Dynamic Behaviors of an Elastically Restrained Beam Carrying a Moving Mass

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Dynamic responses of a simply supported beam with a translational spring carrying a moving mass are studied. Governing equations of motion including all the inertia effects of a moving mass are derived by employing the Galerkin's mode summation method, and solved by using the Runge-Kutta integral method. Numerical solutions for dynamic responses of a beam are obtained for various cases by changing parameters of the spring stiffness, the spring position, the mass ratio and the velocity ratio of a moving mass. Some experiments are conducted to verify the numerical results obtained. Experimental results for the dynamic responses of the test beam have a good agreement with numerical ones.

Key Words : Dynamic Responses, Runge-Kutta Integral Method, Galerkin's Mode Summation Method, Simply Supported Beam, Translational Spring

1. Introduction

The dynamic behaviors of a beam structure carrying moving loads have long been an interesting subject for many researchers. The importance of this subject can easily be recognized in the field of transportation. Guide-ways, bridges, overhead cranes, cable-ways, rails, tunnels and pipelines are the typical examples of the structure to be designed to support moving masses.

The moving mass problems were initially studied without considering the effect of the load inertia, so-called a moving force approximation, by many researchers (Timoshenko, 1922 ; Ayre et al., 1950 ; Fryba, 1972). Ting and his co-researchers (Ting et al., 1974) employed the static Green's function to solve the dynamic responses of beams under a moving mass. Sadiku and Leipholz (1987) studied the dynamics of elastic systems with moving concentrated masses.

Esmailzadeh and Ghorash (1992) investigated the dynamic behavior of a beam taking into account the inertia effect of moving mass. Michaltsos et al. (1996) presented a closed form solution for the dynamic response of a simply supported uniform beam under a moving load of constant mag-

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nitude and velocity by including the effect of its mass. Lee (1996) formulated the equation of motion for a Timoshenko beam acted upon by a concentrated mass moving with a constant speed. He also pointed out the importance of the contact force between the mass and the beam during the motion.

In parallel with the above studies, the dynamic response of a beam on an elastic foundation subjected to moving loads has also been studied. Thambiratram and Zhuge (1996) applied the finite element method to conduct the dynamic analysis of beams on an elastic foundation under moving loads. They presented the effect of the speed of the moving load, the foundation stiffness and the length of the beam on the response of the beam. Lin (1997) performed the vibration analysis of beams traversed by uniform partially distributed moving masses. Foda and Abduljabbar (1998) used the dynamic Green function to investigate the response of a beam structure to a moving mass.

Recently, Michaltsos (2002) derived the equation of motion of a simply supported beam under a moving mass including various inertia effects of the moving mass, and presented the closed form solution.

In this paper, the effect of a moving mass on the dynamic response of a simply supported beam with a spring support was investigated. All the effects of a moving mass such as centrifugal, Coliolis, gravitational, and inertia and the initial deflection of a beam are taken into account in the derivation of the equations of motion. The Runge-Kutta integration technique was used to investigate the dynamic response of a beam. Many different cases are investigated by varying parameter values of the velocity and the magnitude of a moving mass, the stiffness and the position of a spring support. Finally, some experiments are conducted to verify the numerical solutions.

2. Mathematical Model and Formulation

A simply supported beam with a spring support is depicted in Figure 1. The beam is of length L, mass per unit length L, and flexural rigidity EI.

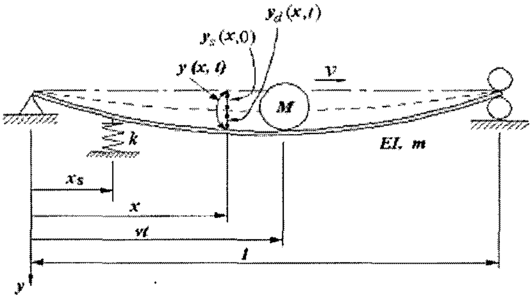


Fig. 1 A mathematical model of an elastically restrained beam subjected to a moving mass

A moving mass of mass with a constant velocity v is travelling on the beam. A translational spring of spring stiffness k is located at the position X_s.

The equation of motion for the model shown in Figure 1 can be written as

$$\begin{aligned} &EI\frac{\partial^4 y(x,t)}{\partial^4 x^4}+m\frac{\partial^2 y(x,t)}{\partial t^2} \\ &=mg+Mg\delta(x-vt) \\ &\quad -M\left[\frac{\partial^2 y(x,t)}{\partial x^2}v^2+2\frac{\partial^2 y(x,t)}{\partial x\partial t}v\right. \\ &\quad \left.+\frac{\partial y(x,t)}{\partial x}a+\frac{\partial^2 y(x,t)}{\partial t^2}\right]\delta(x-vt) \end{aligned} \tag{1}$$

where, g is the gravity, $\delta(x-vt)$ is the Dirac delta function, and a is the acceleration of the moving mass. The solution of the Eq. (1), $y(x,t)$, is assumed to be the sum of the solution for static deflection $y_s(x)$ and that for dynamic deflection $y_d(x,t)$.

$$y(x,t)=y_s(x)+y_d(x,t) \tag{2}$$

Introducing the dimensionless parameter $\xi=x/l$, Eq. (1) can be rewritten as

$$\begin{aligned} &\frac{EI}{l^4}\frac{\partial^4 y(\xi,t)}{\partial\xi^4}+m\frac{\partial^2 y(\xi,t)}{\partial t^2} \\ &=mg+\frac{M}{l}\left[g-\left\{\bar{v}^2\frac{\partial^2 y(\xi,t)}{\partial\xi^2}+2\bar{v}\frac{\partial^2 y(\xi,t)}{\partial\xi\partial t}\right.\right. \\ &\quad \left.\left.+\bar{a}\frac{\partial y(\xi,t)}{\partial\xi}+\frac{\partial^2 y(\xi,t)}{\partial t^2}\right\}\right]\delta(\xi-\bar{v}t) \end{aligned} \tag{3}$$

where, $\bar{v}=v/l$, $\bar{a}=a/l$.

Employing the Galerkin's mode summation method, dimensionless solutions for Eq. (3) are assumed to have the following form

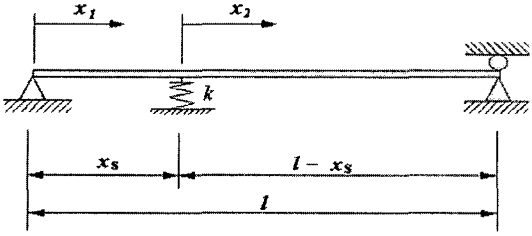


Fig. 2 A new coordinate system for a beam with a spring support

$$y_s(\xi)=\sum_{i=1}^{\infty}A_i\phi_i(\xi) \tag{4}$$

$$y_d(\xi,t)=\sum_{i=1}^{\infty}q_i(t)\,\phi_i(\xi) \tag{5}$$

$$y(\xi,t)=\sum_{i=1}^{\infty}[A_i+q_i(t)]\,\phi_i(\xi) \tag{6}$$

where, the shape function $\phi_i(\xi)$ is determined by the shapes and the boundary conditions of a beam. For the model of having a spring support located at the position of x_s, a new coordinate system shown in Fig. 2 is introduced to express the shape function $\phi_i(\xi)$.

Eqs. (7) and (8) are shape functions that satisfy both the boundary conditions and the shapes of the beam.

$$\begin{aligned}\phi_{i,1}(\xi_1)&=A_{i,1}\sin\beta_i\xi_1+B_{i,1}\cos\beta_i\xi_1\\&\quad+C_{i,1}\sinh\beta_i\xi_1+D_{i,1}\cosh\beta_i\xi_1,\\&\qquad\text{for }(0\le\xi_1\le\xi_s)\end{aligned} \tag{7}$$

$$\begin{aligned}\phi_{i,2}(\xi_2)&=A_{i,2}\sin\beta_i\xi_2+B_{i,2}\cos\beta_i\xi_2\\&\quad+C_{i,2}\sinh\beta_i\xi_2+D_{i,2}\cosh\beta_i\xi_2,\\&\qquad\text{for }(0\le\xi_2\le\xi_s)\end{aligned} \tag{8}$$

where, the eigen-value β_i has the following relationship with the i-th circular natural frequency of the beam.

$$\omega_i^2=\frac{EI}{m}\left(\frac{\beta_i}{l}\right)^4 \tag{9}$$

The associated boundary conditions are

$$\begin{gathered}\phi_{i,1}(0)=0,\ \frac{\partial^2\phi_{i,1}(0)}{\partial\xi_1^2}=0,\\ \phi_{i,1}(\xi_s)=\phi_{i,2}(0)\end{gathered} \tag{10}$$

$$\begin{gathered}\frac{\partial\phi_{i,1}(\xi_s)}{\partial\xi_1}=\frac{\partial\xi_{i,2}(0)}{\partial\xi_2},\\ \frac{\partial^3\phi_{i,1}(\xi_s)}{\partial\xi_1^3}=\frac{kl^3}{EI}+\frac{\partial^3\phi_{i,2}(0)}{\partial\xi_2^3}\end{gathered} \tag{11}$$

$$\begin{gathered}\frac{\partial^2\phi_{i,1}(\xi_s)}{\partial\xi_1^2}=\frac{\partial^2\phi_{i,2}(0)}{\partial\xi_2^2},\ \phi_{i,2}(1-\xi_s)=0,\\ \frac{\partial^2\phi_{i,2}(1-\xi_s)}{\partial\xi_2^2}=0\end{gathered} \tag{12}$$

Substituting Eq. (6) into Eq. (3), eliminating differential terms with respect to time, and performing inner product $\phi_{n,k}(\xi_k)$ yield

$$\begin{aligned}&\sum_{i=1}^{\infty}\left[\int_0^1\phi_{i,k}(\xi_k)\,\phi_{n,k}(\xi_k)\,d\xi_k+\frac{M}{m_kl}\phi_{i,k}(\bar{v}t)\,\phi_{n,k}(\bar{v}t)\right]\ddot{q}_i(t)\\&+\sum_{i=1}^{\infty}\left[\frac{M}{m_kl}2\bar{v}\phi'_{i,k}(\bar{v}t)\,\phi_{n,k}(\bar{v}t)\right]\dot{q}_i(t)\\&+\sum_{i=1}^{\infty}\left[\frac{EI}{ml^4}\beta_i^4\int_0^1\phi_{i,k}(\xi_k)\,\phi_{n,k}(\xi_k)\,d\xi_k\right.\\&\left.+\frac{M}{ml}\bar{v}^2\phi''_{i,k}(\bar{v}t)\,\phi_{n,k}(\bar{v}t)+\frac{M}{ml}\bar{a}\phi'_{i,k}(\bar{v}t)\,\phi_{n,k}(\bar{v}t)\right]q_i(t)\\&=\frac{M}{ml}\left[g-\bar{v}^2\sum_{i=1}^{\infty}A_i\phi''_{i,k}(\bar{v}t)-\bar{a}\sum_{i=1}^{\infty}A_i\phi'_{i,k}(\bar{v}t)\right]\phi_{n,k}(\bar{v}t),\\&\qquad\qquad\text{(for }k=1,2)\end{aligned} \tag{13}$$

From Eq. (13), the equation of motion can be written in matrix form as

$$\begin{aligned}&[M(t)]\{\ddot{\varphi}(t)\}+[C(t)]\{\dot{\varphi}(t)\}\\&+[K(t)]\{\varphi(t)\}=\{f(t)\}\end{aligned} \tag{14}$$

where, the matrix components of Eq. (14) are

$$\begin{aligned}m_{ij}(t)&=\int_0^{\xi_s}\phi_{i,1}(\xi_1)\,\phi_{j,1}(\xi_1)\,d\xi_1\\&\quad+\int_0^{1-\xi_s}\phi_{i,1}(\xi_2)\,\phi_{j,2}(\xi_2)\,d\xi_2+\mu\phi_{i,k}(\bar{v}t)\,\phi_{j,k}(\bar{v}t)\\ c_{ij}(t)&=2\mu\bar{v}\phi'_{i,k}(\bar{v}t)\,\phi_{j,k}(\bar{v}t)\\ k_{ij}(t)&=\omega_i^2\left\{\int_0^{\xi_s}\phi_{i,1}(\xi_1)\,\phi_{i,j}(\xi_i)\,d\xi_1\right.\\&\quad\left.+\int_0^{1-\xi_s}\phi_{i,2}(\xi_2)\,\phi_{j,2}(\xi_2)\,d\xi_2\right\}\\&\quad+\mu\bar{v}^2\phi_{i,k}(\bar{v}t)\,\phi''_{j,k}(\bar{v}t)+\mu_k\bar{a}\phi_{i,k}(\bar{v}t)\,\phi'_{j,k}(\bar{v}t)\\ f_i(t)&=\mu\left[g-\bar{v}^2\sum_{n=1}^{\infty}A_n\phi''_{n,k}(\bar{v}t)-\bar{a}\sum_{n=1}^{\infty}A_n\phi'_{n,k}(\bar{v}t)\right]\phi_{j,k}(\bar{v}t)\end{aligned} \tag{15}$$

where, ξ_s is the dimensionless spring position, $\mu=M/ml$ is the ratio of the moving mass to the mass of the beam.

3. Numerical Results

In order to study the dynamic response of a simply-supported beam with a spring support, the state variable $\varphi(t)$ of Eq. (14) was determined by using Runge-Kutta integration method. For the numerical analyses, dynamic responses were investigated by varying parameters of the mass ratio μ, the dimensionless velocity ratio $v_0 = v/v_{cr}$, the dimensionless spring position ξ, and the dimensionless spring stiffness K.

Figures 3~5 show the dynamic responses of the beam when the dimensionless spring stiffness K and the dimensionless spring position ξ_s vary for $v_0 = \mu = 0.1$.

From these figures, it is found that the maximum dynamic response of the beam increases as the dimensionless spring stiffness K decreases. In Fig. 3 and 4 when $\xi_s = 0.2$ and $\xi_s = 0.5$ respectively, the location of the maximum dynamic deflection shifts to the right side of the beam as the dimensionless spring stiffness K increases.

For $\xi_s = 0.8$ of Fig. 5, however, the location of the maximum deflection shifts slightly to the left side of the beam as K increases.

Figures 6~8 show the dynamic responses of

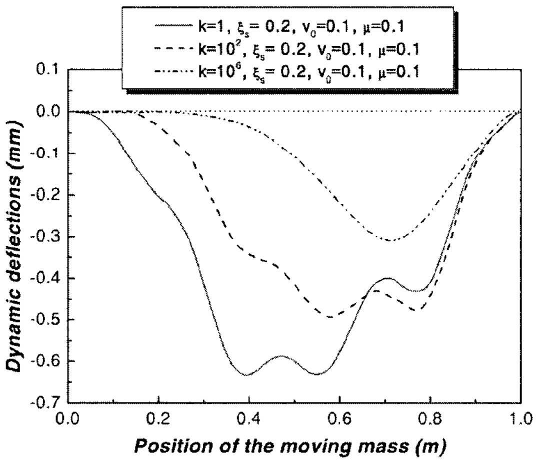


Fig. 3 Dynamic deflections at the position of the moving mass ($v_0 = 0.1$, $\mu = 0.1$, $\xi_s = 0.2$)

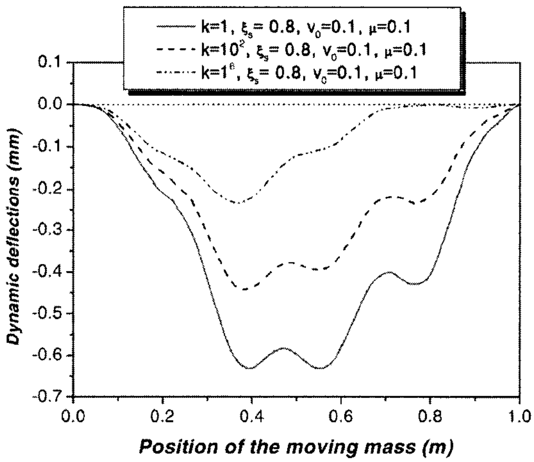


Fig. 5 Dynamic deflections at the position of the moving mass ($v_0 = 0.1$, $\mu = 0.1$, $\xi_s = 0.8$)

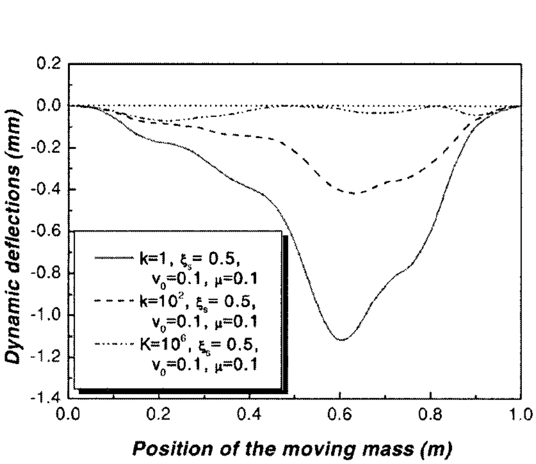


Fig. 4 Dynamic deflections at the position of the moving mass ($v_0 = 0.1$, $\mu = 0.1$, $\xi_s = 0.5$)

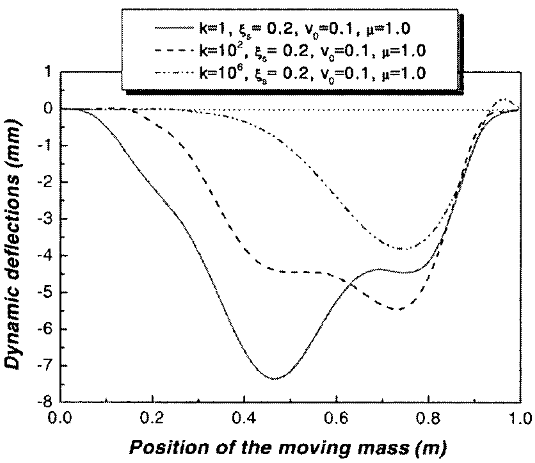


Fig. 6 Dynamic deflections at the position of the moving mass ($v_0 = 0.1$, $\mu = 1.0$, $\xi_s = 0.2$)

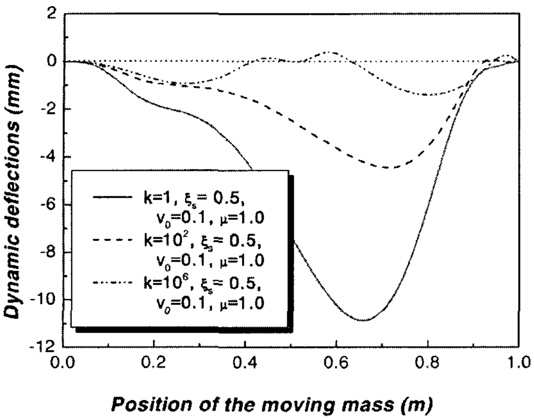


Fig. 7 Dynamic deflections at the position of the moving mass (v_0=0.1, μ=1.0, ξ_s=0.5)

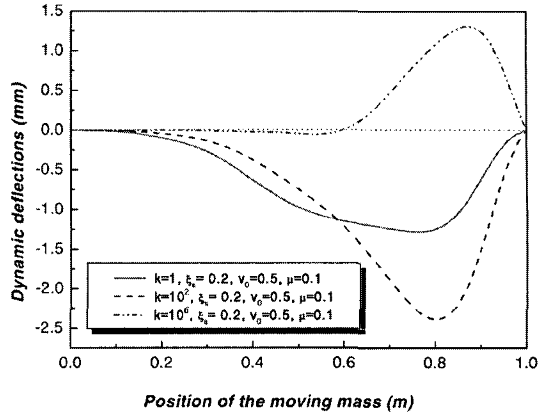


Fig. 9 Dynamic deflections at the position of the moving mass (v_0=0.5, μ=0.1, ξ_s=0.2)

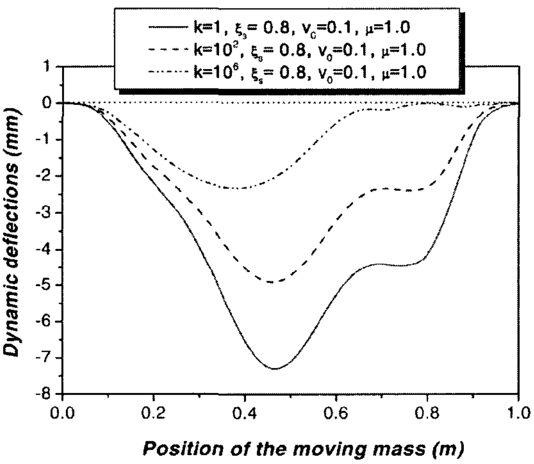


Fig. 8 Dynamic deflections at the position of the moving mass (v_0=0.1, μ=1.0, ξ_s=0.8)

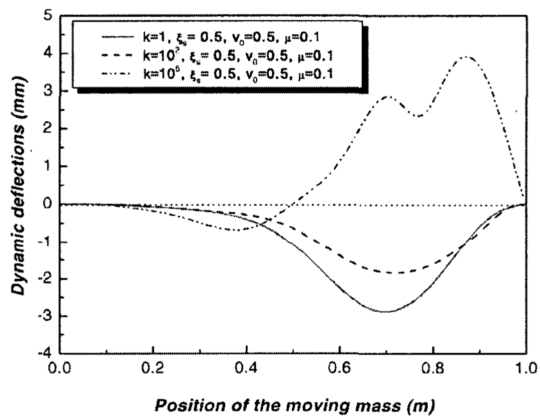


Fig. 10 Dynamic deflections at the position of the moving mass (v_0=0.5, μ=0.1, ξ_s=0.5)

the beam for v_0=0.1 and μ=1.0. In these cases, a similar phenomenon was observed as presented for Figs. 3~5.

The location of the maximum dynamic deflection shifts to the right side of the beam for ξ_s= 0.2 and ξ_s=0.5 while it shifts to the left for ξ_s= 0.8 as the dimensionless spring stiffness K increases. Also it is found that the dynamic response of the beam increases as the dimensionless spring stiffness K decreases.

Figures 9 and 10 show the dynamic deflections of the beam for μ=0.1 and v=0.5. As can be seen in these figures, the maximum dynamic deflection depending on the spring stiffness does not neces-

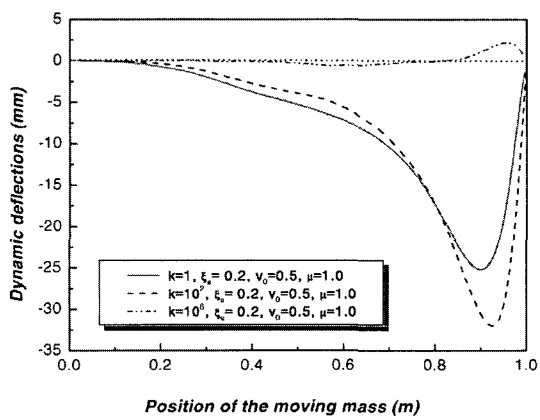


Fig. 11 Dynamic deflections at the position of the moving mass (v_0=0.5, μ=1.0, ξ_s=0.2)

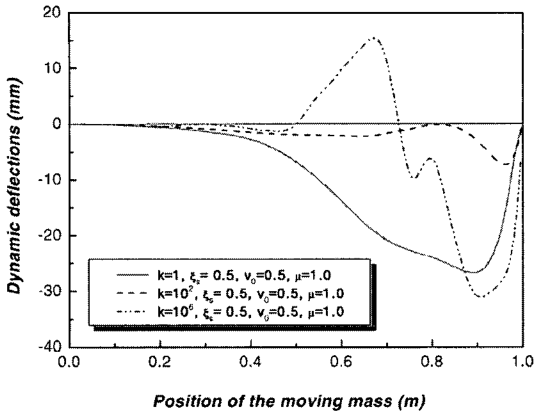


Fig. 12 Dynamic deflections at the position of the moving mass (v_0=0.5, μ=1.0, ξ_s=0.5)

sarily occur with a lower value of spring stiffness.

The dynamic deflections of the beam for μ= 1.0 and v_0=0.5 are shown in Figs. 11 and 12. A similar phenomenon was observed as mentioned for Figs. 9 and 10. Unlikely for a low velocity (v_0=0.1) of the moving mass case, a larger deflection does not necessarily occur with a lower value of spring stiffness for a high velocity (v_0= 0.5) of the moving mass.

4. Experiments

4.1 Experimental setup

An experimental setup was built to verify the dynamic responses of an elastically restrained simply-supported beam carrying a moving mass as shown in Fig. 13.

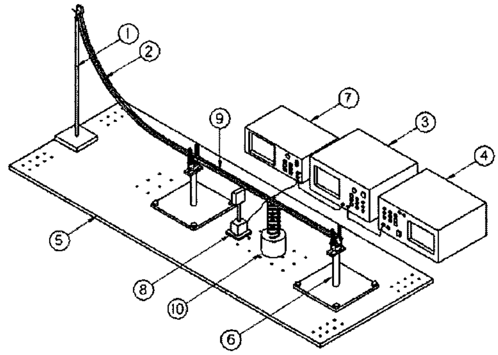
In Fig. 13, number ② is the guide beam attached to the test beam, and number ① is the vertical column supporting the guide beam. The test beam lies on the small bearings of the supporting pole to simulate the simply-supported boundary condition.

Properties of the test beam are presented in Table 1. Three different sizes of a steel ball were used as moving masses. Table 2 shows the specification of moving masses.

Photo. 1 shows the experimental setup.

4.2 Experiments

A series of experiments were conducted to veri-



① Guide beam support column ② Guide beam ③ FFT Analyzer ④ Digital oscilloscope ⑤ Base sheet ⑥ Simply-supported pole ⑦ Laser displacement meter ⑧Laser sensor ⑨ Test beam ⑩ Elastic spring support

Fig. 13 Experimental set up

Table 1 Characteristic properties of the test beam

Beam material	Aluminum 6061
Modules of elasticity (Gpa)	7.07e+10
Density (kg/m³)	2,678.0
Mass (g)	283.0
Length (mm)	1,000.0
Width (mm)	32.0
Thickness (mm)	4.0
Groove width (mm)	10.0
Groove depth (mm)	2.0

Table 2 Specification of the moving mass

Type of moving mass	Weight (g)	Diameter (mm)	Materials
Type 1 : M_1	66.7	25.4	Steel
Type 2 : M_2	150.8	33.3	
Type 3 : M_3	224.8	38.0	

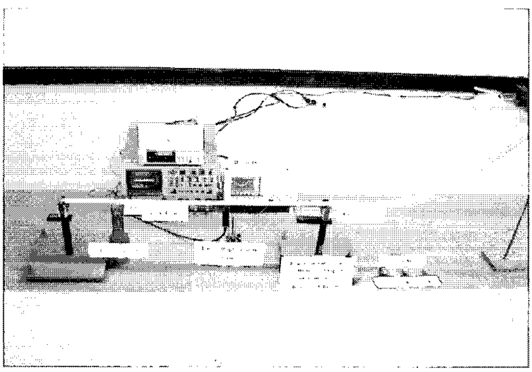


Photo. 1 Photograph of experimental set-up

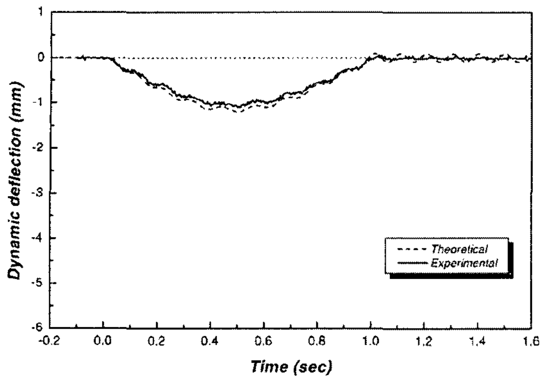


Fig. 14 Comparison theoretical results with experimental ones for beam deflections ($v=1$ (m/s), $M=67$ (g), $\xi_s=0.2$)

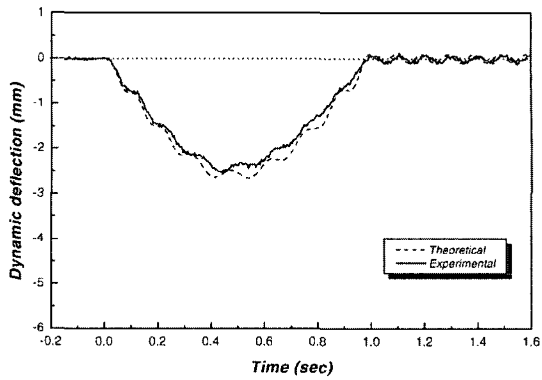


Fig. 16 Comparison theoretical results with experimental ones for beam deflections ($v=1$ (m/s), $M=150.8$ (g), $\xi_s=0.2$

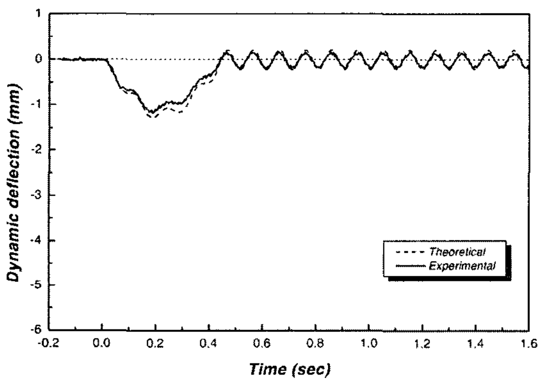


Fig. 15 Comparison theoretical results with experimental ones for beam deflections ($v=2.2$ (m/s), $M=67$ (g), $\xi_s=0.2$)

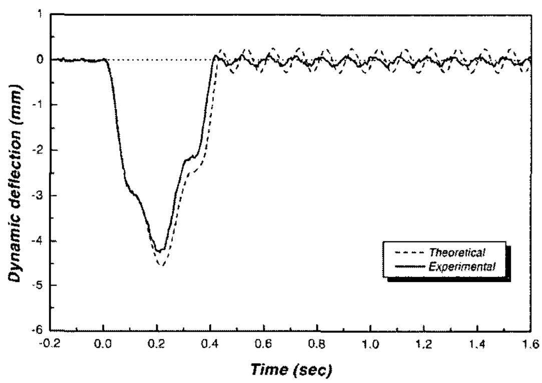


Fig. 17 Comparison theoretical results with experimental ones for beam deflections ($v=2.2$ (m/s), $M=224.8$ (g), $\xi_s=0.8$)

fy the numerical results obtained for dynamic responses of an elastically restrained simply-supported beam with non-symmetric cross section. A spring of spring stiffness $k=404$ (N/m) was chosen for the experimental study. The dynamic deflection of the moving mass was measured by the laser displacement sensor. The sensor is placed at 0.385 (m) from the left end of the test beam. The measured signals for dynamic deflections of the test beam were amplified and displayed on the oscilloscope.

Experiments were performed for various combinations of moving masses and velocities.

Figures 14~17 show the experimental and the numerical results on dynamic responses of the elastically restrained simply-supported beam in the time domain.

A good agreement was obtained as shown in these figures. It is thought that the slight differences between two results are caused by some factors like the friction between the moving mass and the test beam.

5. Conclusions

From the numerical analyses and the experiments on the dynamic responses of an elastically restrained beam, the following conclusions were obtained :

(1) When the velocity of the moving mass is relatively low, the maximum dynamic deflection

of the beam occurs with a lower value of spring stiffness, regardless of spring position. However, the maximum dynamic deflection of the beam does not necessarily occur with a lower value of spring stiffness for a higher velocity of the moving mass.

(2) Numerical results for dynamic responses of the beam considered have a good agreement with experimental ones.

(3) For $\xi_s \leq 0.5$ and a lower velocity of the moving mass, the position of the maximum dynamic deflection shifts to the right side of the beam as the spring stiffness K increases. However, for $\xi_s > 0.5$, the position of the maximum dynamic deflection shifts to the left side of the beam as the spring stiffness K increases.

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