Path Stability of a Crack with an Eigenstrain

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A slightly curved crack with an eigenstrain is considered. Solutions for a slightly curved crack in a linear isotropic material under asymptotic loading as well as for a slightly curved crack in a linear isotropic material with a concentrated force are obtained from perturbation analyses, which are accurate to the first order of the parameter representing the non-straightness. Stress intensity factors for a slightly curved crack with an eigenstrain are obtained from the perturbation solutions by using a body force analogy. Particular attention is given to the crack path stability under mode I loading. A new parameter of crack path stability is proposed for a crack with an eigenstrain. The path stability of a crack with steady state growth in a transforming material and a ferroelectric material is examined.

Key Words : Crack Path Stability, Curved Crack, Eigenstrain

1. Introduction

Curved cracks are frequently observed since the path of the fracture is generally curved under loading of mixed mode type. The problem of curved or kinked cracks in elastic materials has thus been extensively investigated. Several fracture criteria have been proposed in order to determine the favored orientation of the crack extension under loading of mixed mode type. More detailed exposition of the results of studies on the kinked crack in an elastic material by many researchers can be found in Hutchinson and Suo (1992). Curved cracks have been also observed in ferroelectric ceramics (Uchino and Furuta, 1992 ; Tan and Shang, 2000). The problem of curved cracks in solids containing a region in which eigenstrains such as transformation strains and domain switching strains are prescribed, however, has received little attention.

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(Manuscript **Received** February 20, 2006; **Revised** June 12, 2006)

The issue regarding the effect of T-stress on the stability of the crack path has been raised in the literature. Cotterell and Rice (1980) proposed the T-stress theory on the local stability of the crack path. They showed that the straight crack path under mode I loading is stable for negative T-stress and unstable for positive T-stress. Subsequently, this stability concept was extended by Sumi et al. (1985) for the effect of the change of the stability with increasing crack length along a curved trajectory. As a crack in a material with a stress-induced transformation or in a ferroelectric material grows, the stress fields at the advancing crack tip will induce further transformation or domain switching. However, the path stability of the crack with the eigenstrain induced by transformation or domain switching in a zone near the crack tip has not yet been solved in the literature.

The purpose of this study is to investigate the path stability of a crack with an eigenstrain. The asymptotic problem of a slightly curved crack in a linear isotropic material under mechanical loading is analyzed. In order to evaluate stress intensity factors for a slightly curved crack with an eigenstrain, a slightly curved crack in a linear isotropic material with a concentrated force is solved from a perturbation analysis. A perturbation solution of the stress intensity factors for the slightly curved crack with the eigenstrain is obtained by using a body force analogy, which is accurate to the first order of the parameter representing the non-straightness. Of particular interest is the effect of eigenstrains on the crack path stability. The first order solution for the slightly curved crack can be used to predict the path of a fracture. It is shown that for mode I loading, the straight crack path is stable when $T^*<0$ and unstable when $T^*>0$, in which T^* is the stress parameter of crack path stability proposed in this paper. The T^* stresses for a crack with steady state growth in a transforming material as well as in a ferroelectric material under mode I loading are obtained. The results show that although the value of applied T-stress is positive, the straight crack path may be stable for mode I loading.

2. Asymptotic Problem of a Curved Crack

Consider the asymptotic problem of a slightly curved crack in a linear isotropic material as shown in Fig. 1. The crack tip lies at the point $x_1=0$ and $x_2=0$, and the crack surface is described by $x_2=\lambda(x_1)$. Here we are only concerned with the case in which $\lambda(x_1)$ and $\lambda'(x_1)$ are small, and $O(\lambda)=O(\lambda')$. Tractions vanish on the crack surfaces. The remote fields in the asymptotic analysis are given by the near-tip fields for the crack in the linear material.

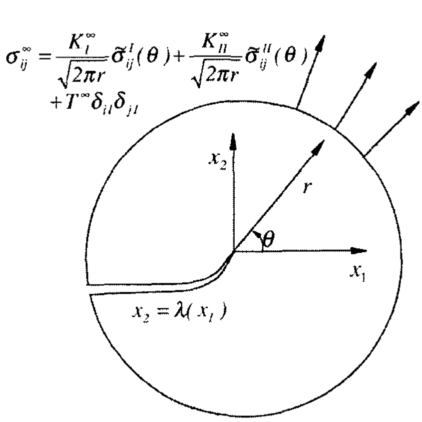


Fig. 1 Asymptotic problem of a slightly curved crack

In isotropic two-dimensional elasticity, stresses are expressed in terms of two complex potentials as follows (Muskhelishivili, 1963)

$$\begin{aligned}&\sigma_{11}+\sigma_{22}=2[\Phi(z)+\overline{\Phi(z)}],\\&\sigma_{22}+i\sigma_{12}=\overline{\Phi(z)}+\Omega(z)+(\bar{z}-z)\Phi'(z).\end{aligned}\tag{1}$$

Here σ_{ij} is the stress. $\Phi(z)$ and $\Omega(z)$ are analytic functions of $z=x_1+ix_2$ and overbar $(\bar{\ })$ denotes the complex conjugate. The analytic functions generating the stress fields at infinity are expressed

$$\begin{aligned}&\Phi(z)=\frac{\overline{K^\infty}}{2\sqrt{2\pi z}}+\frac{1}{4}T^\infty,\\&\Omega(z)=\frac{K^\infty}{2\sqrt{2\pi z}}-\frac{1}{4}T^\infty,\ \text{as}\ z\to\infty,\end{aligned}\tag{2}$$

where K^∞ denotes the complex stress intensity factor defined as $K^\infty=K_I^\infty+iK_{II}^\infty$ and T^∞ is the remote T-stress which is the uniform part of σ_{11} at infinity.

The boundary condition on the crack surfaces leads to the following equation

$$\Phi(z)+\overline{\Phi(z)}+e^{-2i\vartheta}[\overline{\Omega(z)}+(z-\bar{z})\overline{\Phi'(z)}-\overline{\Phi(z)}]=0,\tag{3}$$

where ϑ is the angle of the crack surface, given by $\vartheta=\lambda'(x_1)$ to the first order in λ. Making use of a perturbation expansion in λ, the functions $\Phi(z)$ and $\Omega(z)$ can be written as

$$\begin{aligned}&\Phi(z)=\Phi_0(z)+\Phi_1(z)+O(\lambda^2),\\&\Omega(z)=\Omega_0(z)+\Omega_1(z)+O(\lambda^2).\end{aligned}\tag{4}$$

Here $\Phi_0(z)$ and $\Omega_0(z)$ are analytic functions of zero order, and $\Phi_1(z)$ and $\Omega_1(z)$ are analytic functions of first order λ. Using this perturbation expansion, it can be shown from Eqs. (3) and (4) that the solutions of the zero order in λ which correspond to a straight crack, $\lambda=0$, are

$$\begin{aligned}&\Phi_0(z)=\frac{\overline{K^\infty}}{2\sqrt{2\pi z}}+\frac{1}{4}T^\infty,\\&\Omega_0(z)=\frac{K^\infty}{2\sqrt{2\pi z}}-\frac{1}{4}T^\infty.\end{aligned}\tag{5}$$

Equating all terms of the order λ in Eq. (3) and solving the first order equation, we obtain

$$\Phi_1(z)+\bar{\Omega}_1(z)=\frac{T^\infty}{\pi i\sqrt{z}}\int_{-\infty}^{0}\frac{\sqrt{-t}\lambda'}{t-z}dt,$$
$$\Phi_1(z)-\bar{\Omega}_1(z)=\frac{K_{II}^\infty}{2\pi\sqrt{2\pi}}\int_{-\infty}^{0}\frac{1}{t-z}\left[\frac{\lambda}{\sqrt{-t}}\right]'dt, \quad (6)$$

For the special case in which the magnitude of K_{II}^∞ has the first order of smallness, the first order solution Eq. (6) reduces to

$$\Phi_1(z)=\bar{\Omega}_1(z)=\frac{T^\infty}{2\pi i\sqrt{z}}\int_{-\infty}^{0}\frac{\sqrt{-t}\lambda'}{t-z}dt. \quad (7)$$

Introducing the cylindrical coordinate (r, θ) centered at the tip of curved crack, the mode I and mode II stress intensity factors, K_I and K_{II} are defined as

$$K_I+iK_{II}=\lim_{r\to 0}\sqrt{2\pi r}\left[\sigma_{\theta\theta}(r,\omega)+i\sigma_{r\theta}(r,\omega)\right], \quad (8)$$

where ω is the slope of the crack surface at the crack tip, given by $\omega=\lambda'(0)$ to the first order. Having found the functions $\Phi(z)$ and $\Omega(z)$ for the slightly curved crack as above, the stress intensity factors at the crack tip can be evaluated from Eqs. (1), (5) (6) and (8), which results in

$$K_I=K_I^\infty-\frac{3}{2}\omega K_{II}^\infty+O(\lambda^2),$$
$$K_{II}=K_{II}^\infty+\frac{1}{2}\omega K_I^\infty-\sqrt{\frac{2}{\pi}}T^\infty\int_{-\infty}^{0}\frac{\lambda'(t)}{\sqrt{-t}}dt+O(\lambda^2). \quad (9)$$

3. Curved Crack with a Concentrated Force

Consider a slightly curved crack in a linear isotropic material with a concentrated force as shown in Fig. 2. The concentrated force $p=p_1+ip_2$ is embedded in the material at the point $z=\zeta$, $\zeta=\xi_1+i\xi_2$. The crack surface is described by $x_2=\lambda(x_1)$ and tractions vanish on the crack surfaces. The solution of stress intensity factors due to a point force acting at position at $z=\zeta$ can be used as a weight function in determining stress intensity factors for a curved crack with a distributed body force.

Using the perturbation analysis, the zero order solutions of Eq. (3) are well known as (Suo, 1989)

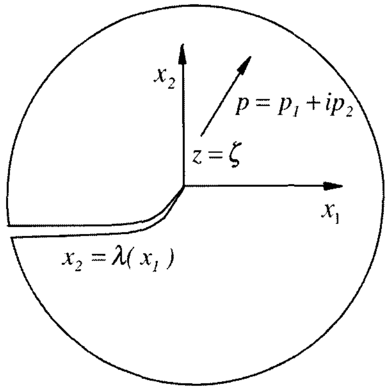


Fig. 2 Curved crack with a concentrated force

$$\Phi_0(z)=\frac{-1}{16\pi(1-v)}\left[\frac{p}{z-\zeta}+\frac{(3-4v)p}{z-\bar{\zeta}}-\frac{\bar{p}(\zeta-\bar{\zeta})}{(z-\bar{\zeta})^2}+\frac{p\sqrt{\zeta}}{\sqrt{z}(z-\zeta)}-\frac{(3-4v)p\sqrt{\bar{\zeta}}}{\sqrt{z}(z-\bar{\zeta})}+\frac{\bar{p}(\zeta-\bar{\zeta})(z+\bar{\zeta})}{2\sqrt{z}\sqrt{\bar{\zeta}}(z-\bar{\zeta})^2}\right],$$
$$\Omega_0(z)=\frac{1}{16\pi(1-v)}\left[\frac{(3-4v)\bar{p}}{z-\zeta}+\frac{\bar{p}}{z-\bar{\zeta}}+\frac{p(\zeta-\bar{\zeta})}{(z-\zeta)^2}-\frac{\bar{p}\sqrt{\bar{\zeta}}}{\sqrt{z}(z-\bar{\zeta})}+\frac{(3-4v)\bar{p}\sqrt{\zeta}}{\sqrt{z}(z-\zeta)}+\frac{p(\zeta-\bar{\zeta})(z+\zeta)}{2\sqrt{z}\sqrt{\zeta}(z-\zeta)^2}\right], \quad (10)$$

where v is the Poisson's ratio. Equating all terms of the order λ in Eq. (3) and solving the first order equation, we obtain

$$\Phi_I(z)+\bar{\Omega}_I(z)=\frac{1}{\pi i\sqrt{z}}\int_{-\infty}^{0}\frac{\sqrt{-t}\left[\lambda(t)g(t)\right]'}{t-z}dt, \quad (11)$$

in which

$$g(t)=\frac{-1}{4\pi(1-v)}\mathrm{Re}\left[\frac{p}{t-\zeta}+\frac{(3-4v)p}{t-\bar{\zeta}}+\frac{p(\zeta-\bar{\zeta})}{(t-\zeta)^2}\right], \quad (12)$$

where Re denotes the real part. The stress intensity factors at the crack tip can be evaluated from Eqs. (1), (10) and (11), which results in

$$K_I=K_I^0-\frac{3}{2}\omega K_{II}^0+O(\lambda^2),$$
$$K_{II}=K_{II}^0+\frac{1}{2}\omega K_I^0-\sqrt{\frac{2}{\pi}}\int_{-\infty}^{0}\frac{\left[\lambda(t)g(t)\right]'}{\sqrt{-t}}dt+O(\lambda^2), \quad (13)$$

where K_I^0 and K_{II}^0 are the zero order stress intensity factors for the crack with $\lambda=0$, given by

$$K_I^0+iK_{II}^0=\frac{1}{4\sqrt{2\pi}(1-v)}\left[\frac{\bar{p}}{\sqrt{\bar{\zeta}}}-\frac{(3-4v)\bar{p}}{\sqrt{\zeta}}+\frac{p(\zeta-\bar{\zeta})}{2\zeta\sqrt{\zeta}}\right]. \quad (14)$$

The stress intensity factors Eq. (13) can be rewritten in the form

$$K_M = h_j^M p_j,\ M=(I,\ II) \qquad (15)$$

in which h_j^M $M=(I,\ II)$ is the weight function for the curved crack, given by

$$h_j^I = h_j^{0I} - \frac{3}{2}\omega h_j^{0II},$$
$$h_j^{II} = h_j^{0II} + \frac{1}{2}\omega h_j^{0I} - \sqrt{\frac{2}{\pi}}\int_{-\infty}^{0}\frac{[\lambda(t)g_j(t)]'}{\sqrt{-t}}dt, \qquad (16)$$

where

$$h_1^{0I} + ih_1^{0II} = \frac{1}{4\sqrt{2\pi}(1-v)}\left[\frac{1}{\sqrt{\bar{\zeta}}} - \frac{(3-4v)}{\sqrt{\zeta}} + \frac{(\zeta-\bar{\zeta})}{2\zeta\sqrt{\zeta}}\right],$$
$$h_2^{0I} + ih_2^{0II} = \frac{i}{4\sqrt{2\pi}(1-v)}\left[\frac{-1}{\sqrt{\bar{\zeta}}} + \frac{(3-4v)}{\sqrt{\zeta}} + \frac{(\zeta-\bar{\zeta})}{2\zeta\sqrt{\zeta}}\right], \qquad (17)$$
$$g_j(t) = \frac{-1}{4\pi(1-v)}Re\left[(\delta_{1j}+i\delta_{2j})\left\{\frac{1}{t-\zeta} + \frac{(3-4v)}{t-\bar{\zeta}} + \frac{(\zeta-\bar{\zeta})}{(t-\zeta)^2}\right\}\right].$$

4. Curved Crack with Eigenstrains

We now consider the problem of a slightly curved crack in the linear material containing a region A in which an eigenstrain ε_{ij}^* is prescribed, as shown in Fig. 3. The remote stress fields at infinity are given by Eq. (3). The incompatibility of the eigenstrain distributed in the region A around the crack tip creates stresses. The stress intensity factor induced by the eigenstrain can be evaluated by a body force analogy. Using the body force analogy, it can be shown that the stress field produced by the eigenstrain distribution is the same as that produced in the body subjected to a certain effective body force field. The stress intensity factors induced by the ε_{ij}^* can be evaluated by the body force analogy. The stress intensity factors can be expressed as

$$K_I = K_I^\infty - \frac{3}{2}\omega K_{II}^\infty + \Delta K_I + O(\lambda^2),$$
$$K_{II} = K_{II}^\infty + \frac{1}{2}\omega K_I^\infty - \sqrt{\frac{2}{\pi}}T^\infty\int_{-\infty}^{0}\frac{\lambda'(t)}{\sqrt{-t}}dt + \Delta K_{II} + O(\lambda^2), \qquad (18)$$

where ΔK_I and ΔK_{II} are the change of the crack tip stress intensity factors induced by the eigenstrain. The explicit form of ΔK_I and ΔK_{II} is expresses as

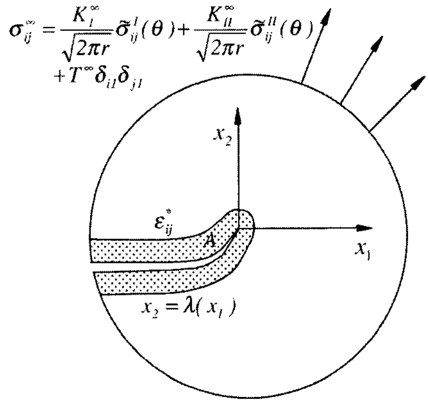


Fig. 3 Curved crack with an eigenstrain

$$\Delta K_M = \frac{E}{1+v}\int_A U_{kl}^M \varepsilon_{kl}^* dA,\ (M=I,\ II). \qquad (19)$$

Here E is the Young's modulus, and U_{kl}^I and U_{kl}^{II} are the near tip weight functions given by

$$U_{kl}^M = \frac{v}{1-2v}h_{j,j}^M\delta_{kl} + \frac{1}{2}(h_{k,l}^M + h_{l,k}^M), \qquad (M=I,\ II), \qquad (20)$$

where $(\ )_{,j} = \partial/\partial\xi_j$.

Making use of Eqs. (19) and (20) together with Eq. (17), we have the change of stress intensity factors in the form

$$\Delta K_I = \Delta K_I^* - \frac{3}{2}\omega\Delta K_{II}^{*0},$$
$$\Delta K_{II} = \Delta K_{II}^* + \frac{1}{2}\omega\Delta K_I^{*0}, - \sqrt{\frac{2}{\pi}}\int_{-\infty}^{0}\frac{[\lambda(t)\Delta T^0(t)]'}{\sqrt{-t}}dt, \qquad (21)$$

in which

$$\Delta K_M^* = \frac{E}{1+v}\int_A U_{kl}^{0M}\varepsilon_{kl}^* dA,\ (M=I,\ II),$$
$$\Delta K_M^{*0} = \frac{E}{1+v}\int_{A_0} U_{kl}^{0M}\varepsilon_{kl}^* dA,\ (M=I,\ II),$$
$$\Delta T^0(t) = \frac{E}{1+v}\lim_{A_\varepsilon\to\infty}\int_{A_0-A_\varepsilon} V_{kl}\varepsilon_{kl}^* dA + \frac{E}{1+v}\frac{3-4v}{8(1-v)}[\{\varepsilon_{11}^*(t,0^+) + \varepsilon_{11}^*(t,0^-)\} - \{\varepsilon_{22}^*(t,0^+) + \varepsilon_{22}^*(t,0^-)\}]. \qquad (22)$$

Here

$$U^{0M}_{kl}=\frac{v}{1-2v}h^{0M}_{j,j}\delta_{kl}+\frac{1}{2}(h^{0M}_{k,l}+h^{0M}_{l,k}),\ (M=I,\ II),$$
$$V_{kl}=\frac{v}{1-2v}g_{j,j}\delta_{kl}+\frac{1}{2}(g_{k,l}+g_{l,k}). \tag{23}$$

A_0 is the area with the prescribed eigenstrain ε^*_{ij} for the straight crack ($\lambda=0$) and A_δ is the circular area with a vanishingly small radius δ and the center at $\zeta=t$. Using Eqs. (17), (22) and (23), it can be shown that $\Delta T^0(t)$ is rewritten in the form :

$$\begin{aligned}\Delta T^0(t)=&\frac{E}{4\pi(1-v^2)}\int_{A_0}\frac{1}{\rho^2}[-2\varepsilon^*_{11}(\cos 4\Theta+\cos 2\Theta)\\&-4\varepsilon^*_{12}\sin 4\Theta+2\varepsilon^*_{22}(\cos 4\Theta-\cos 2\Theta)]dA\\&+\frac{E}{1+v}\frac{3-4v}{8(1-v)}[\{\varepsilon^*_{11}(t,0^+)+\varepsilon^*_{11}(t,0^-)\}\\&-\{\varepsilon^*_{22}(t,0^+)+\varepsilon^*_{22}(t,0^-)\}]\end{aligned} \tag{24}$$

where $\zeta-t=\rho e^{i\Theta}$.

5. Crack Path Stability

The first order solution for the slightly curved crack can be used to predict the path of a fracture as in the work of Cotterell and Rice (1980). We restrict our consideration to the imperfection for which K^∞_{II} differs slightly from zero. It is assumed that the crack propagates in the direction of $K_{II}=0$. Assume the equation to the curved crack path in the form (Karihaloo et al., 1981)

$$\lambda(s)=\begin{cases}\lambda_0+\vartheta_0 s+as^{3/2}+O(b^2), & 0\le s<b\\ \lambda_0, & s<0\end{cases}, \tag{25}$$

where λ_0, ϑ_0 and a are constants, b is a small extension from a preexisting straight crack and $s=x_1+b$. It is noted that the slope of the crack surface at the crack tip is

$$\omega=\vartheta_0+\frac{3}{2}a\sqrt{b}+O(b) \tag{26}$$

When the magnitude of K^∞_{II} has the first order of smallness, it can be shown from Eqs. (18), (21) and (25) that

$$K_{II}=K^*_{II}+\frac{1}{2}\vartheta_0K^*_I+\left(\frac{3}{4}aK^*_I-2\sqrt{\frac{2}{\pi}}T^*\vartheta_0\right)\sqrt{b}+O(b), \tag{27}$$

in which

$$\begin{aligned}K^*_I&=K^\infty_I+\Delta K^{*0}_I,\\K^*_{II}&=K^\infty_{II}+\Delta K^*_{II},\\T^*&=T^\infty+\Delta T^*,\end{aligned} \tag{28}$$

where $\Delta T^*=\Delta T^0(0^-)$. Now, imposing the criterion $K_{II}=0$ along the crack growth path, the constants ϑ_0 and a are determined from the ordered conditions as follows :

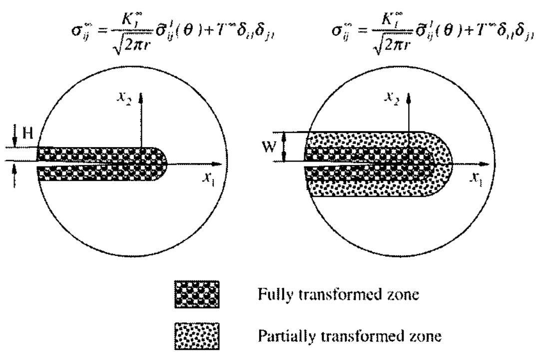
$$\begin{aligned}b^0 &: \vartheta_0=-\frac{2K^*_{II}}{K^*_I},\\ b^{1/2} &: a=\frac{8}{3}\sqrt{\frac{2}{\pi}}\frac{T^*}{K^*_I}\vartheta_0.\end{aligned} \tag{29}$$

The result shows that the slope of crack extension increases as the crack propagates when $T^*>0$, whereas it decreases when $T^*<0$. Thus, for mode I loading, the crack path is stable when $T^*<0$ and unstable when $T^*>0$. Before proceeding, it is worth mentioning that the crack path stability parameter, T^* does not have the meaning of the T-stress for the crack with eigenstrain. The T^* stress, however, reduces to the T stress when there is no region with eigenstrains.

6. T^* Stress

6.1 Elastic material with a transformation zone

Consider a crack with steady state growth in a composite, as shown in Fig. 4. The composite consists of a linear elastic matrix material and particles which undergo a dilatation. The remote stress



(a) (b)

Fig. 4 Steady state crack growth in a composite
(a) Fully transformed zone
(b) Partially transformed zone

fields at infinity are given by

$$\sigma_{ij}^{\infty}=\frac{K_I^{\infty}}{\sqrt{2\pi r}}\tilde{\sigma}_{ij}^{I}(\theta)+T^{\infty}\delta_{i1}\delta_{j1}, \tag{30}$$

where $\tilde{\sigma}_{ij}^{I}(\theta)$ is the universal distribution function of stress. According to Budiansky et al. (1983), the dilatant transformation criterion has the following form :

$$\sigma_m=\sigma_m^c, \tag{31}$$

where σ_m is the mean stress given by $\sigma_m=1/3\sigma_{kk}$ ($k=1, 2, 3$) and σ_m^c is the critical mean stress. In the transformed zone, the eigenstrain tensor induced by the dilatant transformation can be written as

$$\varepsilon_{ij}^{*}=\varepsilon^{*}\delta_{ij}. \tag{32}$$

Here

$$\varepsilon^{*}=\frac{1}{3}(1+v)e^{*}, \tag{33}$$

where e^* is the transformed dilatation of the composite.

For a super-critically transforming material, the composite undergoes the full transformation and the eigenstrain has a constant value in the fully transformed zone, $\varepsilon^*=\varepsilon^T=(1+v)e^T/3$ where e^T is the complete transformed dilatation of the composite. Applying the superposition as shown in Fig. 5, we can obtain the solution for the crack in the material with the critical mean stress σ_m^c under the remote stress fields Eq. (30) using the solution for the crack in the material with the critical mean stress σ_m^{Tc}, given by $\sigma_m^{Tc}=\sigma_m^c-(1+v)T^{\infty}/3$, under the remote stress fields Eq. (30) with $T^{\infty}=0$. Here $\sigma_m^{Tc}>0$ is assumed. Budiansky et al. (1983) solved the problem with $T^{\infty}=0$. Their solution is used in evaluating the T^* stress in this study. The form of the boundary of the fully transformed zone is expressed as

$$r=R(\theta)=H\tilde{R}(\theta) \tag{34}$$

where H is the half height of the wake and

$$\tilde{R}(\theta)=\begin{cases}\frac{8}{3\sqrt{3}}\cos^2\frac{\theta}{2}, & 0\leq|\theta|\leq\frac{\pi}{3}\\ \frac{1}{\sin|\theta|}, & \frac{\pi}{3}\leq|\theta|\leq\pi\end{cases}. \tag{35}$$

Although the form of the transformed zone boundary depends on the redistribution of the stress due to the transformed dilatation, Budiansky et al. (1983) showed that the shape of the transformed zone $\tilde{R}(\theta)$ obtained by using the stress field Eq. (30) is a very good approximation and the half height H of the zone depends on the material parameters. From Eqs. (24), (34) and (35), it can be shown that

$$\Delta T^*=-\frac{E\varepsilon^T}{\pi(1-v^2)}\int_{-\pi}^{\pi}\cos 2\theta\ln\tilde{R}(\theta)\,d\theta. \tag{36}$$

It is noted that ΔT^* does not depend on H. Performing numerically the integral in Eq. (36), we have from Eq. (28)

$$T^*=T^{\infty}-\frac{1.91E\varepsilon^T}{\pi(1-v^2)}. \tag{37}$$

Next, we consider a crack in a composite with both a fully transformed zone and a partially transformed zone, as shown in Fig. 4. The distribution of eigenstrain in the transformed zone is assumed to be expressed as

$$\varepsilon^*=\begin{cases}\varepsilon^T, & r^*\leq H\\ \varepsilon^*(r^*), & H\leq r^*\leq W\end{cases} \tag{38}$$

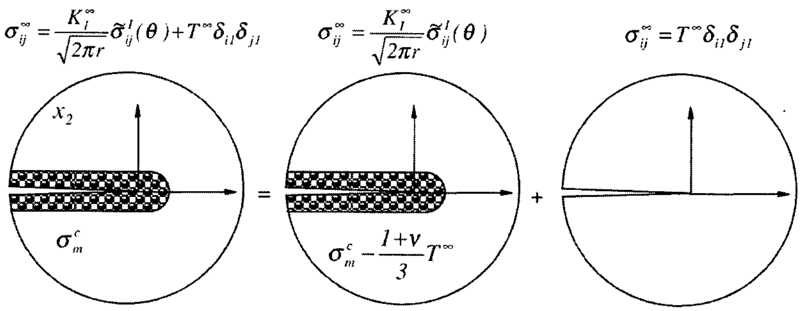


Fig. 5 Application of superposition

Here

$$r^* = \frac{r}{\hat{R}(\theta)}. \qquad (39)$$

$r^* = H$ and $r^* = W$ are the boundaries of fully transformed zone and partially transformed zone, respectively. $\hat{R}(\theta)$ is a dimensionless function describing the shape of the transformed zone boundary. Making use of Eqs. (24), (38) and (39), we obtain

$$\Delta T^* = -\frac{E\varepsilon^T}{\pi(1-v^2)} \int_{-\pi}^{\pi} \cos 2\theta \ln \hat{R}(\theta)\, d\theta. \qquad (40)$$

In obtaining Eq. (40), the relation

$$\int_{A_P} \frac{\varepsilon^*}{r^*} \cos 2\theta dA = \int_H^W \int_{-\pi}^{\pi} \cos 2\theta d\theta \frac{\varepsilon^*(r^*)}{r^*} dr^* = 0, \qquad (41)$$

where A_P is the partially transformed area, has been used.

For a sub-critically transforming material, the composite undergoes the full transformation or partial transformation. Budiansky et al. (1983) solved numerically the problem with $T^\infty = 0$. Their solution showed that the distribution of eigenstrain in the transformed zone is written approximately as Eq. (38) with $\hat{R}(\theta) \approx \tilde{R}(\theta)$. Furthermore, the contribution of the active zone to ΔT^* is less than that of the wake zone. Thus, the predicted result Eq. (37) may be valid approximately for the sub-critically transforming material.

Material parameters for a typical transforming material, for instance, a two phase zirconia system are $E = 200$ GPa, $v = 0.3$ and $e^T = 1.5 \times 10^{-2}$ (Budiansky et al., 1983). The contribution of transformation to T^* is about $\Delta T^* = -870$ MPa. For mode I loading, the straight crack path is thus stable when $T^\infty < 870$ MPa. Typically, the magnitude of ΔT^* due to the dilatant transformation is larger than the applied T^∞.

6.2 Ferroelectric material

Consider a crack with steady state growth in a ferroelectric material, as shown in Fig. 6. The remote stress fields at infinity are given by Eq. (30). Applying the superposition, we can obtain the solution for the crack in the material under the remote stress fields Eq. (30) using the solution

$$\sigma_{ij}^\infty = \frac{K_I^\infty}{\sqrt{2\pi r}} \tilde{\sigma}_{ij}^I(\theta) + T^\infty \delta_{i1}\delta_{j1}$$

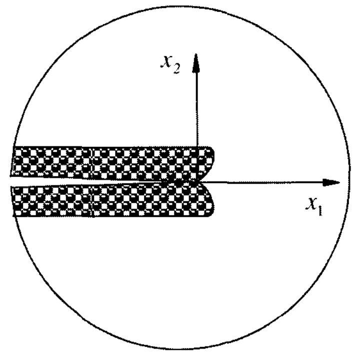


Fig. 6 Steady state crack growth in a ferroelectric material

for the crack in the material under the remote stress fields Eq. (30) with $T^\infty = 0$. Reece and Guiu (2002) solved the problem for the material ideally poled along the positive x_1 direction under $T^\infty = 0$. Their solution is used in evaluating the T^* stress in this study. The form of the boundary of the domain switching zone is expressed as

$$r = R(\theta) = H\hat{R}(\theta), \qquad (42)$$

where H is the half height of the wake and

$$\hat{R}(\theta) = \begin{cases} \dfrac{\sin^2\theta \sin^2 \dfrac{3\theta}{2}}{\sin^3 \dfrac{2\pi}{5} \sin^2 \dfrac{3\pi}{5}}, & 0 \le |\theta| \le \dfrac{2\pi}{5} \\ \dfrac{1}{\sin|\theta|}, & \dfrac{2\pi}{5} \le |\theta| \le \pi \end{cases}. \qquad (43)$$

In obtaining Eq. (43), the switching criterion based on shear stress has been used and $2\tau_C + T^\infty > 0$ where τ_C is the critical shear stress is assumed. In the domain switching zone, the eigenstrain tensor due to the domain switching is given by

$$\varepsilon_{11}^* = -\varepsilon_S^T, \ \varepsilon_{22}^* = \varepsilon_S^T, \ \varepsilon_{11}^* = 0. \qquad (44)$$

Here $\varepsilon_S^T = c\varepsilon_S$ where c is the volume fraction of switching and ε_S is the spontaneous strain associated with domain switching. From Eqs. (24), (42) and (44), it can be shown that

$$\Delta T^* = \frac{E\varepsilon_S^T}{\pi(1-v^2)} \int_{-\pi}^{\pi} \cos 4\theta \ln \hat{R}(\theta)\, d\theta - \frac{3-4v}{2} \frac{E\varepsilon_S^T}{1-v^2}. \qquad (45)$$

Performing numerically the integral in Eq. (45), we have from Eq. (28)

$$T^* = T^\infty - \left(\frac{2.56}{\pi} + \frac{3-4v}{2}\right)\frac{E\varepsilon_s^T}{1-v^2}. \qquad (46)$$

Material parameters for a typical PZT material are $E=77$ GPa, $v=0.25$, $\varepsilon_s=0.005$ and $c=0.1$ (Schaufele and Hardtl, 1996; Reece and Guiu, 2002). Although the value of applied T^∞ stress is positive, the straight crack path is stable for mode I loading when $T^\infty < 75$ MPa.

7. Concluding Remarks

A slightly curved crack with an eigenstrain in a zone attending the crack tip is investigated. Solutions for a slightly curved crack in a linear material subjected to asymptotic loading at infinity as well as for a slightly curved crack in a linear material with a concentrated force are obtained from perturbation analyses, which are accurate to the first order of the parameter representing the non-straightness. Stress intensity factors for a slightly curved crack with an eigenstrain are obtained from the perturbation solutions by using a body force analogy. Of particular interest is the effect of the eigenstrain on the crack path stability. A new crack path stability parameter T^* is proposed for a crack with an eigenstrain. It is shown that for mode I loading, the straight crack path is stable when $T^*<0$ and unstable when $T^*>0$. The path stability of a straight crack with steady state growth in a transforming material as well as in a ferroelectric material is examined. The T^* stresses for the cracks with steady state growth under mode I loading are obtained. The results show that the change of T^* due to the eigenstrains induced by dilatant transformation and domain switching has a negative value. This implies that the straight crack path may be stable under mode I loading when the applied T^∞ stress has a small positive value.

Acknowledgements

This work was supported by Inha University Research Grant (Inha-31429).

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