Exact Evaluation of a Sommerfeld Integral for the Impedance Half-Plane Problem

임피던스 반 평면에 대한 Sommerfeld 적분의 Closed-Form 계산

Il-Suek Koh

고 일 석

Abstract

In this paper, a Sommerfeld integral for an impedance half-plane is considered, which is one of classical problems in electromagnetic theory. First, the integral is evaluated into two series representations which are expressed in terms of exponential integral and Lommel function, respectively. Then based on the Lommel function expansion, an exact, closed-form expression of the integral is formulated, written in terms of incomplete Weber integrals. Additionally, based on the exponential integral expansion, an approximate expression of the integral is obtained. Validity of all formulations derived in this paper is demonstrated through comparisons with a numerical integration of the integral for various situations.

요 약

본 논문에서는 임피던스 반 평면 문제에서 나오는 Sommerfeld 적분의 closed-form을 계산한다. 우선 주어진 Sommerfeld 적분을 두 개의 급수 형태로 표현한다. 급수 중 하나는 exponential integral로 알려진 초월 함수로, 다른 것은 Lommel 함수로 표현이 된다. Lommel 함수 표현으로부터 주어진 Sommerfeld 적분의 closed-form을 incomplete Weber integral로 표현한다. incomplete Weber integral은 여러 분야에서 사용되고 있고 많은 성질들이 알려져 있다. 또 exponential integral 함수 표현으로부터 Sommerfeld 적분을 incomplete Gamma 함수로 근사화한다. 구한 모든 식들은 수치 적분 결과와 비교하여 validation을 한다.

Key words : Sommerfeld Integral, Impedance Half-Plane, Dyadic Green's Function

Ⅰ. Introduction

The problem is classic in the electromagnetic field that an infinitesimal dipole radiates over a dielectric half-plane. The exact solution to the problem was given by Arnold Sommerfeld at 1909[1]. After his work, the problem has been paid a significant attention to by many researchers because of its significant importance to many practical applications ranging from wave propagation model to full-wave analysis of RF circuit[2]. In [3] the historical development of the mathematical formulation is well discussed. The original formulation contains so-called Sommerfeld integrals whose exact, closed-form evaluation is not known yet. Hence many techniques, both analytic and numeric, have been proposed to efficiently calculate the integral, whose detail descriptions can be found in [4].

Since the integrand of the Sommerfeld type integral may be highly oscillatory, and decay very slowly, it is very hard to accurately calculate that kind of integrals

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using a direct numerical integration technique such as Gaussian quadrature. To rectify the difficulty, many numerical methods have been proposed, and well discussed in [5]. One interesting approach is proposed by Lindell *et al.*, which is called as exact image theory[6]~[8]. They modified the original Sommerfeld integrand into a more convenient form for a numerical purpose. The modified integrand decays very fast for any configuration of the source and the observation point. For an impedance boundary problem, a similar formulation is known whose numerical efficiency is verified[4].

In this paper, the Sommerfeld integral occurring in an impedance half-plane problem is exactly evaluated. To formulate a closed form of the integral, first two series representations of the integral are derived, and then based on the series expansions, the closed form is formulated. In Section 2 and 3, the two series expansions are derived, and their convergence properties are investigated. Then in Section 4, the closed form is formulated. All obtained formulations are numerically verified for various cases in Section 5.

Ⅱ. Formulation in Terms of Exponential Integral

When an infinitesimal dipole radiates over an impedance half-plane as seen in Fig. 1, the dyadic Green's function contains several Sommerfeld type integrals[4], whose primitive form is given by

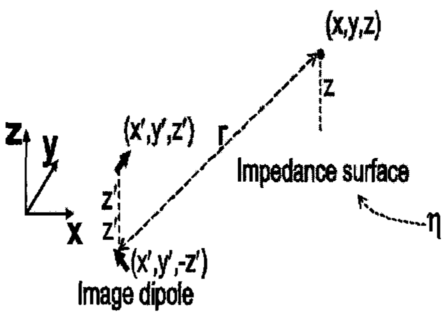
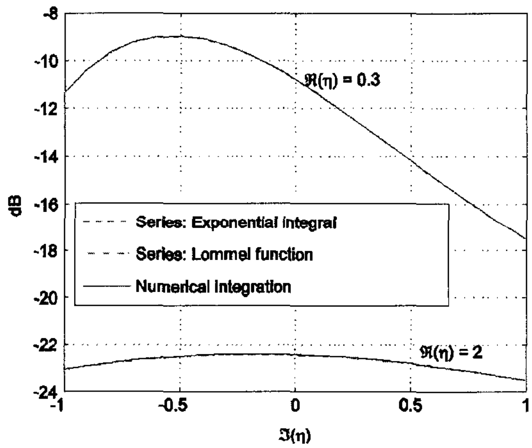


Fig. 1. The geometry of an infinitesimal dipole that radiates above an impedance half plane whose normalized impedance is η.

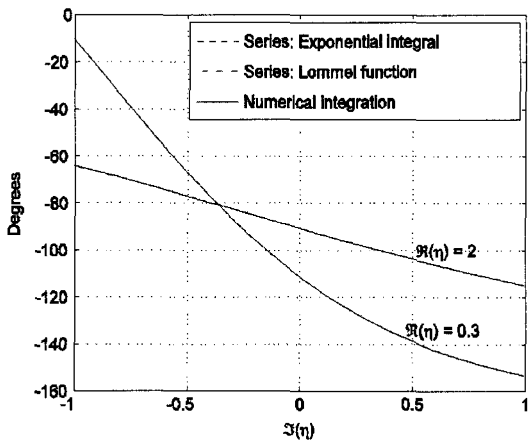
$$\int_0^\infty \frac{k_z-p}{k_z+p} J_0(k_\rho \rho) e^{ik_z(z+z')} \frac{k_\rho}{k_z} dk_\rho \tag{1}$$

where $\rho=\sqrt{(x-x')^2+(y-y')^2}$, $k_z=\sqrt{k_0^2-k_\rho^2}$, $J_0(\cdot)$, is Bessel function of the first kind of zero order, and $p=k_o\eta$ or k_o/η. Here η is the normalized surface impedance. Using the Sommerfeld identity, (1) can be evaluated into two parts as

$$\int_0^\infty \frac{k_z-p}{k_z+p} J_0(k_\rho \rho) e^{ik_z(z+z')} \frac{k_\rho}{k_z} dk_\rho$$
$$= -i\frac{e^{ik_0 r}}{r} - 2p\int_0^\infty \frac{1}{k_z+p} J_0(k_\rho \rho) e^{ik_z(z+z')} \frac{k_\rho}{k_z} dk_\rho \tag{2}$$



(a) Magnitude



(b) Phase

Fig. 2. Comparisons of the series representation and the numerical integration of the Sommerfeld integral as a function of the imaginary part of η with a fixed real part of 0.3 or 2.

where $r=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$. The second integral in (2) can be converted to more convenient forms[4],[9] as

$$\int_0^\infty \frac{1}{k_z+p} J_0(k_\rho \rho) e^{ik_z(z+z')} \frac{k_\rho}{k_z} dk_\rho$$

$$= -i\int_0^\infty \frac{e^{ik_0 R'}}{R'} e^{-p\xi} d\xi \tag{3}$$

$$= e^{-ip(z+z')} \int \frac{e^{ik_0 r}}{r} e^{ip(z+z')} d(z+z') \tag{4}$$

where $R'=\sqrt{(x-x')^2+(y-y')^2+(z-z'+j\xi)^2}$. The second integral in (3) is known as the exact image representation of the Sommefeld integral for an impedance half-plane, whose integrand is Laplace-type, and thus decays very fast. Hence, the integral is appropriate for a numerical computation. Based on the above relations, the Sommerfeld integral in (3) can be exactly evaluated for two cases, $\rho=0$ and $p=k_0$. For $\rho=0$, the second integral in (3) is used to obtain an exact expression given by

$$-i\int_0^\infty \frac{e^{ik_0 R'}}{R'} e^{-p\xi} d\xi = -i\int_0^\infty \frac{e^{ik_0(z+z'+i\xi)}}{z+z'+i\xi} e^{-p\xi} d\xi$$

$$= -E_1[-i(k+p)(z+z')]e^{-ip(z+z')} \tag{5}$$

where $E_1(z)$ is exponential integral[10]. For the other case $p=k_0$, using (4) and a known integral identity, $\int \frac{e^{ik_0[\sqrt{r^2+\xi^2}+\xi]}}{\sqrt{r^2+\xi^2}} d\xi = \mp E_1[-ik_0(\sqrt{r^2+\xi^2}\mp\xi)]$[11], the following equation can be obtained:

$$e^{-ik_0(z+z')} \int \frac{e^{ik_0 r}}{r} e^{ik_0(z+z')} d(z+z')$$

$$= -E_1[-ik_0(r+z+z')]e^{-ik_0(z+z')} \tag{6}$$

Except the two cases, a closed-form evaluation of (3) is not known yet[4].

To begin with, (4) is slightly modified as

$$I = \int \frac{e^{ik_0 r}}{r} e^{ip(z+z)'} d(z+z')$$

$$= \int \frac{e^{ik_0(r+z+z')}}{r} e^{i(p-k_0)(z+z)'} d(z+z')$$

Using a substitution, $\xi=r+z+z'$, the above integral is modified to

$$I = \int \frac{e^{i\frac{1}{2}(k+p)\xi}}{\xi} e^{i\frac{1}{2}(k-p)\frac{\rho^2}{\xi}} d\xi$$

$$= \sum_{n=0}^\infty \frac{1}{n!}\left[i(k-p)\frac{\rho^2}{2}\right]^n \int \frac{e^{i\frac{1}{2}(k+p)\xi}}{\xi^{n+1}} d\xi \tag{7}$$

After some algebraic manipulations, the integral term in (7) can be carried out analytically which is represented in terms of exponential integral[10] as

$$\int \frac{e^{i\frac{1}{2}(k+p)\xi}}{\xi^{n+1}} d\xi$$

$$= \frac{E_{n+1}\left[-\frac{i}{2}(k+p)(r+z+z')\right]}{(r+z+z')^n}$$

where $E_{n+1}(z)=\int_1^\infty \frac{e^{-zt}}{t^{n+1}} dt$ for $R(z)>0$. After inserting the above equation into (7), I can be expressed in a compact form as

$$I = -\sum_{n=0}^\infty \frac{\alpha^n}{n!} E_{n+1}(\beta) \tag{8}$$

where $\alpha=\frac{i}{2}(k-p)(r-z-z')$ and $\beta=-\frac{i}{2}(k+p)(r+z+z')$. For the two cases when the Sommerfeld integral can be analytically evaluated($\alpha=0$), I can be reduced to $-E_1(\beta)$ from (8), which is equivalent to (5) and (6).

To examine the convergence property of (8), the ratio test can be used. If α_n is defined as $\alpha^n/n!E_{n+1}(\beta)$, the test requires the behavior of $\frac{E_{n+2}(\beta)}{E_{n+1}(\beta)}$ as n approaches infinity. Using a series expansion of $E_n(z)$[12], it is easy to show

$$\frac{E_{n+2}(\beta)}{E_{n+1}(\beta)} \to \frac{-\beta}{n+1} \quad \text{for large } n$$

Hence, the ratio of α_{n+1}/α_n becomes zero as n goes to infinity, which indicates that (8) absolutely converges for all values of p and r.

To obtain an approximate but very accurate formulation of (8), the following approximation for the exponential integral can be substituted into (8)[12].

$$E_n(z) \approx \frac{e^{-z}}{z+n}\left[1+\frac{n}{(z+n)^2}+\frac{n(n-2z)}{(z+n)^4}\right]$$

It is valid for a large n. Considering the first term of the above approximation, I can be simply evaluated as

$$I \approx -e^{-\beta}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\frac{1}{\beta+n+1}$$
$$= -\frac{e^{-\beta}}{(-\alpha)^{\beta+1}}\gamma(\beta+1, -\alpha)$$

where $r(\alpha, x)$ is known as incomplete Gamma function[12].

Ⅲ. Formulation in Terms of Lommel Function

In this session, another series representation of (3) is obtained. After substituting $E_n(z) = \frac{1}{(n-1)!}\left[\begin{array}{l}(-z)^{n-1}E_1(z)\\ +e^{-z}\sum_{s=0}^{n-2}(n-s-2)!(-z)^s\end{array}\right]$ for $n>1$[10] into (8), I can be rewritten as

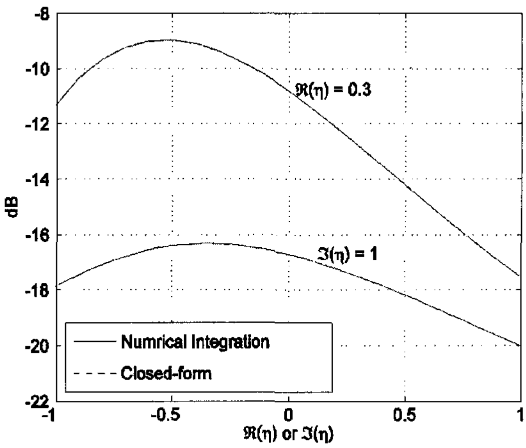
$$I = -E_1(\beta)\sum_{n=0}^{\infty}\frac{(-\alpha\beta)^n}{(n!)^2}$$
$$-e^{-\beta}\sum_{n=1}^{\infty}\sum_{s=0}^{n-1}\frac{(n-s-1)!}{(n!)^2}\alpha^n(-\beta)^s$$

The first summation of the above equation can be simply evaluated as $J_0(2\sqrt{\alpha\beta})$. To evaluate the second double summation, first, the summation terms are rearranged, and the summation direction is changed from horizontal to vertical directions as

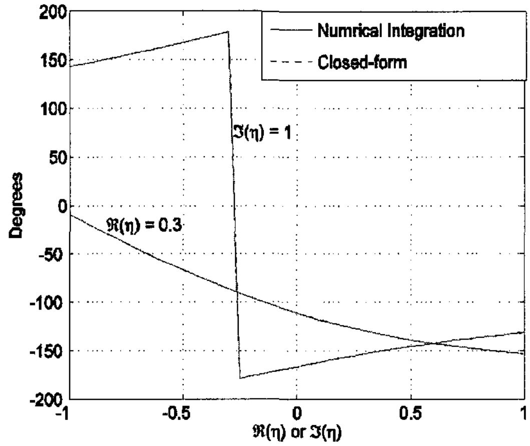
$$I_1 = \sum_{n=1}^{\infty}\sum_{s=0}^{n-1}\frac{(n-s-1)!}{(n!)^2}\alpha^n(-\beta)^\beta$$
$$= \sum_{n=1}^{\infty}\frac{\alpha^n}{n!}\frac{(n-1)!}{n!} + \sum_{n=2}^{\infty}\frac{\alpha^n}{n!}\frac{(n-2)!}{n!}(-\beta) + \cdots$$
$$= \sum_{n=1}^{\infty}\frac{\alpha^n}{nn!}\left[1+\frac{-\alpha\beta}{(n+1)^2}+\cdots\right]$$

The summation in the bracket of the above equation can be written in terms of a generalized hypergeometric function, given by ${}_1F_2(1; n+1, n+1; -\alpha\beta)$[10]. The hypergeometric function can be converted into a transcendental function known as Lommel function[10]. Therefore, the series is rewritten in terms of Lommel function as

$$I = -E_1(\beta)J_0(2\sqrt{\alpha\beta})$$
$$-4e^{-\beta}\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\frac{S_{2n-1,0}(2\sqrt{\alpha\beta})}{(4\beta)^n} \qquad (9)$$



(a) Magnitude



(b) Phase

Fig. 3. Comparisons of the exact formulation and the numerical integration of the Sommerfeld integral as a function of the real and imaginary parts of η.

Using the ratio test, it can be simply shown that the formulated series expansion absolutely converge again for all values of p, and r. Since the Lommel function grows fast with increasing n however, the two serieses are much less efficient for a numerical purpose than the previous series (8).

Ⅳ. Formulation in Terms of Incomplete Weber Integral

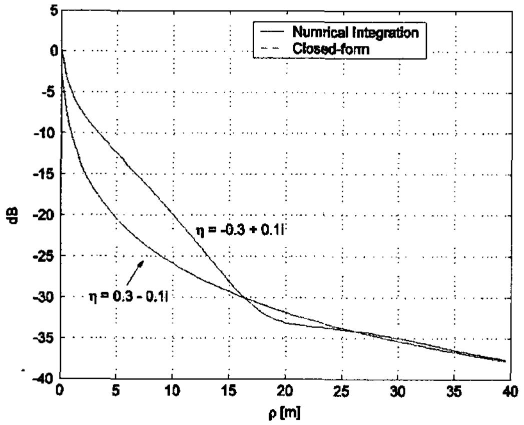
Based on (10) and the integral representation of the Lommel function, I can be evaluated into a closed form.

Using $s_{\mu,\nu}(z)=\frac{\pi}{2}\left[\begin{array}{l} Y_\nu(z)\int_0^z t^\mu J_\nu(t)\,dt- \\ J_\nu(z)\int_0^z t^\mu Y_\nu(t)\,dt \end{array}\right]$, the summation in (10) can be converted into an integral as

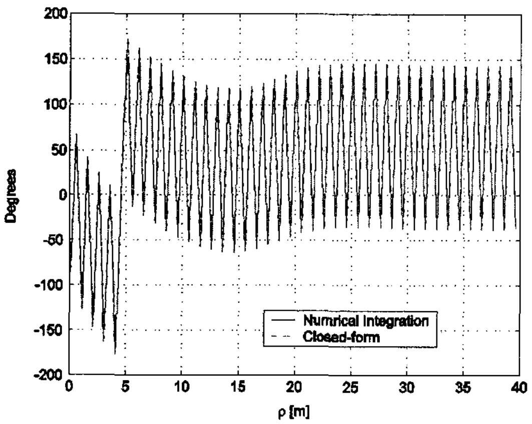
$$I_2=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\frac{S_{2n-1,0}(2\sqrt{\alpha\beta})}{(4\beta)^n}$$
$$=\frac{\pi}{8\beta}\int_0^{2\sqrt{\alpha\beta}} ze^{\frac{z^2}{4\beta}}\left[\begin{array}{l} Y_0(2\sqrt{\alpha\beta})\,J_0(z) \\ -J_0(2\sqrt{\alpha\beta}\,Y_0(z)) \end{array}\right]dz \quad (10)$$

where $Y_\nu(z)$ and $J_\nu(z)$ are Bessel functions of the first and second kind of order ν, respectively.

The integrals in (10) can be expressed in terms of



(a) Magnitude



(b) Phase

Fig. 4. Comparisons of the exact formulation and the numerical integration of the Sommerfeld integral as a function of ρ. For this computation, two η values are considred for $\eta=-0.3+0.1i$.

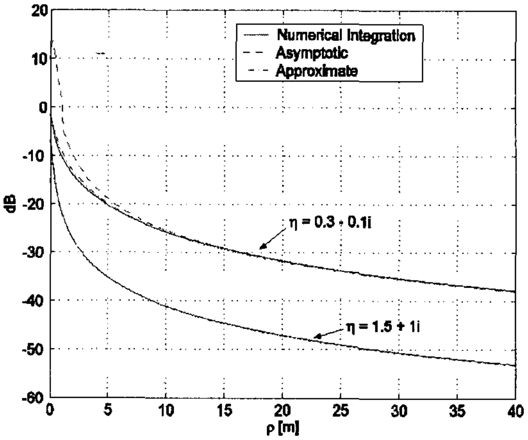


Fig. 5. Comparison of the approximate, asymptotic formulations and the numerical integration of the Sommerfeld integral as a function of ρ.

incomplete Weber integrals[13].

The incomplete Weber integrals are defined as

$$Q_\nu(x,\ z)=\frac{e^x}{(2x)^{\nu+1}}\int_0^z t^{\nu+1}J_\nu(t)\,e^{-\frac{t^2}{4x}}dt$$
$$P_\nu(x,\ z)=\frac{e^x}{(2x)^{\nu+1}}\int_z^L t^{\nu+1}Y_\nu(t)\,e^{-\frac{t^2}{4x}}dt$$

where $R(\nu)>-1$, $L=\infty e^{i\lambda}$, and $|2\lambda-\arg(x)|<\pi/2$. Many properties of the functions can be found in [13]. Hence, (10) can be rewritten as

$$I_2=-\frac{\pi}{4}e^{\beta}\left[\begin{array}{l} Q_0(-\beta,2\sqrt{\alpha\beta})\,Y_0(2\sqrt{\alpha\beta}) \\ +\{P_0(-\beta,2\sqrt{\alpha\beta})-P_0(-\beta,0)\} \\ \cdot J_0(2\sqrt{\alpha\beta}) \end{array}\right]$$

$P_0(-\beta,\ 0)$ can be expressed in terms of exponential integral[10]. Finally, I can be written in a simple closed form as

$$I=\pi\left[\begin{array}{l} Q_0(-\beta,2\sqrt{\alpha\beta})\,Y_0(2\sqrt{\alpha\beta}) \\ +\{P_0(-\beta,2\sqrt{\alpha\beta})\pm i\}\,J_0(2\sqrt{\alpha\beta}) \end{array}\right] \quad (11)$$

where "+" for $\Im(\beta)>0$ and "−" for $\Im(\beta)<0$.

Therefore, the Sommerfeld integral, (3) can be finally expressed in terms of the incomplete Weber function as

$$\int_0^\infty \frac{1}{k_z+p}J_0(k_\rho\rho)e^{ik_z(z+z')}\frac{k_\rho}{k_z}dk_\rho$$

$$= \pi e^{-ip(z+z')} \begin{bmatrix} Q_0(-\beta, 2\sqrt{\alpha\beta}) Y_0(2\sqrt{\alpha\beta}) \\ + \{P_0(-\beta, 2\sqrt{\alpha\beta}) \pm i\} J_0(2\sqrt{\alpha\beta}) \end{bmatrix} \quad (12)$$

V. Numerical Verification

In this section, the formulations, (8), (10), and (12) obtained in the previous sections are verified by comparing the results computed by the formulations, and the direct numerical integration for various cases. For the rest simulations, frequency is fixed to be 300 MHz, $z+z'$=0.1 m is assumed, and for a given η, p is calculated by $k_0 \eta$. First, the validity of the two series representations is examined. Fig. 2 shows plots of results calculated by the two series expansions (8) and (10), and the numerical integration. For this calculation, ρ is fixed at 1 m. In this figure the real part of η is fixed at 0.3 or 2, while the imaginary part is varied from -1 to 1. The figure shows that the series representations are in excellent agreement with the numerical integration results.

To verify the exact representation first, the effect of η is examined. Fig. 3 shows the comparisons of the results of the exact expression and the numerical calculation for two cases: 1) a fixed imaginary part of 1, and varying real part and 2) a fixed real part of 0.3, and varying imaginary part. As seen in the figure the results of the analytical formulation are in excellent agreement with the numerical results for all cases. For this simulation, ρ=1 m is assumed again. The next figure shows the comparisons of results computed by the numerical integration and (12) for two values of η: $0.3-0.1i$, and $-0.3+0.1i$. For a clear comparison, the phase is plotted for a case, $\eta=-0.3+0.1i$. The figures show the formulation (12) can provide exact results for any p, and r.

The final examination is the accuracy of the approximate formulation. Fig. 5 shows a comparison of the approximate, first-order asymptotic expressions and the numerical calculation as a function of the radial distance. For this calculation, two impedances are con-

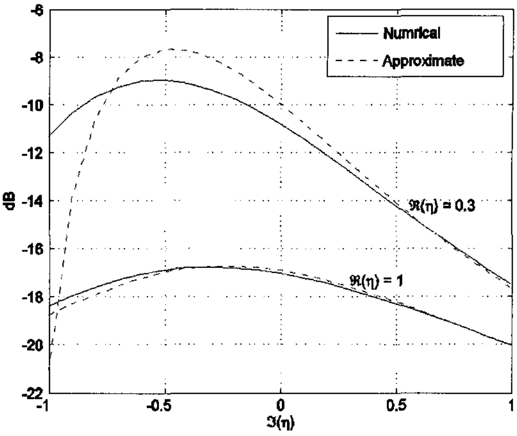


Fig. 6. Magnitudes of the approximate formulation and the numerical integration of the Sommerfeld integral as a function of the imaginary part of η with a fixed real part of 0.3 or 1.

sidered: $\eta=0.3-0.1i$, $1.5+i$. The asymptotic expression can be obtained using the conventional steepest descent method(SDM) given by

$$\int_0^{\infty} \frac{1}{k_z + p} J_0(k_\rho \rho) e^{ik_z(z+z')} \frac{k_\rho}{k_z} dk_\rho \sim -i \frac{e^{ik_0 r}}{r} \frac{1}{k_0 \cos\theta + p}$$

where $\cos\theta=(z+z')/r$. As seen in the Fig. 5, the approximate formulation is more accurate than the asymptotic one over the whole comparison range. Fig. 6 shows the effect of the impedance on the approximate formulation. As seen in the figure, if the real part of impedance is large, the accuracy of the approximate formulation is satisfactory, while the real part is small, the discrepancies between two results becomes large. For this calculation, $z+z'$=0.1 m and ρ=1 mare assumed.

VI. Conclusions

In this paper, two series representations of Sommerfeld integral occurring for an impedance half-plane problem are formulated. Based on one of the series expressions, an exact, and closed-form expression of the integral is obtained, which is written in terms of the incomplete Weber integrals. The obtained representations including the exact one are verified for various cases by comparing results computed by the new formulations and a direct numerical integration technique. The comparisons show the derived equations are valid for any values of the surface impedance(η) and the distance between the source and the observation point (r). Especially, the approximate formulation is uniformly valid from the near- to far-field region.

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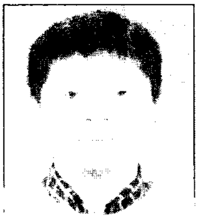
고 일 석



1992년 2월: 연세대학교 전자공학과 (공학사)
1994년 2월: 연세대학교 전자공학과 (공학석사)
1994년 1월~1996년 4월: LG전자 연구원
2002년 8월: The University of Michigan at Ann Arbor 전기 및 전산과 (공학박사)
2004년 3월~현재: 인하대학교 정보통신대학원 교수
[주 관심분야] Theoretical Electromagnetic, Numerical Analysis, Channel Modeling