

THE ASYMPTOTIC STABILITY BEHAVIOR IN A LOTKA-VOLTERRA TYPE PREDATOR-PREY SYSTEM

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ABSTRACT. In this paper, we provide a detailed and explicit procedure of obtaining some regions of attraction for the positive steady state (assumed to exist) of a well known Lotka-Volterra type predator-prey system. Also we obtain the sufficient conditions to ensure that the positive equilibrium point of a well known Lotka-Volterra type predator-prey system with a single discrete delay is globally asymptotically stable.

1. Introduction

We consider the following Lotka-Volterra type predator-prey system of differential equations

$$(1) \quad \begin{aligned} x' &= x(a - y - dz - \alpha x) \\ y' &= y(-b + x + ez - \alpha_1 y) \\ z' &= z(-c + d_1 x - e_1 y - \alpha_2 z). \end{aligned}$$

Here, a, b, c, d, d_1, e, e_1 are positive constants, and $\alpha_1 \geq 0, \alpha_2 \geq 0$, and the prime denotes derivative with respect to t , real variable. The form of (1) suggests that it can model the interaction of two predators whose population sizes are, respectively, $y(t)$ and $z(t)$ and prey whose population size is $x(t)$. Not only do both predators feed on the prey, but the y predator also predate on the z predator. In case $\alpha_1 > 0$, the birth rate of the y predator depends not only on the sizes of the x and z species, but negatively on its own size $y(t)$, and similarly if $\alpha_2 > 0$ for the z predator.

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Also we consider the following Lotka-Volterra type predator-prey system of a mathematical model representing a resource (prey) and two predators system with delay:

$$(2) \quad \begin{aligned} x' &= x(a - y - dz - \alpha x) \\ y' &= y(-b + x(t - \tau) - \alpha_1 y) \\ z' &= z(-c + dx(t - \tau) - \alpha_2 z), \end{aligned}$$

where τ is a positive constant.

In 1998, G. Seifert obtained some sufficient conditions to ensure that the positive equilibrium point of (1) is globally asymptotically stable using a Liapunov function. In this paper, we obtain the sufficient conditions to ensure that the positive equilibrium point of (1) is globally asymptotically stable using new Liapunov function which has some advantage in constructing the Liapunov functional for the delay differential system. Also we obtain the sufficient conditions to ensure that the positive equilibrium point of (2) is globally asymptotically stable using the Liapunov functional which is constructed in a straightforward manner from the Liapunov function which we construct to obtain the sufficient conditions to ensure that the positive equilibrium point of (1) is globally asymptotically stable.

2. The global asymptotic stability of the system of differential equations (1)

We show that under certain conditions, all positive solutions of (1) will approach a positive equilibrium point as $t \rightarrow \infty$; note that in the absence of predators, the first equation of (1) is a simple logistic equation.

We will only be concerned with positive solutions of (1), and use the notation

$$R_+^3 = \{(x, y, z) : x > 0, y > 0, z > 0\}.$$

It is well known that for any solutions $(x(t), y(t), z(t))$, if $(x(t_0), y(t_0), z(t_0)) \in R_+^3$ for some t_0 , then $(x(t), y(t), z(t)) \in R_+^3$ for all $t \geq t_0$. Since the system is autonomous, we usually take t_0 as the so-called initial time.

We first consider a special case of (1) where $d_1 = d, e_1 = e, \alpha_1 = \alpha_2 = 0$.

LEMMA 2.1. *If $d_1 = d, e_1 = e$, and $\alpha_1 = \alpha_2 = 0$, and if*

$$(2.1) \quad cae < d(ae + c - bd) < bdae$$

then (1) will have the unique equilibrium point $(x^*, y^*, z^*) \in R_+^3$ given by

$$\begin{aligned} x^* &= \frac{(ae + c - bd)}{\alpha e} \\ y^* &= \frac{\{d(ae + c - bd) - \alpha ce\}}{\alpha e^2} \\ z^* &= \frac{\{b\alpha e - (ae + c - bd)\}}{\alpha e^2}. \end{aligned}$$

Any solution of (1) satisfies $(x(t), y(t), z(t)) \rightarrow (x^*, y^*, z^*)$ as $t \rightarrow \infty$. Also, the linearization of (1) with respect to (x^*, y^*, z^*) has all eigenvalues with negative real parts.

Proof. If we put $u = \ln \frac{x}{x^*}$, $v = \ln \frac{y}{y^*}$, $w = \ln \frac{z}{z^*}$ into (1) we get

$$\begin{aligned} (2.2) \quad u' &= -\alpha x^*(e^u - 1) - y^*(e^v - 1) - dz^*(e^w - 1) \\ v' &= x^*(e^u - 1) + ez^*(e^w - 1) \\ w' &= dx^*(e^u - 1) - ey^*(e^v - 1). \end{aligned}$$

Define a Liapunov function

$$V(u, v, w) = \frac{x^*}{z^*} \int_0^{u(t)} (e^s - 1) ds + \frac{y^*}{z^*} \int_0^{v(t)} (e^s - 1) ds + \int_0^{w(t)} (e^s - 1) ds.$$

Then we have

$$\begin{aligned} V'_{(2.2)}(u, v, w) &= \frac{x^*}{z^*}(e^u - 1)u'(t) + \frac{y^*}{z^*}(e^v - 1)v'(t) + (e^w - 1)w'(t) \\ &= \frac{x^*}{z^*}(e^u - 1)\{-\alpha x^*(e^u - 1) - y^*(e^v - 1) - dz^*(e^w - 1)\} \\ &\quad + \frac{y^*}{z^*}(e^v - 1)\{x^*(e^u - 1) + ez^*(e^w - 1)\} \\ &\quad + (e^w - 1)\{dx^*(e^u - 1) - ey^*(e^v - 1)\} \\ &= -\frac{\alpha(x^*)^2}{z^*}(e^u - 1)^2. \end{aligned}$$

From a result in the theory of stability (cf. Theorem 1.3, p. 296 in [3]) it follows that $(u(t), v(t), w(t)) \rightarrow M$ as $t \rightarrow \infty$ where M is the largest invariant subset with respect to (2.2) of S ,

$$S = \{(u, v, w) \mid V'_{(2.2)}(u, v, w) = 0\}.$$

If $(u(t), v(t), w(t))$ is a solution of (2.2) and $(u(t), v(t), w(t)) \in M$, then

$$\begin{aligned} (2.3) \quad -y^*(e^v - 1) - dz^*(e^w - 1) &= 0, \\ v' = ez^*(e^w - 1), \end{aligned}$$

and

$$(2.4) \quad w' = -ey^*(e^v - 1).$$

Now we note that

$$(e^w - 1) = -\frac{y^*}{dz^*}(e^v - 1)$$

and

$$(2.5) \quad v' = (ez^*)\left(-\frac{y^*}{dz^*}\right)(e^v - 1) = -\left(\frac{ey^*}{d}\right)(e^v - 1).$$

From (2.4) and (2.5) we have

$$(2.6) \quad w' = dv'.$$

From (2.3) and (2.6) we have

$$-y^*e^v v' - (dz^*)e^w dv' = 0$$

and

$$(y^*e^v + d^2z^*e^w)v' = 0.$$

Thus $v' = 0$, that is, $v' = ez^*(e^w - 1) = 0$ and $w = 0$. By the similar argument we have $w' = 0$ and $v = 0$. Hence $M = \{(0, 0, 0)\}$ and the first part of the lemma follows.

On the other hand, we try to show that the equilibrium (x^*, y^*, z^*) of (1) is locally asymptotically stable.

By linearization of (1) at (x^*, y^*, z^*) , we obtain the system of differential equations

$$(2.7) \quad \begin{aligned} u' &= x(-\alpha u - v - dw), \\ v' &= y(u + 0 \cdot v + ew), \\ w' &= z(du - ey + 0 \cdot w), \end{aligned}$$

where $u = x - x^*$, $v = y - y^*$, $w = z - z^*$, $d_1 = d$, $e_1 = e$, and $\alpha_1 = \alpha_2 = 0$. Thus the Jacobian matrix J of (2,7) at $(0,0)$ is given by

$$J = \begin{pmatrix} -\alpha & -1 & -d \\ 1 & 0 & e \\ d & -e & 0 \end{pmatrix}.$$

Consider $\det(\lambda I - J) = 0$. Then we have

$$\lambda^3 + \alpha\lambda^2 + (d^2 + e^2 + 1)\lambda + \alpha e^2 = 0.$$

Since $\alpha > 0$, $\alpha e^2 > 0$, and $\alpha(d^2 + e^2 + 1) - \alpha e^2 = \alpha d^2 + \alpha > 0$, all eigen values of $\det(\lambda I - J) = 0$ have negative real parts (by the Routh-Hurwitz criterion). Hence, the equilibrium (x^*, y^*, z^*) of (1) is locally asymptotically stable. \square

THEOREM 2.2. *Let (2.1) hold. Then there exists $\delta > 0$ such that for $0 < \alpha_1 < \delta, 0 < \alpha_2 < \delta, |e - e_1| < \delta$, (1) has a unique equilibrium point $(x^*, y^*, z^*) \in R_+^3$ and for such α_1 and α_2 , there exists $\delta_1 > 0, 0 < \delta_1 \leq \delta$ such that for $|d - d_1| < \delta_1, |e - e_1| < \delta_1$ each solution $(x(t), y(t), z(t)) \rightarrow (x^*, y^*, z^*)$ as $t \rightarrow \infty$.*

Proof. The equilibrium point (x^*, y^*, z^*) of (1) now depends on $(\alpha_1, \alpha_2, d_1, e_1)$ and for $(\alpha_1, \alpha_2, d_1, e_1) = (0, 0, d, e)$ reduces to the (x^*, y^*, z^*) as given as in Lemma 2.1, where

$$\begin{aligned} x^* &= \frac{a\alpha_1\alpha_2 + \alpha_1cd + \alpha_2b + ce - bde_1 + aee_1}{\alpha\alpha_1\alpha_2 + d_1e - de_1 + dd_1\alpha_1 + \alpha_2 + \alpha ee_1} \\ y^* &= \frac{\alpha_2(a - \alpha b) + cd + ad_1 - bdd_1 - \alpha ce}{\alpha\alpha_1\alpha_2 + d_1e - de_1 + dd_1\alpha_1 + \alpha_2 + \alpha ee_1} \\ z^* &= \frac{\alpha_1(ad_1 - \alpha c) - ae_1 + bd_1 - c + abe_1}{\alpha\alpha_1\alpha_2 + d_1e - de_1 + dd_1\alpha_1 + \alpha_2 + \alpha ee_1}. \end{aligned}$$

We express (1) in terms of new variables as before:

$$u = \ln \frac{x}{x^*}, \quad v = \ln \frac{y}{y^*}, \quad w = \ln \frac{z}{z^*}.$$

We obtain

$$\begin{aligned} (2.8) \quad u' &= -\alpha x^*(e^u - 1) - y^*(e^v - 1) - dz^*(e^w - 1) \\ v' &= x^*(e^u - 1) - \alpha_1 y^*(e^v - 1) + ez^*(e^w - 1) \\ w' &= d_1 x^*(e^u - 1) - e_1 y^*(e^v - 1) - \alpha_2 z^*(e^w - 1). \end{aligned}$$

Define a Liapunov function

$$\begin{aligned} V(u, v, w) &= x^* \int_0^{u(t)} (e^s - 1) ds + y^* \int_0^{v(t)} (e^s - 1) ds \\ &\quad + z^* \int_0^{w(t)} (e^s - 1) ds. \end{aligned}$$

Then we have

$$\begin{aligned} &V'_{(2.7)}(u, v, w) \\ &= x^*(e^u - 1)u' + y^*(e^v - 1)v' + z^*(e^w - 1)w' \\ &= x^*(e^v - 1)\{-\alpha x^*(e^u - 1) - y^*(e^v - 1) - dz^*(e^w - 1)\} \\ &\quad + y^*(e^v - 1)\{x^*(e^u - 1) - \alpha_1 y^*(e^v - 1) + ez^*(e^w - 1)\} \\ &\quad + z^*(e^w - 1)\{d_1 x^*(e^u - 1) - e_1 y^*(e^v - 1) - \alpha_2 z^*(e^w - 1)\} \\ &= -\alpha(x^*)^2(e^u - 1)^2 - x^*y^*(e^u - 1)(e^v - 1) \\ &\quad - dx^*z^*(e^u - 1)(e^w - 1) + x^*y^*(e^u - 1)(e^v - 1) \\ &\quad - \alpha_1(y^*)^2(e^v - 1)^2 + ey^*z^*(e^v - 1)(e^w - 1) - \alpha_2(z^*)^2(e^w - 1)^2 \\ &\quad + d_1x^*z^*(e^u - 1)(e^w - 1) - e_1y^*z^*(e^v - 1)(e^w - 1) \end{aligned}$$

$$\begin{aligned}
 &= -\alpha(x^*)^2(e^u - 1)^2 - \alpha_1(y^*)^2(e^v - 1)^2 - \alpha_2(z^*)^2(e^w - 1)^2 \\
 &\quad - (d - d_1)x^*z^*(e^u - 1)(e^w - 1) + (e - e_1)y^*z^*(e^v - 1)(e^w - 1) \\
 &\leq -\alpha(x^*)^2(e^u - 1)^2 - \alpha_1(y^*)^2(e^v - 1)^2 - \alpha_2(z^*)^2(e^w - 1)^2 \\
 &\quad + \frac{|d_1 - d|}{2}(x^*)^2(e^u - 1)^2 + \frac{|d_1 - d|}{2}(z^*)^2(e^w - 1)^2 \\
 &\quad + \frac{|e - e_1|}{2}(y^*)^2(e^v - 1)^2 + \frac{|e - e_1|}{2}(z^*)^2(e^w - 1)^2 \\
 &= -(x^*)^2 \left(\alpha - \frac{|d_1 - d|}{2} \right) (e^u - 1)^2 - (y^*)^2 \left(\alpha_1 - \frac{|e - e_1|}{2} \right) (e^v - 1)^2 \\
 &\quad - (z^*)^2 \left(\alpha_2 - \frac{|d_1 - d|}{2} - \frac{|e - e_1|}{2} \right) (e^w - 1)^2.
 \end{aligned}$$

For α_1 and α_2 as above there is clearly a $\delta_1 > 0$ sufficiently small such that the right side of the above inequality is negative definite for $|d_1 - d| < \delta_1$, $|e - e_1| < \delta_1$. Using a well argument, it follows that $(u(t), v(t), w(t)) \rightarrow (0, 0, 0)$ as $t \rightarrow \infty$. This implies that

$$(x(t), y(t), z(t)) \rightarrow (x^*, y^*, z^*) \text{ as } t \rightarrow \infty.$$

□

3. The global asymptotic stability of the system of differential equations (2)

In this section, we determine sufficient conditions for global asymptotic stability of interior equilibrium of a delayed ecological model (2) involving a resource and two predators.

$$\begin{aligned}
 (2) \quad &x' = x(a - y - dz - \alpha x) \\
 &y' = y(-b + x(t - \tau) - \alpha_1 y) \\
 &z' = z(-c + dx(t - \tau) - \alpha_2 z)
 \end{aligned}$$

with initial conditions

$$x(s) = \phi(s) \geq 0, \quad y(0) \geq 0, \quad z(0) \geq 0,$$

where $\tau > 0$ is a constant time delay and $\phi(s)$ is a continuous function from $[-\tau, 0] \rightarrow R$.

In system (2), all parameters are positive constants, $x(t)$ denotes the biomass at time t of resource species and $y(t)$ and $z(t)$ represent the densities of two-predators, respectively.

It is widely known that past history as well as current conditions can influence population dynamics and such interactions have motivated the introduction of delays in population growth models. Sometimes delay can change the dynamics. In delay models with complicated dynamics, the question of global asymptotic stability is very important. Modelling of the ecological interactions involving time delay for population has

been dealt recently by Kuang [5]. More recently, Cao and Freedman [1] considered a general class of models of prey-predator interactions with time delay due to gestation.

In general, the construction of suitable Liapunov functional related the system of delayed differential equations is practically difficult and tedious job. But we construct the Liapunov functional related to (2) in a straightforward manner from the Liapunov function which we construct to obtain the sufficient conditions to ensure that the positive equilibrium point of (1) is globally asymptotically stable.

LEMMA 3.1. *If $a > \alpha b$ and $ad > \alpha c$, then all solutions of (2) are bounded.*

Proof. We can see from the form of the system (2) that its solutions with nonnegative initial conditions are positive, and hence,

$$x'(t) = x(a - y - dz - \alpha x) \leq x(a - \alpha x).$$

It is an elementary fact from ordinary differential equations theory that there exists $T_1 > 0$ such that

$$x(t) \leq \frac{a}{\alpha} \text{ if } t \geq T_1 + 2\tau.$$

Also we have

$$\begin{aligned} y' &= y(-b + x(t - \tau) - \alpha_1 y) \\ &\leq y\left(-b + \frac{a}{\alpha} - \alpha_1 y\right) = y\left\{\left(\frac{a - b\alpha}{\alpha}\right) - \alpha_1 y\right\}. \end{aligned}$$

That is, there exists T_2 such that $T_2 \geq T_1 + \tau$ and $y(t) \leq \left(\frac{a - b\alpha}{\alpha_1 \alpha}\right)$ if $t \geq T_2$.

From (2)

$$\begin{aligned} z'(t) &= z(-c + dx(t - \tau) - \alpha_2 z) \\ &\leq z\left(-c + \frac{ad}{\alpha} - \alpha_2 z\right) = z\left(\frac{ad - \alpha c}{\alpha} - \alpha_2 z\right). \end{aligned}$$

That is, there exists T_3 such that $T_3 \geq T_1 + \tau$ and $z(t) \leq \frac{ad - \alpha c}{\alpha \alpha_2}$ if $t \geq T_3$. \square

LEMMA 3.2. *Let f be a nonnegative function defined on $[0, \infty)$ such that f is integrable on $[0, \infty)$ and is uniformly continuous on $[0, \infty)$. Then*

$$\lim_{t \rightarrow \infty} f(t) = 0.$$

Proof. Suppose f does not approach zero as $t \rightarrow \infty$. Then there is an $\epsilon_0 > 0$ such that for any $M > 0$ there exists T_M such that $T_M \geq M$ and $f(T_M) > \epsilon_0$. This implies that there is an increasing sequence $\{t_n\}$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and $f(t_n) > \epsilon_0$. Suppose that f is uniformly continuous on $[0, \infty)$. Then there is a $\delta > 0$ which depends on $\epsilon_0 > 0$ such that $f(t) > \frac{\epsilon_0}{2}$ for any $t \in (t_n - \delta, t_n + \delta)$ and any n . Now we may assume that the intervals $(t_n - \delta, t_n + \delta)$ does not overlap. Therefore

$$\int_0^\infty f(t)dt \geq \sum_{n=1}^N \int_{t_n-\delta}^{t_n+\delta} f(t)dt \geq N\epsilon_0\delta$$

for any positive integer N and this contradicts the integrability of f on $[0, \infty)$. Hence the lemma follows. \square

THEOREM 3.3. *Suppose that $a > \alpha b$ and $ad > \alpha c$ and there is a unique positive equilibrium point (x^*, y^*, z^*) of (2), where*

$$x^* = \frac{\alpha\alpha_1\alpha_2 + \alpha_1cd + \alpha_2b}{\alpha\alpha_1\alpha_2 + d^2\alpha_1 + \alpha_2}, \quad y^* = \frac{\alpha_2(a - \alpha b) + d(c - bd)}{\alpha\alpha_1\alpha_2 + d^2\alpha_1 + \alpha_2},$$

$$z^* = \frac{\alpha_2(ad - \alpha c) + bd - c}{\alpha\alpha_1\alpha_2 + d^2\alpha_1 + \alpha_2}.$$

Moreover, if

$$\left(M_y + d^2M_z + \frac{\alpha_1M_yy^*}{2} + \frac{\alpha\alpha_2M_z}{2} \right) \tau < \alpha,$$

$$\frac{M_yx^*\tau}{2} < 1, \quad \text{and} \quad \frac{dM_zx^*\tau}{2} < 1,$$

then the unique positive equilibrium point (x^*, y^*, z^*) of (2) is globally asymptotically stable, where

$$M_x = \frac{a}{\alpha}, \quad M_y = \frac{a - \alpha b}{\alpha\alpha_1}, \quad M_z = \frac{ad - \alpha c}{\alpha\alpha_2}.$$

Proof. If we put $u = \ln \frac{x}{x^*}$, $v = \ln \frac{y}{y^*}$, $w = \ln \frac{z}{z^*}$ into (2), we get

$$(3.1) \quad \begin{aligned} u'(t) &= -\alpha x^*(e^{u(t)} - 1) - y^*(e^{v(t)} - 1) - dz^*(e^{w(t)} - 1) \\ v'(t) &= x^*(e^{u(t-\tau)} - 1) - \alpha_1 y^*(e^{v(t)} - 1) \\ w'(t) &= dx^*(e^{u(t-\tau)} - 1) - \alpha_2 z^*(e^{w(t)} - 1). \end{aligned}$$

Define a Liapunov functional

$$\begin{aligned}
 V(u, v, w) &= x^* \int_0^{u(t)} (e^s - 1) ds + y^* \int_0^{v(t+\tau)} (e^s - 1) ds \\
 &\quad + z^* \int_0^{w(t+\tau)} (e^s - 1) ds.
 \end{aligned}$$

Then we have

$$\begin{aligned}
 (3.2) \quad &V'_{(3.1)}(u, v, w) \\
 &= x^*(e^{u(t)} - 1)u'(t) + y^*(e^{v(t+\tau)} - 1)v'(t + \tau) \\
 &\quad + z^*(e^{w(t+\tau)} - 1)w'(t + \tau) \\
 &= x^*(e^{u(t)} - 1)\{-\alpha x^*(e^{u(t)} - 1) - y^*(e^{v(t)} - 1) - dz^*(e^{w(t)} - 1)\} \\
 &\quad + y^*(e^{v(t+\tau)} - 1)\{x^*(e^{u(t)} - 1) - \alpha_1 y^*(e^{v(t+\tau)} - 1)\} \\
 &\quad + z^*(e^{w(t+\tau)} - 1)\{dx^*(e^{u(t)} - 1) - \alpha_2 z^*(e^{w(t+\tau)} - 1)\} \\
 &= -\alpha(x^*)^2(e^{u(t)} - 1)^2 - x^*y^*(e^{u(t)} - 1)(e^{v(t)} - 1) \\
 &\quad - dx^*z^*(e^{u(t)} - 1)(e^{w(t)} - 1) + x^*y^*(e^{u(t)} - 1)(e^{v(t+\tau)} - 1) \\
 &\quad - \alpha_1(y^*)^2(e^{v(t+\tau)} - 1)^2 + dx^*z^*(e^{u(t)} - 1)(e^{w(t+\tau)} - 1) \\
 &\quad - \alpha_2(z^*)^2(e^{w(t+\tau)} - 1)^2 \\
 &= -\alpha(x^*)^2(e^{u(t)} - 1)^2 - \alpha_1(y^*)^2(e^{v(t+\tau)} - 1)^2 \\
 &\quad - \alpha_2(z^*)^2(e^{w(t+\tau)} - 1)^2 + x^*y^*(e^{u(t)} - 1)\{e^{v(t+\tau)} - e^{v(t)}\} \\
 &\quad + dx^*y^*(e^{u(t)} - 1)\{e^{w(t+\tau)} - e^{w(t)}\} \\
 &= -\alpha(x^*)^2(e^{u(t)} - 1)^2 - \alpha_1(y^*)^2(e^{v(t+\tau)} - 1)^2 \\
 &\quad - \alpha_2(z^*)^2(e^{w(t+\tau)} - 1)^2 + x^*y^*(e^{u(t)} - 1) \int_t^{t+\tau} de^{v(s)} \\
 &\quad + dx^*z^*(e^{u(t)} - 1) \int_t^{t+\tau} de^{w(s)} \\
 &= -\alpha(x^*)^2(e^{u(t)} - 1)^2 - \alpha_1(y^*)^2(e^{v(t+\tau)} - 1)^2 \\
 &\quad - \alpha_2(z^*)^2(e^{w(t+\tau)} - 1)^2 + x^*y^*(e^{u(t)} - 1) \int_t^{t+\tau} e^{v(s)}v'(s)ds \\
 &\quad + dx^*z^*(e^{u(t)} - 1) \int_t^{t+\tau} e^{w(s)}w'(s)ds \\
 &= -\alpha(x^*)^2(e^{u(t)} - 1)^2 - \alpha_1(y^*)^2(e^{v(t+\tau)} - 1)^2 \\
 &\quad - \alpha_2(z^*)^2(e^{w(t+\tau)} - 1)^2 \\
 &\quad + x^*y^*(e^{u(t)} - 1) \int_t^{t+\tau} \frac{y(s)}{y^*} \{x^*(e^{u(s-\tau)} - 1) - \alpha_1 y^*(e^{w(s)} - 1)\} ds \\
 &\quad + dx^*z^*(e^{u(t)} - 1) \int_t^{t+\tau} \frac{z(s)}{z^*} \{dx^*(e^{u(s-\tau)} - 1) - \alpha_2 z^*(e^{w(s)} - 1)\} ds.
 \end{aligned}$$

By lemma 3.1

$$0 \leq x(t) \leq \frac{a}{\alpha}, \quad 0 \leq y(t) \leq \frac{a - \alpha b}{\alpha \alpha_1}, \quad 0 \leq z(t) \leq \frac{ad - \alpha c}{\alpha \alpha_2}$$

eventually for all large t . That is, there exists $T^* > 0$ such that if $t \geq T^*$, then

$$x(t) \leq M_x, \quad y(t) \leq M_y, \quad \text{and} \quad z(t) \leq M_z.$$

From (3.2) if $t \geq T^* + \tau$, then we have

(3.3)

$$\begin{aligned} & V'_{(3.1)}(u, v, w) \\ \leq & -\alpha(x^*)^2(e^{u(t)} - 1)^2 - \alpha_1(y^*)^2(e^{v(t+\tau)} - 1)^2 \\ & -\alpha_2(z^*)^2(e^{w(t+\tau)} - 1)^2 \\ & + (x^*)^2 M_y |e^{u(t)} - 1| \int_t^{t+\tau} |e^{u(s-\tau)} - 1| ds \\ & + \alpha_1 x^* y^* M_y |e^{u(t)} - 1| \int_t^{t+\tau} |e^{v(s)} - 1| ds \\ & + d^2 (x^*)^2 M_z |e^{u(t)} - 1| \int_t^{t+\tau} |e^{u(s-\tau)} - 1| ds \\ & + d\alpha_2 x^* z^* M_z |e^{u(t)} - 1| \int_t^{t+\tau} |e^{w(s)} - 1| ds \\ \leq & -\alpha(x^*)^2(e^{u(t)} - 1)^2 - \alpha_1(y^*)^2(e^{v(t+\tau)} - 1)^2 \\ & -\alpha_2(z^*)^2(e^{w(t+\tau)} - 1)^2 + \frac{(x^*)^2 M_y}{2} (e^{u(t)} - 1)^2 \tau \\ & + \frac{(x^*)^2 M_y}{2} \int_{t-\tau}^t (e^{u(s)} - 1)^2 ds + \frac{\alpha_1 M_y y^* (x^*)^2}{2} (e^{u(t)} - 1)^2 \tau \\ & + \frac{\alpha_1 M_y x^* (y^*)^2}{2} \int_{t-\tau}^t (e^{v(s+\tau)} - 1)^2 ds + \frac{d^2 M_z (x^*)^2}{2} (e^{u(t)} - 1)^2 \tau \\ & + \frac{d^2 M_z (x^*)^2}{2} \int_{t-\tau}^t (e^{u(s)} - 1)^2 ds + \frac{d\alpha_2 M_z (x^*)^2}{2} (e^{u(t)} - 1)^2 \tau \\ & + \frac{d\alpha_2 M_z x^* (z^*)^2}{2} \int_{t-\tau}^t (e^{w(s+\tau)} - 1)^2 ds. \end{aligned}$$

Define a Liapunov functional $U(u, v, w)$ as following:

$$\begin{aligned} U(u, v, w) = & V(u, v, w) + \frac{M_y (x^*)^2}{2} \int_{t-\tau}^t \left(\int_p^t (e^{u(s)} - 1)^2 ds \right) dp \\ & + \frac{\alpha_1 M_y x^* (y^*)^2}{2} \int_{t-\tau}^t \left(\int_p^t (e^{v(s+\tau)} - 1)^2 ds \right) dp \\ & + \frac{d^2 (x^*)^2 M_z}{2} \int_{t-\tau}^t \left(\int_p^t (e^{u(s)} - 1)^2 ds \right) dp \end{aligned}$$

$$+ \frac{d\alpha_2 M_z x^*(z^*)^2}{2} \int_{t-\tau}^t \left(\int_p^t (e^{w(s+\tau)} - 1)^2 ds \right) dp.$$

if $t \geq T^* + \tau$. Then, from (3.3), we have

$$\begin{aligned} & U'_{(3.1)}(u, v, w) \\ \leq & -\alpha(x^*)^2(e^{u(t)} - 1)^2 - \alpha_1(y^*)^2(e^{v(t+\tau)} - 1)^2 \\ & -\alpha_2(z^*)^2(e^{w(t+\tau)} - 1)^2 + \frac{M_y(x^*)^2}{2}(e^{u(t)} - 1)^2\tau \\ & + \frac{M_y(x^*)^2}{2} \int_{t-\tau}^t (e^{u(s)} - 1)^2 ds \\ & + \frac{\alpha_1 M_y y^*(x^*)^2}{2} (e^{u(t)} - 1)^2\tau \\ & + \frac{\alpha_1 M_y x^*(y^*)^2}{2} \int_{t-\tau}^t (e^{v(s+\tau)} - 1)^2 ds + \frac{d^2 M_z (x^*)^2}{2} (e^{u(t)} - 1)^2\tau \\ & + \frac{d^2 M_z (x^*)^2}{2} \int_{t-\tau}^t (e^{u(s)} - 1)^2 ds + \frac{d\alpha_2 M_z (x^*)^2}{2} (e^{u(t)} - 1)^2\tau \\ & + \frac{d\alpha_2 M_z x^*(z^*)^2}{2} \int_{t-\tau}^t (e^{w(s+\tau)} - 1)^2 ds + \frac{M_y(x^*)^2}{2} (e^{u(t)} - 1)^2\tau \\ & - \frac{M_y(x^*)^2}{2} \int_{t-\tau}^t (e^{u(s)} - 1)^2 ds + \frac{\alpha_1 M_y x^*(y^*)^2}{2} (e^{v(t+\tau)} - 1)^2\tau \\ & - \frac{\alpha_1 M_y x^*(y^*)^2}{2} \int_{t-\tau}^t (e^{v(s+\tau)} - 1)^2 ds + \frac{d^2 M_z (x^*)^2}{2} (e^{u(t)} - 1)^2\tau \\ & - \frac{d^2 M_z (x^*)^2}{2} \int_{t-\tau}^t (e^{u(s)} - 1)^2 ds + \frac{d\alpha_2 M_z x^*(z^*)^2}{2} (e^{w(t+\tau)} - 1)^2\tau \\ & - \frac{d\alpha_2 M_z x^*(z^*)^2}{2} \int_{t-\tau}^t (e^{w(s+\tau)} - 1)^2 ds \end{aligned}$$

if $t \geq T^* + \tau$. That is,

$$\begin{aligned} U'_{(3.1)}(u, v, w) \leq & -(x^*)^2 A(e^{u(t)} - 1)^2 - (y^*)^2 B(e^{v(t+\tau)} - 1)^2 \\ & - (z^*)^2 C(e^{w(t+\tau)} - 1)^2 \end{aligned}$$

if $t \geq T^* + \tau$, where

$$A = \left\{ \alpha - \left(M_y + d^2 M_z + \frac{\alpha_1 M_y y^*}{2} + \frac{d\alpha_2 M_z}{2} \right) \tau \right\},$$

$$B = \left\{ \alpha_1 - \frac{\alpha_1 M_y x^* \tau}{2} \right\} \text{ and } C = \left\{ \alpha_2 - \frac{d\alpha_2 M_z x^* \tau}{2} \right\}.$$

Thus

$$\begin{aligned}
 & U(u(t), v(t), w(t)) + (x^*)^2 A \int_{T^* + \tau}^t (e^{u(s)} - 1)^2 ds \\
 & + (y^*)^2 B \int_{T^* + \tau}^t (e^{v(s+\tau)} - 1)^2 ds \\
 & + (z^*)^2 C \int_{T^* + \tau}^t (e^{w(s+\tau)} - 1)^2 ds \\
 = & U(u(t), v(t), w(t)) \\
 & + A \int_{T^* + \tau}^t (x(s) - x^*)^2 ds \\
 & + B \int_{T^* + \tau}^t (y(s + \tau) - y^*)^2 ds \\
 & + C \int_{T^* + \tau}^t (z(s + \tau) - z^*)^2 ds \\
 \leq & U(u(T^* + \tau), v(T^* + \tau), z(T^* + \tau)).
 \end{aligned}$$

Also from lemma 3.1 we note that $x(t), y(t)$, and $z(t)$ are uniformly continuous on $[T^*, \infty)$. By lemma 3.2 $(x(t), y(t), z(t)) \rightarrow (x^*, y^*, z^*)$ as $t \rightarrow \infty$. Hence the proof is complete. \square

REMARK 3.4. It is not difficult to find the suitable constants $a, b, c, d, \alpha, \alpha_1, \alpha_2, \tau$ which satisfy the conditions in Theorem 3.3 if τ is small enough. For example, we can choose the constants such that $a = b = \frac{1}{2}$, $c = \frac{1}{3}$, $d = \frac{2}{3}$, $\alpha = \alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{4}{5}$, $0 \leq \tau \leq \frac{1}{3}$. Also we can choose the constants such that $a = 2$, $b = \frac{1}{2}$, $c = \frac{1}{6}$, $d = \frac{1}{3}$, $\alpha = 1$, $\alpha_1 = 2$, $\alpha_2 = \frac{3}{2}$, $0 \leq \tau \leq \frac{5}{7}$

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