x) = (t_i, s_i)$ and $A(y) = (t_j, s_j)$. Thus $\mu_A(xy) \geq t_i = \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq t_i = \nu_A(x) \vee \nu_A(y)$. Also $\mu_A(x^{-1}) \geq t_i = \mu_A(x)$ and $\nu_A(x^{-1}) \leq t_i = \nu_A(x)$. Hence, in all, $A \in \mathrm{IFG}(G)$.

($\Rightarrow$) : Suppose $A \in \mathrm{IFG}(G)$. Then, by Corollary 2.2, $A^{(t_0,s_0)}, \ldots, A^{(t_r,s_r)}$ are the only level subgroups of A, where $\{(t_0, s_0), (t_1, s_1), \cdots, (t_r, s_r)\} = I_m(A)$, $t_0 > t_1 > \cdots > t_r$ and $s_0 < s_1 < \cdots < s_r$. Furthermore, the level subgroups form a chain $C(A) : A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_r,s_r)}$. Thus clearly, $A^{(t_0,s_0)} = (e)$ and $A^{(t_r,s_r)} = G$.

Suppose $C(A)$ is maximal and let $G_i = A^{(t_i,s_i)}$. Then the necessary condition holds. Assume that $C(A)$ is not maximal. Then we redefine $C(A)$ by introducing subgroups of G. Let us call this chain as $G_0 \subset G_1 \subset \cdots \subset G_s$, where $G_0 = A^{(t_0,s_0)} = (e)$ and $G_s = A^{(t_r,s_r)} = G$. Then for each G_i between $A^{(t_0,s_0)}(= G_0)$ and $A^{(t_1,s_1)}(= G_j$ for some $j)$, $A(\hat{G_i}) = (t_1, s_1)$. Similarly, for each G_k between $A^{(t_i,s_i)}$ and $A^{(t_{i+1},s_{i+1})}$, $A(\hat{G_k}) = (t_{i+1}, s_{i+1})$ and $A(\hat{G_s}) = (t_r, s_r)$. Thus $A(G_0) = (t_0, s_0)$, $A(\hat{G_1}) = \cdots = A(\hat{G_j}) = (t_1, s_1)$, $A(\hat{G}_{j+1}) = \cdots = A(\hat{G_k}) = (t_r, s_r), \cdots, A(\hat{G_s}) = (t_r, s_r)$, where $\hat{G_1} = G_1 - G_0$, $\hat{G_2} = G_2 - G_1, \cdots, \hat{G_s} = G_s - G_{s-1}$, $t_0 > t_1 > \cdots > t_r$ and $s_0 < s_1 < \cdots < s_r$. This completes the proof. ■

The following is the immediate result of Theorem 3.4.

Corollary 3.4. Let G be a cyclic p-group of order p^r and let $A \in \mathrm{IFS}(G)$. Then $A \in \mathrm{IFG}(G)$ if and only if for each $x \in G$ with $O(x) = p^i$, $A(x) = (t_i, s_i)$, where

$i = 0, 1, \cdots, r$, $t_0 > t_1 > \cdots > t_r$ and $s_0 < s_1 < \cdots < s_r$.

Remark 3.5. We can also prove this Corollary by using Proposition 3.1.

References

[1] K.Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1986), 87-96.

[2] Baldev Banerjee and Dhiren Kr. Basnet, Intuitionistic fuzzy subrings and ideals, J.Fuzzy Mathematics 11(1)(2003), 139-155.

[3] P.Bhattacharya, Fuzzy subgroups : Some characterizations, J.Math. Anal. Appl.128(1987), 241-252.

[4] R.Biswas, Intuitionistic fuzzy subgroups, Mathematical Forum x(1989)37-46.

[5] D. Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88(1997), 81-89.

[6] D. Çoker and A.Haydar Es, On fuzzy compactness in intuitionistic fuzzy topological spaces, J. Fuzzy Math. 3(1995), 899-909.

[7] P.S.Das, Fuzzy groups and level subgroups, J. Math. Anal. Appl. 84(1981), 264-269.

[8] V.N.Dixit, R.Kumar and N.Ajmal, Level subgroups and union of fuzzy subgroups, Fuzzy sets and systems 37(1990), 359-371.

[9] V.N.Dxit, S.K.Bhambri and P.Kumar, Union of fuzzy subgroups, Fuzzy Sets and Systems 78(1996), 121-123.

[10] H.Gürçay, D. Çoker and A.Haydar Es, On fuzzy continuity in intuitionistic fuzzy topological spaces, J. Fuzzy Math. 5(1997), 365-378.

[11] K. Hur, H.W.Kang and H.K.Song, Intuitionistic fuzzy subgroups and subrings, Honam Mathematical J. 25 (1) (2003), 19-41.

[12] K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy subgroupoids, International Journal of Fuzzy Logic and Intelligent Systems 3 (1) (2003), 72-77.

[13] K.Hur, J.H.Kim and J.H.Ryou, Intuitionistic fuzzy topological spaces, J.Korea Soc. Math. Educ. Ser. B : Pure Appl. Math. 11 (2) (2004), 243-265.

[14] K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy subgroups and cosets, Honam Mathematical J. 26(1) (2004), 17-41.

[15] K.Hur, Y.B.Jun and J.H.Ryou, Intuitionistic fuzzy topological groups, Honam Math.J.26(2)(2004),163-192.

[16] K.Hur, S.Y.Jang and H.W.Kang, Intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets, Honam Math.J.26(4)(2004),559-587.

[17] S.J.Lee and E.P.Lee, The category of intuitionistic fuzzy topological spaces, Bull. Korean Math. Soc. 37(1)(2000), 63-76.

[18] Wang-jin Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems 8(1982), 133-139.

[19] A.Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35(1971), 512-517.

[20] L.A.Zadeh, Fuzzy sets, Inform. and Control 8(1965), 338-353.

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