Bull. Korean Math. Soc. **43** (2006), No. 3, pp. 461–470

FUZZY SET THEORY APPLIED TO IMPLICATIVE IDEALS IN BCK-ALGEBRAS

Young Bae Jun and Seok Zun Song

Abstract. As a continuation of [4], characterizations of fuzzy implicative ideals are given. An extension property for fuzzy implicative ideals is established. We prove that the family of fuzzy implicative ideals is a completely distributive lattice. Using level subsets of a BCK-algebra X with respect to a fuzzy set $\bar{A}$ in X, we construct a fuzzy implicative ideal of X containing $\bar{A}$.

1. Introduction

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. For the general development of BCK-algebras, the ideal theory and its fuzzification play an important role. The notion of implicative ideals in a BCK-algebra is first introduced by K. Iséki [1] in 1975, and then the fuzzification of implicative ideals is discussed in [8] by O. G. Xi. In 1995, Y. B. Jun, S. M. Hong and E. H. Roh [4] developed the fuzzy implicative ideals in BCK-algebras. This paper is a continuation of the paper [4]. We first give characterizations of fuzzy implicative ideals, and establish an extension property for fuzzy implicative ideals. We prove that the family of fuzzy implicative ideals is a completely distributive lattice. Given a fuzzy set $\bar{A}$ in a BCK-algebra X, we construct a fuzzy implicative ideal of X containing $\bar{A}$ by using level subsets of X with respect to $\bar{A}$.

2. Preliminaries

By a *BCK-algebra*, we mean an algebra $(X;*,0)$ of type (2,0) satisfying the following conditions:

Received August 2, 2002.
2000 Mathematics Subject Classification: 06F35, 03G25, 03B52.
Key words and phrases: (fuzzy) ideal, (fuzzy) implicative ideal.

(K1) $((x * y) * (x * z)) * (z * y) = 0$,
(K2) $(x * (x * y)) * y = 0$,
(K3) $x * x = 0$,
(K4) $0 * x = 0$,
(K5) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$. One can define a partial order relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$ for all $x \in X$. The following are true in a BCK-algebra X.

(P1) $x * 0 = x$,
(P2) $(x * y) * z = (x * z) * y$.

A nonempty subset A of a BCK-algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in A$,
(I2) $\forall x, y \in X,\ x * y \in A,\ y \in A \Rightarrow x \in A$.

For further details of BCK-algebras we refer to [6]. We now review the fuzzy logic concepts. A function from a set X into $[0, 1]$ is called a *fuzzy set* in X. We place a bar over a symbol to denote a fuzzy set so $\bar{A}$, $\bar{B}$, $\bar{X}, \ldots$ all represent fuzzy set in a set.

Let X be a BCK-algebra. A fuzzy set $\bar{A} : X \to [0, 1]$ is called a *fuzzy ideal* of X if it satisfies

(F1) $\bar{A}(0) \geq \bar{A}(x),\ \forall x \in X$,
(F2) $\bar{A}(x) \geq \min\{\bar{A}(x * y),\ \bar{A}(y)\},\ \forall x, y \in X$.

LEMMA 2.1. [3] *Every fuzzy ideal of a BCK-algebra X is order reversing, that is, if $\bar{A}$ is a fuzzy ideal of X and $x \leq y$ in X, then $\bar{A}(x) \geq \bar{A}(y)$.*

The following is a characterization of a fuzzy ideal of a BCK-algebra.

LEMMA 2.2. [7, Theorem 3.3] *A fuzzy set $\bar{A}$ in a BCK-algebra X is a fuzzy ideal of X if and only if it satisfies*

(F3) $\forall x, y, z \in X,\ x * y \leq z \Rightarrow \bar{A}(x) \geq \min\{\bar{A}(y),\ \bar{A}(z)\}$.

LEMMA 2.3. [8, Theorem 5] *A fuzzy set $\bar{A}$ in a BCK-algebra X is a fuzzy implicative ideal of X if and only if for any $t \in [0, 1]$, $U(\bar{A}; t) := \{x \in X \mid \bar{A}(x) \geq t\}$ is an implicative ideal of X when $U(\bar{A}; t) \neq \emptyset$.*

3. Fuzzy implicative ideals

In what follows, let X denote a BCK-algebra unless otherwise specified. Let $\bar{A}$ be a fuzzy set in X. For any $w \in X$, we consider the set

$$\uparrow\bar{A}(w) := \{x \in X \mid \bar{A}(w) \leq \bar{A}(x)\}.$$

Obviously, $w \in \uparrow\bar{A}(w)$. If $\bar{A}$ is a fuzzy ideal of X, then $0 \in \uparrow\bar{A}(w)$ by (F1). The following is our question: *For a fuzzy set $\bar{A}$ in X satisfying* (F1), *is $\uparrow\bar{A}(w)$ an ideal of X?* But the following example provides a negative answer, that is, there exists an element $w \in X$ such that $\uparrow\bar{A}(w)$ is not an ideal of X.

EXAMPLE 3.1. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Let $\bar{A}$ be a fuzzy set in X defined by $\bar{A}(0) = 0.9$, $\bar{A}(a) = 0.7$, $\bar{A}(b) = 0.3$, $\bar{A}(c) = 0.2$, and $\bar{A}(d) = 0.5$. Note that $\bar{A}$ is not a fuzzy ideal of X because $\bar{A}(b) < \min\{\bar{A}(b * d),\ \bar{A}(d)\}$. Then $\uparrow\bar{A}(d) = \{0, a, d\}$ is not an ideal of X since $b * d = 0 \in \uparrow\bar{A}(d)$ and $d \in \uparrow\bar{A}(d)$, but $b \notin \uparrow\bar{A}(d)$. Note that $\uparrow\bar{A}(b) = \{0, a, b, d\}$ is an ideal of X.

We give conditions for the set $\uparrow\bar{A}(w)$ to be an ideal.

THEOREM 3.2. *Let $w \in X$. If $\bar{A}$ is a fuzzy ideal of X, then $\uparrow\bar{A}(w)$ is an ideal of X.*

Proof. Recall that $0 \in \uparrow\bar{A}(w)$. Let $x, y \in X$ be such that $x * y \in \uparrow\bar{A}(w)$ and $y \in \uparrow\bar{A}(w)$. Then $\bar{A}(w) \leq \bar{A}(x * y)$ and $\bar{A}(w) \leq \bar{A}(y)$. Since $\bar{A}$ is a fuzzy ideal of X, it follows from (F2) that

$$\bar{A}(x) \geq \min\{\bar{A}(x * y),\ \bar{A}(y)\} \geq \bar{A}(w),$$

so that $x \in \uparrow\bar{A}(w)$. Therefore $\uparrow\bar{A}(w)$ is an ideal of X. □

THEOREM 3.3. *Let $\bar{A}$ be a fuzzy set in X and $w \in X$. Then*

(i) *If $\uparrow\bar{A}(w)$ is an ideal of X, then $\bar{A}$ satisfies the following implication*

$$\text{(3.1)} \qquad \forall x, y, z \in X,\ \bar{A}(x) \leq \min\{\bar{A}(y * z),\ \bar{A}(z)\} \Rightarrow \bar{A}(x) \leq \bar{A}(y).$$

(ii) *If $\bar{A}$ satisfies* (F1) *and* (3.1), *then $\uparrow\bar{A}(w)$ is an ideal of X.*

Proof. (i) Assume that $\uparrow\bar{A}(w)$ is an ideal of X for each $w \in X$. Let x, y, $z \in X$ be such that $\bar{A}(x) \leq \min\{\bar{A}(y * z), \bar{A}(z)\}$. Then $y * z \in \uparrow\bar{A}(x)$ and $z \in \uparrow\bar{A}(x)$. It follows from (I2) that $y \in \uparrow\bar{A}(x)$, that is, $\bar{A}(x) \leq \bar{A}(y)$.

(ii) Suppose that $\bar{A}$ satisfies (F1) and (3.1). For each $w \in X$, let $x, y \in X$ be such that $x*y \in \uparrow\bar{A}(w)$ and $y \in \uparrow\bar{A}(w)$. Then $\bar{A}(x*y) \geq \bar{A}(w)$ and $\bar{A}(y) \geq \bar{A}(w)$, which imply that $\bar{A}(w) \leq \min\{\bar{A}(x*y), \bar{A}(y)\}$. Using (3.1), we have $\bar{A}(w) \leq \bar{A}(x)$ and so $x \in \uparrow\bar{A}(w)$. Since $\bar{A}$ satisfies (F1), it follows that $0 \in \uparrow\bar{A}(w)$. Therefore $\uparrow\bar{A}(w)$ is an ideal of X. □

DEFINITION 3.4. [1] A nonempty subset A of X is called an *implicative ideal* of X if it satisfies (I1) and

(I3) $\forall x, y, z \in X,\ (x * y) * z \in A,\ y * z \in A \Rightarrow x * z \in A.$

Note that every implicative ideal is an ideal, but not converse.

DEFINITION 3.5. [8] A fuzzy set $\bar{A} : X \to [0, 1]$ is called a *fuzzy implicative ideal* of X (it is called a *fuzzy positive implicative ideal* in [3]) if it satisfies (F1) and

(F4) $\bar{A}(x * z) \geq \min\{\bar{A}((x * y) * z), \bar{A}(y * z)\},\ \forall x, y, z \in X.$

Note that every fuzzy implicative ideal is a fuzzy ideal, but not converse (see [3]). We provide a condition for a fuzzy ideal to be a fuzzy implicative ideal.

THEOREM 3.6. *If $\bar{A}$ is a fuzzy ideal of X satisfying*

(F5) $\bar{A}(x * y) \geq \min\{\bar{A}(((x * y) * y) * z), \bar{A}(z)\},\ \forall x, y, z \in X,$

then $\bar{A}$ is a fuzzy implicative ideal of X.

Proof. Using (K1) and (P2), we have

$$((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z,\ \forall x, y, z \in X.$$

Since $\bar{A}$ is order reversing, it follows from (F5) that

$$\begin{aligned} \bar{A}(x * z) &\geq \min\{\bar{A}(((x * z) * z) * (y * z)), \bar{A}(y * z)\} \\ &\geq \min\{\bar{A}((x * y) * z), \bar{A}(y * z)\} \end{aligned}$$

for all $x, y, z \in X$. Hence $\bar{A}$ is a fuzzy implicative ideal of X. □

THEOREM 3.7. *Let $w \in X$. If $\bar{A}$ is a fuzzy implicative ideal of X, then $\uparrow\bar{A}(w)$ is an implicative ideal of X.*

Proof. Recall that $0 \in \uparrow\bar{A}(w)$. Let $x, y, z \in X$ be such that $(x*y)*z \in \uparrow\bar{A}(w)$ and $y*z \in \uparrow\bar{A}(w)$. Then $\bar{A}(w) \leq \bar{A}((x*y)*z)$ and $\bar{A}(w) \leq \bar{A}(y*z)$. Since $\bar{A}$ is a fuzzy implicative ideal of X, it follows from (F4) that

$$\bar{A}(x * z) \geq \min\{\bar{A}((x * y) * z), \bar{A}(y * z)\} \geq \bar{A}(w),$$

so that $x * z \in \uparrow\bar{A}(w)$. Hence $\uparrow\bar{A}(w)$ is an implicative ideal of X. □

As a generalization of Lemma 2.2, we have the following results.

THEOREM 3.8. *If a fuzzy set $\bar{A}$ in X is a fuzzy ideal of X, then*
(F6) $\forall x, w_1, w_2, \ldots, w_n \in X$, $\prod_{i=1}^{n} x * w_i = 0 \Rightarrow \bar{A}(x) \geq \min\{\bar{A}(w_i) \mid i = 1, 2, \ldots, n\}$, *where* $\prod_{i=1}^{n} x * w_i = (\cdots((x * w_1) * w_2) * \cdots) * w_n$.

Proof. The proof is by induction on n. Let $\bar{A}$ be a fuzzy ideal of X. Lemmas 2.1 and 2.2 show that the condition (F6) is valid for $n = 1, 2$. Assume that $\bar{A}$ satisfies the condition (F6) for $n = k$, that is,

$$\forall x, w_1, w_2, \ldots, w_k \in X, \prod_{i=1}^{k} x * w_i = 0$$
$$\Rightarrow \bar{A}(x) \geq \min\{\bar{A}(w_i) \mid i = 1, 2, \ldots, k\}.$$

Let $x, w_1, w_2, \ldots, w_k, w_{k+1} \in X$ be such that $\prod_{i=1}^{k+1} x * w_i = 0$. Then

$$\bar{A}(x * w_1) \geq \min\{\bar{A}(w_j) \mid j = 2, 3, \ldots, k+1\}.$$

Since $\bar{A}$ is a fuzzy ideal of X, it follows from (F2) that

$$\begin{aligned} \bar{A}(x) &\geq \min\{\bar{A}(x * w_1), \bar{A}(w_1)\} \\ &\geq \min\{\bar{A}(w_1), \min\{\bar{A}(w_j) \mid j = 2, 3, \ldots, k+1\}\} \\ &= \min\{\bar{A}(w_i) \mid i = 1, 2, \ldots, k+1\}. \end{aligned}$$

This completes the proof. □

Now we consider the converse of Theorem 3.8.

THEOREM 3.9. *Let $\bar{A}$ be a fuzzy set in X satisfying the condition* (F6). *Then $\bar{A}$ is a fuzzy ideal of X.*

Proof. Note that $(\cdots((0 * \underbrace{x) * x) * \cdots) * x}_{n \text{ times}} = 0$ for all $x \in X$. It follows from (F6) that $\bar{A}(0) \geq \bar{A}(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $x * y \leq z$. Then

$$0 = (x * y) * z = (\cdots(((x * y) * z) * \underbrace{0) * \cdots) * 0}_{n-2 \text{ times}},$$

and so $\bar{A}(x) \geq \min\{\bar{A}(y), \bar{A}(z), \bar{A}(0)\} = \min\{\bar{A}(y), \bar{A}(z)\}$. Hence, by Lemma 2.2, we conclude that $\bar{A}$ is a fuzzy ideal of X. □

LEMMA 3.10. [3, Theorems 2 and 3] *Let $\bar{A}$ be a fuzzy ideal of X. Then*

- *$\bar{A}$ is a fuzzy implicative ideal of X if and only if it satisfies*

(F9) $\bar{A}(x * y) \geq \bar{A}\big((x * y) * y\big)$, $\forall x, y \in X$.

- *$\bar{A}$ is a fuzzy implicative ideal of X if and only if it satisfies*

(F10) $\bar{A}\big((x * z) * (y * z)\big) \geq \bar{A}\big((x * y) * z\big)$, $\forall x, y, z \in X$.

THEOREM 3.11. *If $\bar{A}$ is a fuzzy implicative ideal of X, then*

(F7) $\forall x, y, a, b \in X$, $\big((x*y)*y\big)*a \leq b \Rightarrow \bar{A}(x*y) \geq \min\{\bar{A}(a), \bar{A}(b)\}$.

(F8) $\forall x, y, z, a, b \in X$, $\big((x * y) * z\big) * a \leq b \Rightarrow \bar{A}\big((x * z) * (y * z)\big) \geq \min\{\bar{A}(a), \bar{A}(b)\}$.

Proof. Let $x, y, a, b \in X$ be such that $\big((x * y) * y\big) * a \leq b$. Using Lemma 2.2, we have $\bar{A}\big((x * y) * y\big) \geq \min\{\bar{A}(a), \bar{A}(b)\}$. It follows that

$$\begin{aligned} \bar{A}(x * y) &\geq \min\big\{\bar{A}((x * y) * y),\ \bar{A}(y * y)\big\} && \text{[by (F4)]} \\ &= \min\big\{\bar{A}((x * y) * y),\ \bar{A}(0)\big\} && \text{[by (K3)]} \\ &= \bar{A}\big((x * y) * y\big) && \text{[by (F1)]} \\ &\geq \min\{\bar{A}(a),\ \bar{A}(b)\}. \end{aligned}$$

Now let $x, y, z, a, b \in X$ be such that $\big((x * y) * z\big) * a \leq b$, that is,

$$\big(((x * y) * z) * a\big) * b = 0.$$

Since $\bar{A}$ is a fuzzy implicative ideal of X, it follows from Lemmas 2.2 and 3.10 that

$$\bar{A}\big((x * z) * (y * z)\big) \geq \bar{A}\big((x * y) * z\big) \geq \min\{\bar{A}(a),\ \bar{A}(b)\}.$$

This completes the proof. □

We now give conditions for a fuzzy set to be a fuzzy implicative ideal.

THEOREM 3.12. *Let $\bar{A}$ be a fuzzy set in X satisfying the condition* (F7). *Then $\bar{A}$ is a fuzzy implicative ideal of X.*

Proof. We first prove that $\bar{A}$ is a fuzzy ideal of X. Let $x, y, z \in X$ be such that $x * y \leq z$. Then

$$\big(((x * 0) * 0) * y\big) * z = (x * y) * z = 0, \text{ that is, } ((x * 0) * 0) * y \leq z,$$

which implies from (P1) and (F7) that $\bar{A}(x) = \bar{A}(x * 0) \geq \min\{\bar{A}(y), \bar{A}(z)\}$. Therefore, by Lemma 2.2, we know that $\bar{A}$ is a fuzzy ideal of X. Note that $\big(((x*y)*y)*((x*y)*y)\big)*0 = 0$ for all $x, y \in X$. Using (F7) and (F1), we have

$$\bar{A}(x * y) \geq \min\big\{\bar{A}((x * y) * y),\ \bar{A}(0)\big\} = \bar{A}\big((x * y) * y\big),$$

and so $\bar{A}$ is a fuzzy implicative ideal of X by Lemma 3.10. □

THEOREM 3.13. *Let $\bar{A}$ be a fuzzy set in X satisfying* (F8). *Then $\bar{A}$ is a fuzzy implicative ideal of X.*

Proof. Let $x, y, a, b \in X$ be such that $((x * y) * y) * a \leq b$, that is,

$$(((x * y) * y) * a) * b = 0.$$

Then

$$\begin{aligned} \bar{A}(x * y) &= \bar{A}((x * y) * 0) = \bar{A}((x * y) * (y * y)) && \text{[by (P1) and (K3)]} \\ &\geq \min\{\bar{A}(a), \bar{A}(b)\}, && \text{[by (F8)]} \end{aligned}$$

and so $\bar{A}$ is a fuzzy implicative ideal of X by Theorem 3.12. □

COROLLARY 3.14. *If $\bar{A}$ is a fuzzy implicative ideal of X, then*

(F11) $\bar{A}((x * z) * (y * z)) \geq \min\{\bar{A}(w_i) \mid i = 1, 2, \ldots, n\}$ *whenever* $\prod_{i=1}^{n}((x * y) * z) * w_i = 0$ *for all* $x, y, z, w_1, \ldots, w_n \in X$.

Proof. Let $x, y, z, w_1, \ldots, w_n \in X$ be such that $\prod_{i=1}^{n}((x*y)*z)*w_i = 0$. Then

$$\begin{aligned} \bar{A}((x * z) * (y * z)) &\geq \bar{A}((x * y) * z) && \text{[by Lemma 3.10]} \\ &\geq \min\{\bar{A}(w_i) \mid i = 1, 2, \ldots, n\}. && \text{[by Theorem 3.8]} \end{aligned}$$

This completes the proof. □

THEOREM 3.15. (Extension property for fuzzy implicative ideals) *Let $\bar{A}$ and $\bar{B}$ be fuzzy ideals of X such that $\bar{A}(0) = \bar{B}(0)$ and $\bar{A} \subseteq \bar{B}$, that is, $\bar{A}(x) \leq \bar{B}(x)$ for all $x \in X$. If $\bar{A}$ is a fuzzy implicative ideal of X, then so is $\bar{B}$.*

Proof. Assume that $\bar{A}$ is a fuzzy implicative ideal of X. For any $x, y, z \in X$, we have

$$\begin{aligned} & \bar{B}(((x * z) * (y * z)) * ((x * y) * z)) \\ = \;& \bar{B}(((x * z) * ((x * y) * z)) * (y * z)) && \text{[by (P2)]} \\ = \;& \bar{B}(((x * ((x * y) * z)) * z) * (y * z)) && \text{[by (P2)]} \\ \geq \;& \bar{A}(((x * ((x * y) * z)) * z) * (y * z)) && \text{[since } \bar{A} \subseteq \bar{B}\text{]} \\ \geq \;& \bar{A}(((x * ((x * y) * z)) * y) * z) && \text{[by Lemma 3.10]} \\ = \;& \bar{A}(((x * y) * ((x * y) * z)) * z) && \text{[by (P2)]} \\ = \;& \bar{A}(((x * y) * z) * ((x * y) * z)) && \text{[by (P2)]} \\ = \;& \bar{A}(0) = \bar{B}(0). && \text{[by (K3) and assumption]} \end{aligned}$$

It follows from (F1) and (F2) that

$$\begin{aligned} & \bar{B}((x * z) * (y * z)) \\ \geq \;& \min\{\bar{B}(((x * z) * (y * z)) * ((x * y) * z)), \bar{B}((x * y) * z)\} \\ \geq \;& \min\{\bar{B}(0), \bar{B}((x * y) * z)\} \\ = \;& \bar{B}((x * y) * z) \end{aligned}$$

for all $x, y, z \in X$. Hence, by Lemma 3.10, $\bar{B}$ is a fuzzy implicative ideal of X. □

For a family $\{\bar{A}_i \mid i \in \Lambda\}$ of fuzzy sets in X, define the join $\bigvee_{i\in\Lambda} \bar{A}_i$ and the meet $\bigwedge_{i\in\Lambda} \bar{A}_i$ as follows:

$$\Big(\bigvee_{i\in\Lambda} \bar{A}_i\Big)(x) = \sup_{i\in\Lambda} \bar{A}_i(x) \quad \text{and} \quad \Big(\bigwedge_{i\in\Lambda} \bar{A}_i\Big)(x) = \inf_{i\in\Lambda} \bar{A}_i(x)$$

for all $x \in X$, where Λ is any index set.

THEOREM 3.16. *The family of fuzzy implicative ideals of X is a completely distributive lattice with respect to the meet and the join.*

Proof. Let $\{\bar{A}_i \mid i \in \Lambda\}$ be a family of fuzzy implicative ideals of X. Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering in $[0,1]$, it is sufficient to show that $\bigvee_{i\in\Lambda} \bar{A}_i$ and $\bigwedge_{i\in\Lambda} \bar{A}_i$ are fuzzy implicative ideals of X. For any $x \in X$, we have

$$\Big(\bigvee_{i\in\Lambda} \bar{A}_i\Big)(0) = \sup_{i\in\Lambda} \bar{A}_i(0) \geq \sup_{i\in\Lambda} \bar{A}_i(x) = \Big(\bigvee_{i\in\Lambda} \bar{A}_i\Big)(x)$$

and

$$\Big(\bigwedge_{i\in\Lambda} \bar{A}_i\Big)(0) = \inf_{i\in\Lambda} \bar{A}_i(0) \geq \inf_{i\in\Lambda} \bar{A}_i(x) = \Big(\bigwedge_{i\in\Lambda} \bar{A}_i\Big)(x).$$

Let $x, y, z \in X$. Then

$$\begin{aligned}
\Big(\bigvee_{i\in\Lambda} \bar{A}_i\Big)(x * z) &= \sup_{i\in\Lambda} \bar{A}_i(x * z) \\
&\geq \sup_{i\in\Lambda} \min\{\bar{A}_i((x * y) * z),\ \bar{A}_i(y * z)\} \\
&\geq \min\{\sup_{i\in\Lambda} \bar{A}_i((x * y) * z),\ \sup_{i\in\Lambda} \bar{A}_i(y * z)\} \\
&= \min\Big\{\Big(\bigvee_{i\in\Lambda} \bar{A}_i\Big)((x * y) * z),\ \Big(\bigvee_{i\in\Lambda} \bar{A}_i\Big)(y * z)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\Big(\bigwedge_{i\in\Lambda} \bar{A}_i\Big)(x * z) &= \inf_{i\in\Lambda} \bar{A}_i(x * z) \\
&\geq \inf_{i\in\Lambda} \min\{\bar{A}_i((x * y) * z),\ \bar{A}_i(y * z)\} \\
&= \min\{\inf_{i\in\Lambda} \bar{A}_i((x * y) * z),\ \inf_{i\in\Lambda} \bar{A}_i(y * z)\} \\
&= \min\Big\{\Big(\bigwedge_{i\in\Lambda} \bar{A}_i\Big)((x * y) * z),\ \Big(\bigwedge_{i\in\Lambda} \bar{A}_i\Big)(y * z)\Big\}.
\end{aligned}$$

Hence $\bigvee_{i\in\Lambda} \bar{A}_i$ and $\bigwedge_{i\in\Lambda} \bar{A}_i$ are fuzzy implicative ideals of X. This completes the proof. □

THEOREM 3.17. *Let $\bar{A}$ be a fuzzy set in X. Then a fuzzy set $\bar{A}^*$ in X defined by*

$$\bar{A}^*(x) = \sup\{t \in [0,1] \mid x \in \langle U(\bar{A};t)\rangle\},\ \forall x \in X$$

is the least fuzzy implicative ideal of X that contains $\bar{A}$, where $\langle U(\bar{A};t)\rangle$ means the least implicative ideal of X containing $U(\bar{A};t)$.

Proof. For any $s \in \mathrm{Im}(\bar{A}^*)$, let $s_n = s - \frac{1}{n}$ for any $n \in \mathbb{N}$. Let $x \in U(\bar{A}^*;s)$. Then $\bar{A}^*(x) \geq s$, which implies that

$$\sup\{t \in [0,1] \mid x \in \langle U(\bar{A};t)\rangle\} \geq s > s - \frac{1}{n} = s_n,\ \forall n \in \mathbb{N}.$$

Hence there exists $t^* \in \{t \in [0,1] \mid x \in \langle U(\bar{A};t)\rangle\}$ such that $t^* > s_n$. Thus $U(\bar{A};t^*) \subseteq U(\bar{A};s_n)$ and so $x \in \langle U(\bar{A};t^*)\rangle \subseteq \langle U(\bar{A};s_n)\rangle$ for all $n \in \mathbb{N}$. Consequently $x \in \bigcap_{n\in\mathbb{N}} \langle U(\bar{A};s_n)\rangle$. On the other hand, if $x \in \bigcap_{n\in\mathbb{N}} \langle U(\bar{A};s_n)\rangle$, then $s_n \in \{t \in [0,1] \mid x \in \langle U(\bar{A};t)\rangle\}$ for any $n \in \mathbb{N}$. Therefore

$$s - \frac{1}{n} = s_n \leq \sup\{t \in [0,1] \mid x \in \langle U(\bar{A};t)\rangle\} = \bar{A}^*(x),\ \forall n \in \mathbb{N}.$$

Since n is arbitrary, it follows that $s \leq \bar{A}^*(x)$ so that $x \in U(\bar{A}^*;s)$. Hence $U(\bar{A}^*;s) = \bigcap_{n\in\mathbb{N}} \langle U(\bar{A};s_n)\rangle$, which is an implicative ideal of X. Using Lemma 2.3, we conclude that $\bar{A}^*$ is a fuzzy implicative ideal of X. We now prove that $\bar{A}^*$ contains $\bar{A}$. For any $x \in X$, let $s \in \{t \in [0,1] \mid x \in U(\bar{A};t)\}$. Then $x \in U(\bar{A};s)$ and so $x \in \langle U(\bar{A};s)\rangle$. Thus $s \in \{t \in [0,1] \mid x \in \langle U(\bar{A};t)\rangle\}$, which implies that

$$\{t \in [0,1] \mid x \in U(\bar{A};t)\} \subseteq \{t \in [0,1] \mid x \in \langle U(\bar{A};t)\rangle\}.$$

It follows that

$$\begin{aligned}\bar{A}(x) &= \sup\{t \in [0,1] \mid x \in U(\bar{A};t)\}\\ &\leq \sup\{t \in [0,1] \mid x \in \langle U(\bar{A};t)\rangle\} = \bar{A}^*(x),\end{aligned}$$

which shows that $\bar{A}^*$ contains $\bar{A}$. Finally let $\bar{B}$ be a fuzzy implicative ideal of X containing $\bar{A}$. Let $x \in X$. If $\bar{A}^*(x) = 0$, then clearly $\bar{A}^*(x) \leq \bar{B}(x)$. Assume that $\bar{A}^*(x) = s \neq 0$. Then $x \in U(\bar{A}^*;s) = \bigcap_{n\in\mathbb{N}} \langle U(\bar{A};s_n)\rangle$, and so $x \in \langle U(\bar{A};s_n)\rangle$ for all $n \in \mathbb{N}$. It follows that

$$\bar{B}(x) \geq \bar{A}(x) \geq s_n = s - \frac{1}{n},\ \forall n \in \mathbb{N}$$

so that $\bar{B}(x) \geq s = \bar{A}^*(x)$ since n is arbitrary. This shows that $\bar{A}^* \subseteq \bar{B}$. The proof is complete. □

References

[1] K. Iséki, *On ideals in BCK-algebras,* Math. Sem. Notes Kobe Univ. **3** (1975), no. 1, 1–12.

[2] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras,* Math. Japon. **23** (1978), no. 1, 1–26.

[3] Y. B. Jun, S. M. Hong, J. Meng, and X. L. Xin, *Characterizations of fuzzy positive implicative ideals in BCK-algebras,* Math. Japon. **40** (1994), no. 3, 503–507.

[4] Y. B. Jun, S. M. Hong, and E. H. Roh, *Fuzzy implicative ideals in BCK-algebras,* Tamkang J. Math. **26** (1995), no. 2, 119–124.

[5] Y. B. Jun and E. H. Roh, *Fuzzy commutative ideals of BCK-algebras,* Fuzzy Sets and Systems **64** (1994), no. 3, 401–405.

[6] J. Mang and Y. B. Jun, *BCK-algebras,* Kyungmoonsa, Seoul, Korea, 1994.

[7] J. Meng, X. L. Xin, and Y. S. Pu, *Quotient BCK-algebras induced by a fuzzy ideal,* Southeast Asian Bull. Math. **23** (1999), no. 2, 243–251.

[8] O. G. Xi, *Fuzzy BCK-algebra,* Math. Japon. **36** (1991), no. 5, 935–942.

YOUNG BAE JUN, DEPARTMENT OF MATHEMATICS EDUCATION (AND RINS), GYEONGSANG NATIONAL UNIVERSITY, CHINJU 660-701, KOREA
E-mail: ybjun@gnu.ac.kr

SEOK ZUN SONG, DEPARTMENT OF MATHEMATICS, CHEJU NATIONAL UNIVERSITY, CHEJU 690-756, KOREA
E-mail: szsong@cheju.ac.kr