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ON THE WEAK LAW OF LARGE NUMBERS FOR SEQUENCES OF BANACH SPACE VALUED RANDOM ELEMENTS

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Abstract. We establish a weak law of large numbers for sequence of random elements with values in p-uniformly smooth Banach space. Our result is more general and stronger than some well-known ones.

1. Introduction and notations

Recently, the weak law of large numbers (w.l.l.n.) in Banach space has been studied by many authors (see [1], [3], [6]). The aim of this paper is to establish a weak law of large numbers for sequence of random elements in p-uniformly smooth Banach space. Our result is more general and stronger than some well-known ones in [2] (for details see below).

Let us begin with some definitions and notations. A real separable Banach space $\mathcal{X}$ is said to be p-uniformly smooth $(1 \leqslant p \leqslant 2)$ if

$$\rho_*(\tau) = \sup\{\frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} - 1; \|x\| = 1; \|y\| = \tau\} \leqslant C\tau^p$$

for some constant C.

Theorem 1.1. (see [7]) (Assouad, Hoffmann Jϕrgensen) *A real Banach space $\mathcal{X}$ is p-uniformly smooth if and only if there exists a positive K such that for all $x, y \in \mathcal{X}$ we have*

$$\|x+y\|^p + \|x-y\|^p \leqslant 2\|x\|^p + K\|y\|^p.$$

Theorem 1.2. (see [7]) (Assouad) *A real separable Banach space $\mathcal{X}$ is p-uniformly smooth $(1 \leqslant p \leqslant 2)$ if and only if all $q \geqslant 1$, there*

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exists a positive constant C such that for all $\mathcal{X}$ - valued martingale $\{M_n, \mathcal{F}_n, n \geqslant 1\}$ we have

$$E\|M_n\|^q \leqslant CE(\sum_{i=1}^{n} \|dM_i\|^p)^{q/p} \quad (\textit{with } \ dM_i = M_i - M_{i-1}).$$

(Marcinkiewicz-Zygmund inequality)

In [1], Adler, Rosalsky, and Volodin have taken notion about martingale type p Banach spaces: A real separable Banach space $\mathcal{X}$ is said to be martingale type p $(1 \leqslant p \leqslant 2)$ if there exists a finite constant C such that for all martingale $\{S_n, n \geqslant 1\}$ with values in $\mathcal{X}$ then

$$\sup_{n\geqslant 1} E\|S_n\|^p \leqslant C\sum_{i=1}^{\infty} E\|S_n - S_{n-1}\|^p.$$

By Marcinkiewicz-Zygmund inequality we derive that a p-uniformly smooth Banach space is a martingale type p Banach space.

In this paper we assume that $\mathcal{X}$ is a p-uniformly smooth Banach space $(1 \leqslant p \leqslant 2)$, $\{X_n, n \geqslant 1\}$ is a sequence of random elements with values in $\mathcal{X}$, $(\mathcal{F}_n)$ is a sequence of σ - algebras such that X_n - $\mathcal{F}_n$ measurable for all $n = 1, 2, \ldots$.

2. Results

The main aim of this paper is to prove the following result.

THEOREM 2.1. *Let $(S_n = \sum_{i=1}^{n} X_i)$ be a sequence of random elements with values in $\mathcal{X}$ and (b_n) a sequence of positive numbers with $b_n \uparrow \infty$ as $n \longrightarrow \infty$. Then writing $X_{ni} = X_i I_{(\|X_i\| \leqslant b_n)}$, $1 \leqslant i \leqslant n$, we have that*

(2.1) $$b_n^{-1} S_n \xrightarrow{P} 0 \quad \textit{as } n \longrightarrow \infty, \qquad \textit{if}$$

(2.2) $$\sum_{i=1}^{n} P(\|X_i\| > b_n) \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty,$$

(2.3) $$b_n^{-1} \sum_{i=1}^{n} E(X_{ni}/\mathcal{F}_{i-1}) \xrightarrow{P} 0 \quad \textit{as} \quad n \longrightarrow \infty,$$

(2.4) $$b_n^{-p} \sum_{i=1}^{n} E\|X_{ni} - E(X_{ni}/\mathcal{F}_{i-1})\|^p \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty.$$

Proof. Let $S_{nn} = \sum_{i=1}^{n} X_{ni}$. Then we have

$$P\left(\frac{S_{nn}}{b_n} \neq \frac{S_n}{b_n}\right) \leqslant P\left(\bigcup_{i=1}^{n}\{X_{ni} \neq X_i\}\right) \leqslant \sum_{i=1}^{n} P(X_{ni} \neq X_i) = \sum_{i=1}^{n} P(\|X_i\| > b_n) \longrightarrow 0, \quad n \longrightarrow \infty \quad \text{(by (2.2))},$$

and it suffices to show that $b_n^{-1} S_{nn} \xrightarrow{P} 0$ as $n \longrightarrow \infty$. By (2.3), we have

$$b_n^{-1} \sum_{i=1}^{n} E(X_{ni}/\mathcal{F}_{i-1}) \xrightarrow{P} 0 \quad \text{as} \quad n \longrightarrow \infty.$$

So that it suffices to prove that

$$b_n^{-1} \sum_{i=1}^{n} [X_{ni} - E(X_{ni}/\mathcal{F}_{i-1})] \xrightarrow{P} 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Let $Y_{nk} = \sum_{i=1}^{n} [X_{ni} - E(X_{ni}/\mathcal{F}_{i-1})]$, $\quad 1 \leqslant k \leqslant n, \quad n = 1, 2, \ldots$.

By noting that $(Y_{nk},\ \mathcal{F}_k\ ;\ 1 \leqslant k \leqslant n)$ is a martingale on $\mathcal{X}$, we have, for arbitrary $\varepsilon > 0$

$$\begin{aligned}
&P\left(\|b_n^{-1} \sum_{i=1}^{n} [X_{ni} - E(X_{ni}/\mathcal{F}_{i-1})]\| > \varepsilon\right) \\
&\leqslant \varepsilon^{-p} b_n^{-p} E\| \sum_{i=1}^{n} [X_{ni} - E(X_{ni}/\mathcal{F}_{i-1})]\|^p \qquad \text{(Markov's inequality)} \\
&= \varepsilon^{-p} b_n^{-p} E\|Y_{nn}\|^p \\
&\leqslant \varepsilon^{-p} b_n^{-p} C E\left(\sum_{i=1}^{n} \|Y_{ni} - Y_{n(i-1)}\|^p\right) \text{(Marcinkiewicz-Zygmund inequality)} \\
&= C\varepsilon^{-p} b_n^{-p} \sum_{i=1}^{n} E\|X_{ni} - E(X_{ni}/\mathcal{F}_{i-1})\|^p \to 0 \text{ as } n \to \infty \quad \text{(by (2.4))},
\end{aligned}$$

which completes the proof. □

In the case, when $\mathcal{X} = \mathbb{R}$ then $p = 2$ and $E|X_{ni} - E(X_{ni}/\mathcal{F}_{i-1})|^2 = EX_{ni}^2 - E[E(X_{ni}/\mathcal{F})]^2$. The following corollary follows immediately from theorem 2.1.

COROLLARY 2.2. *Let* $(S_n = \sum_{i=1}^n X_i)$ *be a sequence of random variables and* (b_n) *a sequence of positive numbers with* $b_n \uparrow \infty$ *as* $n \longrightarrow \infty$. *Then writing* $X_{ni} = X_i I_{(|X_i| \leqslant b_n)}$, $1 \leqslant i \leqslant n$, *we have that*

$$(2.1)' \qquad b_n^{-1} S_n \xrightarrow{P} 0 \quad \text{as } n \longrightarrow \infty, \qquad \text{if}$$

$$(2.2)' \qquad \sum_{i=1}^n P(|X_i| > b_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

$$(2.3)' \qquad b_n^{-1} \sum_{i=1}^n E(X_{ni}/\mathcal{F}_{i-1}) \xrightarrow{P} 0 \quad \text{as } n \longrightarrow \infty,$$

$$(2.4)' \qquad b_n^{-2} \sum_{i=1}^n \{EX_{ni}^2 - E(X_{ni}/\mathcal{F}_{i-1})^2\} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

The below example shows that the above corollary is stronger than the theorem 2.13 in [2] which considered the same problem under the assumption:$(S_n = \sum_{i=1}^n X_i, \ ; \ \mathcal{F}_n)$ is a martingale.

Let (Y_i) be a sequence of independent and identically distributed random variables such that

$$P(Y_i = -1) = P(Y_i = 1) = \frac{1}{2}.$$

Then $EY_i = 0$ ($\forall i = 1, 2, \ldots$) and

$$\frac{1}{n} \sum_{i=1}^n Y_i \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Put

$$X_i = Y_i + \frac{1}{i}\ .$$

Then $EX_i = \frac{1}{i}$ ($\forall i = 1, 2 \cdots$) and

$$\frac{1}{n} \sum_{i=1}^n X_i = \frac{1}{n} \sum_{i=1}^n Y_i + \frac{1}{n} \sum_{i=1}^n \frac{1}{i} \xrightarrow{P} 0 + 0 = 0 \text{ as } n \to \infty.$$

Thus, $(S_n = \sum_{i=1}^n X_i)$ satisfies the condition $(2.1)'$ and it also satisfies the conditions $(2.2)'$, $(2.3)'$, $(2.4)'$ (with $b_n = n$). (Because (X_i) is the sequence of independent random variables and in this case, the conditions $(2.2)'$, $(2.3)'$, $(2.4)'$ are necessary as well as sufficient for the condition $(2.1)'$ (see [5], p. 290)). But $(S_n = \sum_{i=1}^n X_i, \ ; \ \mathcal{F}_n)$ is not a martingale. ($\mathcal{F}_n$ denote the σ- algebra generated by $(X_i; \ 1 \leqslant i \leqslant n)$). It shows that the martingale condition of $(S_n = \sum_{i=1}^n X_i; \ \mathcal{F}_n)$ in the theorem 2.13 of [2] is too strong.

Let (X_n), X be random elements in $\mathcal{X}$. The sequence $\{X_n, n \geqslant 1\}$ is said to be stochastically dominated by X if there exists a constant $C > 0$ such that $P\{\|X_n\| \geqslant t\} \leqslant CP\{\|X\| \geqslant t\}$ for all nonnegative real numbers t and for all $n \geqslant 1$. In this case, we write $(X_n) \prec X$.

LEMMA 2.3. (see [4]) *Assume* $f_n : \mathbb{R} \longrightarrow \mathbb{R}^+$ *satisfy* : $0 \leqslant f_n \leqslant 1$; $n = 1, 2, \ldots$ *and* $\sup\limits_{n\in\mathbb{N}}(xf_n(x)) \longrightarrow 0$ *as* $x \longrightarrow \infty$. *Then*

$$\sup_{n\in\mathbb{N}} \left(\frac{1}{y} \int_0^y x f_n(x) dx \right) \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty.$$

COROLLARY 2.4. *Let* $(S_n = \sum\limits_{i=1}^{n} X_i,\ \mathcal{F}_n)$ *be a martingale on* $\mathcal{X}$. *If* $(X_n) \prec Y$ *with* $Y \in L^1(\Omega, \mathcal{F}, P)$, *then*

$$n^{-1} S_n \xrightarrow{P} 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{2.5}$$

Proof. We'll prove that (X_n) satisfies all conditions of Theorem 2.1. Let $X_{ni} = X_i I_{(\|X_i\| \leqslant n)}$, $\quad 1 \leqslant i \leqslant n$.

At first we have

$$\sum_{i=1}^{n} P(\|X_i\| \geqslant n) \leqslant CnP(\|Y\| \geqslant n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{2.6}$$

Next, for arbitrary $\varepsilon > 0$ we have

$$\begin{aligned}
& P(\|n^{-1} \sum_{i=1}^{n} E(X_{ni}/\mathcal{F}_{i-1})\| > \varepsilon) \\
& \leqslant \varepsilon^{-1} n^{-1} E\| \sum_{i=1}^{n} E(X_{ni}/\mathcal{F}_{i-1})\| \\
& = \varepsilon^{-1} n^{-1} E\| \sum_{i=1}^{n} E\left[(X_i - X_i I_{(\|X_i\| > n)}/\mathcal{F}_{i-1}\right] \| \\
& \leqslant \varepsilon^{-1} n^{-1} E \sum_{i=1}^{n} E(\|X_i\| I_{(\|X_i\| > n)}/\mathcal{F}_{i-1})
\end{aligned}$$

$$= \varepsilon^{-1} n^{-1} \sum_{i=1}^{n} E\|X_i\| I_{(\|X_i\| > n)} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{2.7}$$

$$\left(\because E\|X_i\| = \int_0^\infty P(\|X_i\| \geqslant x) dx \leqslant \int_0^\infty CP(\|Y\| \geqslant x) dx = CE\|Y\| < \infty \right).$$

At the end, we have

$$
\begin{aligned}
& n^{-p}\sum_{i=1}^{n} E\|X_{ni} - E(X_{ni}/\mathcal{F}_{i-1})\|^p \\
& \leqslant n^{-p}\sum_{i=1}^{n} E\,[\,2\|X_{ni}\|^p + KE(\|X_{ni}\|^p/\mathcal{F}_{i-1})] \\
& = n^{-p}C'\sum_{i=1}^{n} E\|X_{ni}\|^p \\
& = C'n^{-p}\int_0^n P(\|X_i\|^p \geqslant x)dx \\
& \leqslant C'n^{-p}\int_0^n px^{p-1}P(\|X_i\|^p \geqslant x)dx \\
& \leqslant C'n^{-p}n\int_0^n px^{p-2}xP(\|Y\| \geqslant x)dx \\
& \leqslant C'pn^{-1}\int_0^n xP(\|Y\| \geqslant x)dx \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty \qquad (2.8)
\end{aligned}
$$

(The last inequality follows from Lemma 2.2 with $f_n(x) = P(\|Y\| \geqslant x)$, $n = 1, 2, \ldots$).

Combining (2.6), (2.7) and (2.8) we get (2.5) and which completes the proof. □

COROLLARY 2.5. *Let* $(S_n = \sum_{i=1}^{n} X_i,\ \mathcal{F}_n\ )$ *be a martingale on* $\mathcal{X}$. *If* $\sup_n E\|X_n\|^p < \infty$, *then*

$$
n^{-\frac{1}{q}}S_n \xrightarrow{P} 0 \ \text{ as } \ n \longrightarrow \infty \quad (\text{with } \ 0 < q < p). \qquad (2.9)
$$

Proof. We'll prove that (X_n) satisfies all conditions of Theorem 2.1 with $b_n = n^{\frac{1}{q}}$.

Let $X_{ni} = X_i I_{(\|X_i\| \leqslant n^{\frac{1}{q}})}$. Then we have

$$
\sum_{i=1}^{n} P(\|X_i\| > n^{\frac{1}{q}})
$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{n} n^{-\frac{p}{q}} E\|X_i\|^p \\
(2.10) \qquad &= n^{-\frac{p}{q}} \sum_{i=1}^{n} E\|X_i\|^p \\
&\leqslant n^{-\frac{p}{q}+1} \sup_n E\|X_n\|^p \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

For arbitrary $\varepsilon > 0$ we have

$$
\begin{aligned}
&P\left(n^{-\frac{1}{q}} \|\sum_{i=1}^{n} E(X_{ni}/\mathcal{F}_{i-1})\| > \varepsilon\right) \\
&\leqslant \varepsilon^{-p} n^{-\frac{p}{q}} E\|\sum_{i=1}^{n} E(X_{ni}/\mathcal{F}_{i-1}\|^p \\
&\leqslant \varepsilon^{-p} n^{-\frac{p}{q}} E \sum_{i=1}^{n} E\left(\|X_i\|^p I_{(\|X_i\| \leqslant n^{\frac{1}{q}})} / \mathcal{F}_{i-1}\right) \\
(2.11) \qquad &\leqslant \varepsilon^{-p} n^{1-\frac{p}{q}} E(\|X_i\|^p I_{(\|X_i\| > n^{\frac{1}{q}}} / \mathcal{F}_{i-1}) \xrightarrow{P} 0 \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

At the end we have

$$
\begin{aligned}
&n^{-\frac{p}{q}} \sum_{i=1}^{n} E\|X_{ni} - E(X_{ni}/\mathcal{F}_{i-1})\|^p \\
&\leqslant n^{-\frac{p}{q}} C \sum_{i-1}^{n} E\|X_{ni}\|^p \\
&= C n^{-\frac{p}{q}} \sum_{i=1}^{n} \left[E\|X_i\|^p - E\|X_i\|^p I_{(\|X_i\| > n^{\frac{1}{q}})}\right] \\
(2.12) \qquad &\leqslant C n^{-\frac{p}{q}} \sum_{i=1}^{n} E\|X_i\|^p \\
&\leqslant C n^{1-\frac{p}{q}} \left(\sup_n E\|X_n\|^p\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

Combining (2.10), (2.11) and (2.12) we get (2.9) and which completes the proof. □

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