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Proximities and two types uniformities

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Abstract

In a strictly two-sided, commutative biquantale, we introduce the notion of $(L,\odot)$-proximity spaces. We investigate the relations among $(L,\odot)$-proximity spaces, Hutton $(L,\otimes)$-uniform spaces, $(L,\odot)$ uniform spaces, enriched $(L,\odot)$-topological spaces and enriched $(L,\odot)$-interior spaces.

Key words :
$(L,\odot)$-proximity spaces. Hutton $(L,\otimes)$-uniform spaces, $(L,\odot)$-uniform spaces, enriched $(L,\odot)$-topological spaces, enriched $(L,\odot)$-interior spaces

1. Introduction

Recently, Gutiérrez García et al.[2] introduced L-valued Hutton unifomity where a quadruple $(L,\le,\otimes,*)$ is defined by a GL-monoid $(L,*)$ dominated by $\otimes$, a cl-quasi-monoid $(L,\le,\otimes)$. Kubiak et al.[14] studied the relationships between the categories of $I(L)$-uniform spaces and L-uniform spaces. Kim et al. [9-11] introduced the notion of Hutton $(L,\otimes)$-uniformities, $(L,\odot)$-uniformities, enriched $(L,\odot)$-topologies and enriched $(L,\odot)$-interior spaces.

In this paper, we introduce the notion of $(L,\odot)$-proximity spaces. We investigate the relations among $(L,\odot)$-proximity spaces, Hutton $(L,\otimes)$-uniform spaces, $(L,\odot)$ uniform spaces, enriched $(L,\odot)$-topological spaces and enriched $(L,\odot)$-interior spaces.

2. Preliminaries

Definition 2.1. [4-7, 12,16] A triple $(L,\le,\odot)$ is called a *strictly two-sided, commutative biquantale* (stsc-biquantale, for short) iff it satisfies the following properties:

(L1) $L=(L,\le,\vee,\wedge,\top,\bot)$ is a completely distributive lattice where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(L2) $(L,\odot)$ is a commutative semigroup;

(L3) $a=a\odot\top$, for each $a\in L$;

(L4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i)\odot b=\bigvee_{i\in\Gamma}(a_i\odot b).$$

(L5) $\odot$ is distributive over arbitrary meets, i.e.

$$(\bigwedge_{i\in\Gamma} a_i)\odot b=\bigwedge_{i\in\Gamma}(a_i\odot b).$$

A mapping $n:L\to L$ is called a *strong negation* , denoted by $n(a)=a^*$, if it satisfies the following conditions:

(N1) $n(n(a))=a$ for each $a\in L$.

(N2) If $a\le b$ for each $a,b\in L$, then $n(a)\ge n(b)$.

In this paper, we assume that $(L,\le,\odot,\oplus,^*)$ is a stac-biquantale with a strong negation * which is defined by

$$x\oplus y=(x^*\odot y^*)^*.$$

Lemma 2.2. [12] Let $(L,\le,\odot,\oplus,^*)$ be a biquantale with a strong negation * which is defined by

$$x\oplus y=(x^*\odot y^*)^*.$$

For each $x,y,z\in L$, $\{y_i\mid i\in\Gamma\}\subset L$, we have the following properties.

(1) If $y\le z$, $(x\odot y)\le(x\odot z)$.
(2) If $y\le z$, $(x\oplus y)\le(x\oplus z)$.
(3) If $x\odot y\le x\wedge y$.
(4) $0^*=1$, $1^*=0$ and $x\vee y\le x\oplus y$.
(5) $\bigwedge_{i\in\Gamma} y_i^*=(\bigvee_{i\in\Gamma} y_i)^*$.
(6) $\bigvee_{i\in\Gamma} y_i^*=(\bigwedge_{i\in\Gamma} y_i)^*$.
(7) $x\oplus(\bigwedge_{i\in\Gamma} y_i)=\bigwedge_{i\in\Gamma}(x\oplus y_i)$.

All algebraic operations on L can be extended pointwisely to the set L^X as follows: for all $x\in X$, $\lambda,\mu\in L^X$ and $\alpha\in L$,

(1) $\lambda\le\mu$ iff $\lambda(x)\le\mu(x)$;
(2) $(\lambda\odot\mu)(x)=\lambda(x)\odot\mu(x)$;
(3) $1_X(x)=\top$, $\alpha\odot 1_X(x)=\alpha$ and $1_\emptyset(x)=\bot$;
(4) $(\alpha\to\lambda)(x)=\alpha\to\lambda(x)$ and $(\lambda\to\alpha)(x)=\lambda(x)\to\alpha$;
(5) $(\alpha\odot\lambda)(x)=\alpha\odot\lambda(x)$.

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Definition 2.3. [9-11] Let $\Omega(X)$ be a subset of $(L^X)^{(L^X)}$ such that

(O1) $\lambda \leq \phi(\lambda)$, for each $\lambda \in L^X$,
(O2) $\phi(\bigvee_{i\in\Gamma} \lambda_i) = \bigvee_{i\in\Gamma} \phi(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$,
(O3) $\alpha \odot \phi(\lambda) = \phi(\alpha \odot \lambda)$, for each $\lambda \in L^X$.

For $\phi, \phi_1, \phi_2, \phi_3 \in \Omega(X)$, we define, for all $\lambda \in L^X$,

$$\phi^{-1}(\lambda) = \bigwedge\{\rho \in L^X \mid \phi(\rho^*) \leq \lambda^*\},$$

$$\phi_1 \circ \phi_2(\lambda) = \phi_1(\phi_2(\lambda)),$$

$$\phi_1 \otimes \phi_2(\lambda) = \bigwedge\{\phi_1(\lambda_1) \odot \phi_2(\lambda_2) \mid \lambda = \lambda_1 \odot \lambda_2\}.$$

Definition 2.4. [9-11] A nonempty subset $\mathbf{U}$ of $\Omega(X)$ is called a Hutton $(L, \otimes)$-*uniformity* on X if it satisfies the following conditions:

(U1) If $\phi \leq \psi$ with $\phi \in \mathbf{U}$ and $\psi \in \Omega(X)$, then $\psi \in \mathbf{U}$.
(U2) For each $\phi, \psi \in \mathbf{U}$, $\phi \otimes \psi \in \mathbf{U}$.
(U3) For each $\phi \in \mathbf{U}$, there exists $\psi \in \mathbf{U}$ such that $\psi \circ \psi \leq \phi$.
(U4) For each $\phi \in \mathbf{U}$, there exists $\phi^{-1} \in \mathbf{U}$.

The pair $(X, \mathbf{U})$ is said to be a *Hutton* $(L, \otimes)$-*uniform space.*

Definition 2.5. [9-11] Let $E(X \times X) = \{u \in L^{X\times X} \mid u(x, x) = \top\}$ be a subset of $L^{X\times X}$. A nonempty subset $\mathbf{D}$ of $E(X \times X)$ is called an $(L, \odot)$-*uniformity* on X if it satisfies the following conditions:

(D1) If $u \leq v$ with $u \in \mathbf{D}$ and $v \in E(X \times X)$, then $v \in \mathbf{D}$.
(D2) For each $u, v \in \mathbf{D}$, $u \odot v \in \mathbf{D}$.
(D3) For each $u \in \mathbf{D}$, there exists $v \in \mathbf{D}$ such that $v \circ v \leq u$ where

$$v \circ v(x, y) = \bigvee_{z\in X} (v(x, z) \odot v(z, y)).$$

(D4) For each $u \in \mathbf{D}$, there exists $u^s \in \mathbf{U}$ where $u^s(x, y) = u(y, x)$.

The pair $(X, \mathbf{D})$ is said to be an $(L, \odot)$-*uniform space.*

Theorem 2.6. [9-11] We define two mappings $\Gamma : E(X \times X) \to \Omega(X)$ and $\Lambda : \Omega(X) \to E(X \times X)$ as follows:

$$\Gamma(u)(\lambda)(y) = \bigvee_{x\in X} \lambda(x) \odot u(x, y).$$

$$\Lambda(\phi)(x, y) = \phi(1_{\{x\}})(y).$$

Then for $u, u_1, u_2 \in E(X \times X)$ and $\phi, \phi_1, \phi_2 \in \Omega(X)$, we have the following properties:

(1) $\Gamma(u_1 \odot u_2) \leq \Gamma(u_1) \otimes \Gamma(u_2)$.
(2) $\Gamma(u)^{-1} = \Gamma(u^s)$.
(3) $\Gamma(u_1 \circ u_2) = \Gamma(u_2) \circ \Gamma(u_1)$.
(4) $\Gamma(\alpha \odot u) = \alpha \odot \Gamma(u)$.
(5) $\Lambda(\phi_1) \odot \Lambda(\phi_2) = \Lambda(\phi_1 \otimes \phi_2)$.
(6) $\Lambda(\phi)^s = \Lambda(\phi^{-1})$.
(7) $\Lambda(\phi_1) \circ \Lambda(\phi_2) = \Lambda(\phi_2 \circ \phi_1)$.
(8) $\Lambda(\alpha \odot \phi) = \alpha \odot \Lambda(\phi)$.

Theorem 2.7. [9-11] Let $\mathbf{D}$ be an $(L, \odot)$-uniform space. We define a subset $\mathbf{U_D}$ of $\Omega(X)$ as follows:

$$\mathbf{U_D} = \{\phi \in \Omega(X) \mid \exists u \in \mathbf{D}, \Gamma(u) \leq \phi\}.$$

Then $\mathbf{U_D}$ is a Hutton $(L, \otimes)$- uniformity on X.

Theorem 2.8. [9-11] Let $\mathbf{U}$ be a Hutton $(L, \otimes)$-uniformity on X. We define a subset $\mathbf{D_U}$ of $E(X \times X)$ as follows:

$$\mathbf{D_U} = \{u \in E(X \times X) \mid \exists \phi \in \mathbf{U}, \Lambda(\phi) \leq u\}.$$

Then:

(1) $\mathbf{D_U}$ is an $(L, \odot)$-uniformity on X.
(2) $\mathbf{D_{U_D}} = \mathbf{D}$ and $\mathbf{U_{D_U}} = \mathbf{U}$.

Definition 2.9. [10] A subset $\mathbf{T}$ of L^X is called an $(L, \odot)$-*topology* on X if it satisfies the following conditions:

(T1) $1_X, 1_\emptyset \in \mathbf{T}$.
(T2) If $\lambda_1, \lambda_2 \in \mathbf{T}$, then $\lambda_1 \odot \lambda_2 \in \mathbf{T}$.
(T3) If $\lambda_1, \lambda_2 \in \mathbf{T}$, then $\lambda_1 \wedge \lambda_2 \in \mathbf{T}$.
(T4) If $\lambda_i \in \mathbf{T}$ for all $i \in \Gamma$, then $\bigvee_{i\in\Gamma} \lambda_i \in \mathbf{T}$

The pair $(X, \mathbf{T})$ is called an $(L, \odot)$-*topological space.*

An $(L, \odot)$-topological space is called *enriched* iff it satisfies:

(E) If $\lambda \in \mathbf{T}$, then $\alpha \odot \lambda \in \mathbf{T}$.

Definition 2.10. [10] A function $\mathbf{I} : L^X \to L^X$ is called an $(L, \odot)$-*interior operator* on X iff $\mathbf{I}$ satisfies the following conditions:

(I1) $\mathbf{I}(1_X) = 1_X$.
(I2) $\mathbf{I}(\lambda) \leq \lambda$.
(I3) $\mathbf{I}(\lambda \odot \mu) \geq \mathbf{I}(\lambda) \odot \mathbf{I}(\mu)$.
(I4) $\mathbf{I}(\lambda \wedge \mu) = \mathbf{I}(\lambda) \wedge \mathbf{I}(\mu)$.

The pair $(X, \mathbf{I})$ is called an $(L, \odot)$-*interior space.*

An $(L, \odot)$-interior space $(X, \mathbf{I})$ is called *topological* if

(T) $\mathbf{I}(\mathbf{I}(\lambda)) \geq \mathbf{I}(\lambda)$, $\forall \lambda \in L^X$.

An $(L, \odot)$-interior space $(X, \mathbf{I})$ is called *enriched* if

(E) $\mathbf{I}(\alpha \odot \lambda) \geq \alpha \odot \mathbf{I}(\lambda)$, $\forall \alpha \in L, \lambda \in L^X$.

An $(L, \odot)$-interior space $(X, \mathbf{I})$ is called *principle* if

(P) $\mathbf{I}(\bigwedge_{i\in\Gamma} \lambda_i) = \bigwedge_{i\in\Gamma} \mathbf{I}(\lambda_i)$, $\forall i \in \Gamma, \lambda_i \in L^X$.

Theorem 2.11. [10] (1) Let $(X, \mathbf{T})$ be an enriched $(L, \odot)$-topological space. Define a map $\mathbf{I_T} : L^X \to L^X$ as follows:

$$\mathbf{I_T}(\lambda) = \bigvee\{\rho \in L^X \mid \rho \leq \lambda, \ \rho \in \mathbf{T}\}.$$

Then $\mathbf{I_T}$ is an enriched topological $(L, \odot)$-interior operator on X induced by $\mathbf{T}$.

(2) Let $(X, \mathbf{I})$ be an enriched topological $(L, \odot)$-interior space. Define a subset $\mathbf{T_I}$ of L^X by

$$\mathbf{T_I} = \{\lambda \in L^X \mid \lambda \leq \mathbf{I}(\lambda)\}.$$

Then $\mathbf{T_I}$ is an enriched $(L, \odot)$-topology on X induced by $\mathbf{I}$.

(3) $\mathbf{I_{T_I}} = \mathbf{I}$ and $\mathbf{T_{I_T}} = \mathbf{T}$.

Theorem 2.12. [10] Let $\mathbf{U}$ be a Hutton $(L,\otimes)$-uniformity on X. We define a mapping $\mathbf{I_U}: L^X \to L^X$ as follows:

$$\mathbf{I_U}(\lambda) = \bigvee\{\rho \in L^X \mid \exists\phi \in \mathbf{U}, \phi(\rho) \le \lambda\}.$$

Then:

(1) $\mathbf{I_U}$ is an enriched topological $(L,\odot)$-interior operator on X.

(2) $\mathbf{T_{I_U}}$ is an enriched $(L,\odot)$-topology induced by $\mathbf{U}$.

Theorem 2.13. [10] Let $\mathbf{D}$ be an $(L,\odot)$-uniformity on X. We define a mapping $\mathbf{I_D}: L^X \to L^X$ as follows:

$$\mathbf{I_D}(\lambda) = \bigvee\{\rho \in L^X \mid \exists u \in \mathbf{D}, \Gamma(u)(\rho) \le \lambda\}.$$

Then:

(1) $\mathbf{I_D}$ is an enriched topological $(L,\odot)$-interior operator on X.

(2)

$$\mathbf{I_D}(\lambda) = \bigvee\{\alpha \odot 1_{\{y\}} \mid \exists u \in \mathbf{D}, \alpha \odot u(y,-) \le \lambda\}.$$

(3) $\mathbf{T_{I_D}}$ is an enriched $(L,\odot)$-topology induced by $\mathbf{D}$. Moreover, $\mathbf{I_{U_D}} = \mathbf{I_D}$.

(4) If $\mathbf{U}$ is a Hutton $(L,\otimes)$-uniformity on X, then $\mathbf{I_{D_U}} = \mathbf{I_U}$.

3. Proximities and two types uniformities

Definition 3.1. A subset δ of $L^X \times L^X$ is called an $(L,\odot)$-*proximity* on X if it satisfies the following conditions:

(P1) $(1_X, 1_\emptyset) \notin \delta$.

(P2) If $(\lambda,\rho) \notin \delta$ and $\mu \le \lambda$, then $(\mu,\rho) \notin \delta$.

(P3) If $(\lambda,\rho_i) \notin \delta$ for $i = 1,2$, then $(\lambda, \rho_1 \vee \rho_2) \notin \delta$.

(P4) If $(\lambda,\rho) \notin \delta$, then $\lambda \le \rho^*$.

(P5) If $(\lambda_i,\rho_i) \notin \delta$ for $i = 1,2$, then $(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \notin \delta$.

(P6) If $(\lambda,\rho) \notin \delta$, there exists $\gamma \in L^X$ such that $(\lambda,\gamma) \notin \delta$ and $(\gamma^*,\rho) \notin \delta$.

(P7) If $(\lambda,\rho) \in \delta$, then $(\rho,\lambda) \in \delta$

The pair (X,δ) is said to be an $(L,\odot)$-*proximity space*.

An $(L,\odot)$-proximity space is called *enriched* if it satisfies

(E) If $(\lambda,\rho) \notin \delta$ and $\alpha \in L$, then $(\alpha \odot \lambda, \alpha^* \oplus \rho) \notin \delta$.

An $(L,\odot)$-proximity space is called *principle* if it satisfies

(P) If $(\lambda_j,\rho) \notin \delta$ for all $j \in J$, then $(\bigvee_{j\in J}\lambda_j, \rho) \notin \delta$.

Theorem 3.2. Let (X,δ) be an (resp. principle, enriched) $(L,\odot)$- proximity space. Define a function $\mathbf{I}_\delta: L^X \to L^X$ by

$$\mathbf{I}_\delta(\lambda) = \bigvee\{\rho \in L^X \mid (\rho,\lambda^*) \notin \delta\}.$$

Then $(X,\mathbf{I}_\delta)$ is a (resp. principle, enriched) topological $(L,\odot)$-interior space.

Proof. (1) (I1) Since $(1_X, 1_\emptyset) \notin \delta$, $\mathbf{I}_\delta(1_X) = 1_X$.

(I2) Since $(\rho,\lambda^*) \notin \delta$, by (P4), $\rho \le \lambda$. Thus, $\mathbf{I}_\delta(\lambda) \le \lambda$.

(I3) From (L4) and (P5), we have:

$$\begin{aligned}
&\mathbf{I}_\delta(\lambda_1) \odot \mathbf{I}_\delta(\lambda_2)\\
&= \Big\{\bigvee\{\rho_1 \in L^X \mid (\rho_1,\lambda_1^*) \notin \delta\}\Big\}\\
&\quad \odot\Big\{\bigvee\{\rho_2 \in L^X \mid (\rho_2,\lambda_2^*) \notin \delta\}\Big\}\\
&= \bigvee\{\rho_1 \odot \rho_2 \in L^X \mid (\rho_1,\lambda_1^*) \notin \delta, (\rho_2,\lambda_2^*) \notin \delta\}\\
&\le \bigvee\{\rho_1 \odot \rho_2 \in L^X \mid (\rho_1 \odot \rho_2, \lambda_1^* \oplus \lambda_2^*) \notin \delta\}\\
&\le \mathbf{I}_\delta(\lambda_1 \odot \lambda_2).
\end{aligned}$$

(I4) Since $(\rho_1,\lambda_1^*) \notin \delta, (\rho_2,\lambda_2^*) \notin \delta$ implies $(\rho_1 \wedge \rho_2, \lambda_1^* \vee \lambda_2^*) \notin \delta$ from (P2) and (P3), by (L1), we have

$$\begin{aligned}
&\mathbf{I}_\delta(\lambda_1) \wedge \mathbf{I}_\delta(\lambda_2)\\
&= \Big\{\bigvee\{\rho_1 \in L^X \mid (\rho_1,\lambda_1^*) \notin \delta\}\Big\}\\
&\quad \wedge\Big\{\bigvee\{\rho_2 \in L^X \mid (\rho_2,\lambda_2^*) \notin \delta\}\Big\}\\
&= \bigvee\{\rho_1 \wedge \rho_2 \in L^X \mid (\rho_1,\lambda_1^*) \notin \delta, (\rho_2,\lambda_2^*) \notin \delta\}\\
&\le \bigvee\{\rho_1 \wedge \rho_2 \in L^X \mid (\rho_1 \wedge \rho_2, \lambda_1^* \vee \lambda_2^*) \notin \delta\}\\
&\le \mathbf{I}_\delta(\lambda_1 \wedge \lambda_2).
\end{aligned}$$

(T) Let $(\rho,\lambda^*) \notin \delta$ be given. Then there exists $\gamma \in L^X$ such that $(\rho,\gamma) \notin \delta$ and $(\gamma^*,\lambda^*) \notin \delta$ So, $\mathbf{I}_\delta(\lambda) \ge \gamma^*$. Thus, $(\rho,\gamma) \notin \delta$ and $\mathbf{I}_\delta(\lambda)^* \le \gamma$ implies $(\rho, \mathbf{I}_\delta(\lambda)^*) \notin \delta$. Now, $(\rho,\lambda^*) \notin \delta$ implies $(\rho, \mathbf{I}_\delta(\lambda)^*) \notin \delta$. Hence

$$\begin{aligned}
\mathbf{I}_\delta(\lambda) &\le \bigvee\{\rho \in L^X \mid (\rho,\lambda^*) \notin \delta\}\\
&\le \bigvee\{\rho \in L^X \mid (\rho,\mathbf{I}_\delta(\lambda)^*) \notin \delta\}\\
&= \mathbf{I}_\delta(\mathbf{I}_\delta(\lambda)).
\end{aligned}$$

(E) If δ is enriched, by (L4),

$$\begin{aligned}
\alpha \odot \mathbf{I}_\delta(\lambda) &= \alpha \odot \Big\{\bigvee\{\rho \in L^X \mid (\rho,\lambda^*) \notin \delta\}\Big\}\\
&\le \bigvee\{\alpha \odot \rho \in L^X \mid (\alpha \odot \rho, \alpha^* \oplus \lambda^*) \notin \delta\}\\
&\le \mathbf{I}_\delta(\alpha \odot \lambda).
\end{aligned}$$

(P) If (X,δ) is principle, we will show that $\mathbf{I}_\delta(\bigwedge\lambda_i) = \bigwedge\mathbf{I}_\delta(\lambda_i)$.

Suppose $\mathbf{I}_\delta(\bigwedge_{i\in\Gamma}\lambda_i) \not\ge \bigwedge_{i\in\Gamma}\mathbf{I}_\delta(\lambda_i)$. By a completely distributive lattice L and the definition of $\mathbf{I}_\delta(\lambda_i)$, there exists ρ_i with $(\rho_i,\lambda_i^*) \notin \delta$ for each $i \in \Gamma$ such that $\mathbf{I}_\delta(\bigwedge_{i\in\Gamma}\lambda_i) \not\ge \bigwedge_{i\in\Gamma}\rho_i$.

On the other hand, since $(\rho_i,\lambda_i^*) \notin \delta$ for each $i \in \Gamma$ and (X,δ) is principle, $(\bigwedge_{i\in\Gamma}\rho_i, \bigvee_{i\in\Gamma}\lambda_i^*) \notin \delta$. So, $\mathbf{I}_\delta(\bigwedge_{i\in\Gamma}\lambda_i) \ge \bigwedge_{i\in\Gamma}\rho_i$. It is a contradiction. Hence $\mathbf{I}_\delta(\bigwedge\lambda_i) \ge \bigwedge\mathbf{I}_\delta(\lambda_i)$. By (I4), since $\lambda \le \mu$ implies $\mathbf{I}_\delta(\lambda) \le \mathbf{I}_\delta(\mu)$, $\mathbf{I}_\delta(\bigwedge\lambda_i) \le \bigwedge\mathbf{I}_\delta(\lambda_i)$.

□

Example 3.3. Let X be a set. Define two subsets δ_i of $L^X \times L^X$ as follows:

$$(\lambda,\rho) \notin \delta_1 \text{ iff } \lambda = 1_\emptyset \text{ or } \rho = 1_\emptyset,$$

$$(\lambda,\rho) \notin \delta_2 \text{ iff } \lambda \leq \rho^*.$$

(1) Let $(\lambda_1 \odot \lambda_2, \mu_1 \oplus \mu_2) \in \delta_1$. Since $\lambda_1 \odot \lambda_2 \neq 1_\emptyset$ and $\mu_1 \oplus \mu_2 \neq 1_\emptyset$ imply $\lambda_1 \neq 1_\emptyset, \lambda_2 \neq 1_\emptyset$ $\mu_1 \neq 1_\emptyset$ or $\mu_2 \neq 1_\emptyset$, we have $(\lambda_1,\mu_1) \in \delta_1$ or $(\lambda_2,\mu_2) \in \delta_1$. We easily show δ_1 is a principal $(L,\odot)$-proximity on X. For each $\alpha \neq \top$ with $\alpha \odot \lambda \neq 1_\emptyset$, since $\alpha^* \oplus 1_\emptyset) \notin \delta_1 \geq \alpha^* \odot 1_X \neq 1_\emptyset$ from Lemma 2.2(4), δ is not enriched because

$$(\lambda, 1_\emptyset) \notin \delta_1, \quad (\alpha \odot \lambda, \alpha^* \oplus 1_\emptyset) \in \delta_1.$$

(2) Since $\lambda_1 \leq \rho_2^*$ and $\lambda_2 \leq \rho_2^*$ implies $\lambda_1 \odot \lambda_2 \leq (\rho_1 \oplus \rho_2)^*$, we have $(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \notin \delta_2$. Since $(\lambda,\lambda^*) \notin \delta_2$ and $(\lambda,\rho) \notin \delta_2$, δ_2 satisfies (P5) and (P6). Other cases are easy. Hence δ_2 is a principal enriched $(L,\odot)$-proximity on X

(3) We can obtain $\mathbf{I}_{\delta_1}, \mathbf{I}_{\delta_2} : L^X \to L^X$ as follows:

$$\mathbf{I}_{\delta_1}(\lambda) = \begin{cases} 1_X, & \text{if } \lambda = 1_X, \\ 1_\emptyset, & \text{otherwise}, \end{cases}$$

$$\mathbf{I}_{\delta_2}(\lambda) = \lambda, \quad \forall \lambda \in L^X.$$

Since $\alpha \odot 1_X \not\leq \mathbf{I}_{\delta_1}(\alpha \odot 1_X) = 1_\emptyset$ for $\alpha \notin \{\top, \bot\}$, $\mathbf{I}_{\delta_1}$ is not enriched. But $\mathbf{I}_{\delta_2}$ is enriched because

$$\mathbf{I}_{\delta_2}(\alpha \odot \lambda) = \alpha \odot \lambda = \alpha \odot \mathbf{I}_{\delta_2}(\lambda).$$

(4) We can obtain $(L,\odot)$-topologies $\mathbf{T}_{\mathbf{I}_{\delta_1}}, \mathbf{T}_{\mathbf{I}_{\delta_2}}$ as follows:

$$\mathbf{T}_{\mathbf{I}_{\delta_1}} = \{1_X, 1_\emptyset\}, \quad \mathbf{T}_{\mathbf{I}_{\delta_2}} = L^X$$

Theorem 3.4. Let $\mathbf{U}$ be a Hutton $(L,\otimes)$-uniformity on X. We define

$(\lambda,\rho) \notin \delta_{\mathbf{U}}$ iff $\exists \phi \in \mathbf{U}, \ \phi(\lambda) \leq \rho^*$.

Then $\delta_{\mathbf{U}}$ is an enriched $(L,\odot)$-proximity on X such that $\mathbf{I}_{\delta_{\mathbf{U}}} = \mathbf{I}_{\mathbf{U}}$.

Proof. (1) (P1) Since $\phi(1_X) = 1_X$, we have $(1_X, 1_\emptyset) \notin \delta_{\mathbf{U}}$. (P2) and (P4) are obvious.

(P3) Since $(\lambda,\rho_i) \notin \delta_{\mathbf{U}}$ iff $\exists \phi_i \in \mathbf{U}, \phi_i(\lambda) \leq \rho_i^*$ for $i = 1,2$, there exists $\phi_1 \otimes \phi_2 \in \mathbf{U}$ such that, by $\phi_1 \otimes \phi_2 \leq \phi_i$ for $i = 1,2$,

$$\phi_1 \otimes \phi_2(\lambda) \leq \phi_1(\lambda) \wedge \phi_2(\lambda) \leq \rho_1^* \wedge \rho_2^*.$$

So, $(\lambda, \rho_1 \vee \rho_2) \notin \delta_{\mathbf{U}}$.

(P5) Since $(\lambda_i,\rho_i) \notin \delta_{\mathbf{U}}$ iff $\exists \phi_i \in \mathbf{U}, \phi_i(\lambda_i) \leq \rho_i^*$ for $i = 1,2$, there exists $\phi_1 \otimes \phi_2 \in \mathbf{U}$ such that

$$\phi_1 \otimes \phi_2(\lambda_1 \odot \lambda_2) \leq \phi_1(\lambda_1) \odot \phi_2(\lambda_2) \leq \rho_1^* \odot \rho_2^*.$$

So, $(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \notin \delta_{\mathbf{U}}$.

(P6) Since $(\lambda,\rho) \notin \delta_{\mathbf{U}}$ iff $\exists \phi \in \mathbf{U}, \phi(\lambda) \leq \rho^*$, there exists $\psi \in \mathbf{U}$ with $\psi \circ \psi \leq \phi$ such that

$$\psi(\lambda) \leq \psi(\lambda), \quad \psi \circ \psi(\lambda) \leq \phi(\lambda) \leq \rho^*.$$

So, there exists $\psi(\lambda)^*$ such that

$$(\lambda, \psi(\lambda)^*) \notin \delta_{\mathbf{U}}, \ (\psi(\lambda), \rho) \notin \delta_{\mathbf{U}}.$$

(P7) $(\lambda,\rho) \notin \delta_{\mathbf{U}}$ iff $\exists \phi \in \mathbf{U}, \phi(\lambda) \leq \rho^*$ iff $\exists \phi^{-1} \in \mathbf{U}, \phi^{-1}(\rho) \leq \lambda^*$ iff $(\rho,\lambda) \notin \delta_{\mathbf{U}}$.

(E) Let $(\lambda,\rho) \notin \delta_{\mathbf{U}}$. Then $\exists \phi \in \mathbf{U}, \phi(\lambda) \leq \rho^*$. It implies

$$\phi(\alpha \odot \lambda) = \alpha \odot \phi(\lambda) \leq \alpha \odot \rho^* = (\alpha^* \oplus \rho)^*.$$

Thus, $(\alpha \odot \lambda, \alpha^* \oplus \rho) \notin \delta_{\mathbf{U}}$. Hence $\delta_{\mathbf{U}}$ is an enriched $(L,\odot)$-proximity on X.

By the definition of $\delta_{\mathbf{U}}$, $\mathbf{I}_{\delta_{\mathbf{U}}} = \mathbf{I}_{\mathbf{U}}$.

□

Theorem 3.5. Let $\mathbf{D}$ be an $(L,\odot)$-uniformity on X. We define

$(\lambda,\rho) \notin \delta_{\mathbf{D}}$ iff $\exists u \in \mathbf{D}, \Gamma(u)(\lambda) \leq \rho^*$.

Then $\delta_{\mathbf{D}}$ is an enriched $(L,\odot)$-proximity on X such that $\mathbf{I}_{\delta_{\mathbf{D}}} = \mathbf{I}_{\mathbf{D}}$.

Proof. (1) (P1) Since $\Gamma(u)(1_X) \leq 1_X$, we have $\mathbf{I}_{\mathbf{D}}(1_X) = 1_X$. (P2) is obvious.

(P3) Since $(\lambda,\rho_i) \notin \delta_{\mathbf{D}}$ iff $\exists u_i \in \mathbf{D}, \Gamma(u_i)(\lambda) \leq \rho_i^*$ for $i = 1,2$, there exists $\Gamma(u_1) \otimes \Gamma(u_2) \in \mathbf{U}_{\mathbf{D}}$ such that, by Theorems 2.6-7,

$$\begin{aligned}\Gamma(u_1 \odot u_2)(\lambda) &\leq (\Gamma(u_1) \otimes \Gamma(u_2))(\lambda) \\ &\leq \Gamma(u_1)(\lambda) \wedge \Gamma(u_2)(\lambda) \leq \rho_1^* \wedge \rho_2^*.\end{aligned}$$

So, $(\lambda, \rho_1 \vee \rho_2) \notin \delta_{\mathbf{D}}$.

(P4) Since $\rho \leq \Gamma(u)(\rho) \leq \lambda$, $\mathbf{I}_{\mathbf{D}}(\lambda) \leq \lambda$ for all $\lambda \in L^X$.

(P5) Since $(\lambda_i,\rho_i) \notin \delta_{\mathbf{D}}$ iff $\exists u_i \in \mathbf{D}, \Gamma(u_i)(\lambda_i) \leq \rho_i^*$ for $i = 1,2$, there exists $u_1 \odot u_2 \in \mathbf{D}$ such that

$$\begin{aligned}\Gamma(u_1 \odot u_2)(\lambda_1 \odot \lambda_2) &\leq (\Gamma(u_1) \otimes \Gamma(u_2))(\lambda_1 \odot \lambda_2) \\ &\leq \Gamma(u_1)(\lambda_1) \odot \Gamma(u_1)(\lambda_2) \\ &\leq \rho_1^* \odot \rho_2^*.\end{aligned}$$

So, $(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \notin \delta_{\mathbf{D}}$.

(P6) Since $(\lambda,\rho) \notin \delta_{\mathbf{D}}$ iff $\exists u \in \mathbf{D}, \Gamma(u)(\lambda) \leq \rho^*$, there exists $v \in \mathbf{D}$ with $v \circ v \leq u$ such that, by Theorem 2.6(3), $\Gamma(v)(\lambda) \leq \Gamma(v)(\lambda)$ and

$$\Gamma(v) \circ \Gamma(v)(\lambda) = \Gamma(v \circ v)(\lambda) \leq \Gamma(u)(\lambda) \leq \rho^*.$$

So, there exists $\Gamma(v)(\lambda)^*$ such that

$$(\lambda, \Gamma(v)(\lambda)^*) \notin \delta_{\mathbf{D}}, \ (\Gamma(v)(\lambda), \rho) \notin \delta_{\mathbf{D}}.$$

(P7) Let $(\lambda,\rho) \notin \delta_{\mathbf{D}}$. Then there exists $u \in \mathbf{D}, \Gamma(u)(\lambda) \leq \rho^*$. So, $\Gamma(u)^{-1}(\rho) \leq \lambda^*$. By Theorem

2.6(2), for $u \in \mathbf{D}$, there exists $u^s \in \mathbf{D}$ with $\Gamma(u)^{-1} = \Gamma(u^s)$ such that $\Gamma(u^s)(\rho) \leq \lambda^*$. Hence $(\rho, \lambda) \notin \delta_{\mathbf{D}}$.

(⇐) Let $(\lambda, \rho) \notin \delta_{\mathbf{D}}$. Then $\exists u \in \mathbf{D}, \Gamma(u)(\lambda) \leq \rho^*$. By Theorem 2.6(4), we have

$$\Gamma(u)(\alpha \odot \lambda) = \alpha \odot \Gamma(u)(\lambda) \leq \alpha \odot \rho^* = (\alpha^* \oplus \rho)^*.$$

Thus, $(\alpha \odot \lambda, \alpha^* \oplus \rho) \notin \delta_{\mathbf{D}}$.

By the definition of $\delta_{\mathbf{D}}$, $\mathbf{I}_{\delta_{\mathbf{D}}} = \mathbf{I}_{\mathbf{D}}$.

□

Example 3.6. Let $X = \{x, y, z\}$ be a set and $([0,1], \odot)$ a biquantale defined by $x \odot y = \max\{0, x + y - 1\}$ (ref.[4-6,14]). Define $\phi \in \Omega(X)$ as follows:

$$\phi(1_{\{x\}}) = \phi(1_{\{y\}}) = 1_{\{x,y\}}, \quad \phi(1_{\{z\}}) = 1_{\{z\}}$$

Then $\mathbf{U} = \{\psi \in \Omega(X) \mid \phi \leq \psi\}$ is a Hutton $(L, \otimes)$-uniformity on X (Example in reference [9]). We obtain an enriched $(L, \odot)$ proximity on X as follows, for each $\alpha \in L$,

$$(\lambda, \rho) \notin \delta_{\mathbf{U}} \ \text{iff} \begin{cases} \lambda \leq \alpha \odot 1_{\{x,y\}}, \ \rho \leq \alpha^* \oplus 1_{\{z\}}, \\ \lambda \leq \alpha \oplus 1_{\{z\}}, \ \rho \leq \alpha^* \odot 1_{\{x,y\}}, \\ \lambda \leq \alpha \odot 1_{\{z\}}, \ \rho \leq \alpha^* \oplus 1_{\{x,y\}}, \\ \lambda \leq \alpha \oplus 1_{\{x,y\}}, \ \rho \leq \alpha^* \odot 1_{\{z\}}, \end{cases}$$

For each $\lambda \in L^X$, by $\mathbf{I}_{\delta_{\mathbf{U}}} = \mathbf{I}_{\mathbf{U}}$ of Theorem 3.4,

$$\mathbf{I}_{\mathbf{U}}(\lambda)(x) = \mathbf{I}_{\mathbf{U}}(\lambda)(y) = \lambda(x) \wedge \lambda(y), \quad \mathbf{I}_{\mathbf{U}}(\lambda)(z) = \lambda(z).$$

We obtain

$$\mathbf{T}_{\mathbf{I}_{\mathbf{U}}} = \mathbf{T}_{\mathbf{I}_{\delta_{\mathbf{U}}}}$$

$$= \{\alpha \odot 1_X, \lambda \in L^X \mid \lambda(x) = \lambda(y) = a, \lambda(y) = b, \forall a, b, \alpha \in L\}.$$

We obtain an $(L, \odot)$-uniformity $\mathbf{D}_{\mathbf{U}} = \{u \in E(X \times X) \mid \Lambda(\phi) \leq u\}$. For each $\lambda \in L^X$, by Theorems 2.13 and 3.5,

$$(\mathbf{I}_{\delta_{\mathbf{D}_{\mathbf{U}}}} = \mathbf{I}_{\mathbf{D}_{\mathbf{U}}})(\lambda) = (\alpha \odot 1_{\{x\}}) \vee (\alpha \odot 1_{\{y\}}) \vee (\beta \odot 1_{\{z\}}).$$

where $\alpha = \lambda(x) \wedge \lambda(y)$, $\beta = \lambda(z)$. Hence $\mathbf{I}_{\mathbf{D}_{\mathbf{U}}} = \mathbf{I}_{\mathbf{U}} = \mathbf{I}_{\delta_{\mathbf{U}}} = \mathbf{I}_{\delta_{\mathbf{D}_{\mathbf{U}}}}$.

Theorem 3.7. Let δ be a principle, enriched $(L, \odot)$-proximity on X. We define $c_\delta : L^X \to L^X$ as follows:

$$c_\delta(\lambda) = \bigwedge\{\rho \in L^X \mid (\rho^*, \lambda) \notin \delta\}.$$

Then:
(1) (C1) $c_\delta(1_\emptyset) = 1_\emptyset$.
(C2) $\lambda \leq c_\delta(\lambda)$.
(C3) $c_\delta(\lambda \oplus \mu) \leq c_\delta(\lambda) \oplus c_\delta(\mu)$.
(C4) $c_\delta(c_\delta(\lambda)) = c_\delta(\lambda)$, $\forall \lambda \in L^X$.
(C5) $c_\delta(\alpha \odot \lambda) = \alpha \odot c_\delta(\lambda)$, $\forall \alpha \in L, \lambda \in L^X$.
(C6) $c_\delta(\bigvee_{j \in J} \lambda) = \bigvee_{j \in J} c_\delta(\lambda_j)$.
(2) $c_\delta \in \Omega(X)$ such that $c_\delta^{-1} = c_\delta$.
(3) $c_\delta(\lambda^*) = (\mathbf{I}_\delta(\lambda))^*$ for all $\lambda \in L^X$.
(4) c_δ has a right adjoint mapping $\mathbf{I}_\delta$ satisfying $c_\delta(\lambda) \leq \mu$ iff $\lambda \leq \mathbf{I}_\delta(\mu)$, for each $\lambda, \mu \in L^X$. Furthermore, $\lambda \leq \mathbf{I}_\delta(c_\delta(\lambda))$ and $c_\delta(\mathbf{I}_\delta(\mu)) \leq \mu$.

Proof. (C1) and (C2) are obvious.

(C3) Suppose there exist $\lambda_1, \lambda_2 \in L^X$ such that

$$c_\delta(\lambda_1 \oplus \lambda_2) \not\leq c_\delta(\lambda_1) \oplus c_\delta(\lambda_2)$$

By the definition of c_δ and Lemma 2.2(7), there exist $\rho_i \in L^X$ with $(\rho^*, \lambda_i) \notin \delta$ for $i = 1, 2$ such that

$$c_\delta(\lambda_1 \oplus \lambda_2) \not\leq \rho_1 \oplus \rho_2$$

Since $(\rho_i^*, \lambda_i) \notin \delta$ for $i = 1, 2$, $(\rho_1^* \odot \rho_2^*, \lambda_1 \oplus \lambda_2) \notin \delta$. Then $c_\delta(\lambda_1 \oplus \lambda_2) \leq \rho_1 \oplus \rho_2$. It is a contradiction. Hence c_δ holds (C3).

(C4) Suppose there exists $\lambda \in L^X$ such that $c_\delta(c_\delta(\lambda)) \not\leq c_\delta(\lambda)$. By the definition of $c_\delta(\lambda)$, there exist $\rho \in L^X$ with $(\rho^*, \lambda) \notin \delta$ such that $c_\delta(c_\delta(\lambda)) \not\leq \rho$. On the other hand, since $(\rho^*, \lambda) \notin \delta$, there exists $\gamma \in L^X$ with $(\rho^*, \gamma) \notin \delta$ and $(\gamma^*, \lambda) \notin \delta$. It implies $c_\delta(\gamma) \leq \rho$ and $c_\delta(\lambda) \leq \gamma$. Hence $c_\delta(c_\delta(\lambda)) \leq c_\delta(\gamma) \leq \rho$. It is a contradiction. Hence $c_\delta \circ c_\delta \leq c_\delta$. By the definition of c_δ, $c_\delta \circ c_\delta \geq c_\delta$.

(C5)

$$\begin{aligned} \alpha \odot c_\delta(\lambda) &= \alpha \odot \bigwedge\{\rho \in L^X \mid (\rho^*, \lambda) \notin \delta\} \\ &= \bigwedge\{\alpha \odot \rho \mid (\alpha^* \oplus \rho^*, \alpha \odot \lambda) \notin \delta\} \\ &= c_\delta(\alpha \odot \lambda). \end{aligned}$$

(C6) Suppose $c_\delta(\bigvee_{j \in J} \lambda_j) \not\leq \bigvee_{j \in J} c_\delta(\lambda_j)$. Since L is a completely distributive lattice, for each $j \in J$, there exists $\rho_j \in L^X$ with $(\rho_j^*, \lambda_j) \notin \delta$ such that

$$c_\delta(\bigvee_{j \in J} \lambda_j) \not\leq \bigvee_{j \in J} \rho_j.$$

Since δ is principle, $(\rho_j^*, \lambda_j) \notin \delta$ implies $(\bigwedge_{j \in J} \rho_j^*, \bigvee_{j \in J} \lambda_j) \notin \delta$. Hence $c_\delta(\bigvee_{j \in J} \lambda_j) \leq \bigvee_{j \in J} \rho_j$. It is a contradiction. Hence $c_\delta(\bigvee_{j \in J} \lambda_j) \leq \bigvee_{j \in J} c_\delta(\lambda_j)$.

Since $\lambda \leq \mu$ implies $c_\delta(\lambda) \leq c_\delta(\mu)$, we have $c_\delta(\bigvee_{j \in J} \lambda_j) \geq \bigvee_{j \in J} c_\delta(\lambda_j)$.

(2) By (1), $c_\delta \in \Omega(X)$. Let $(\rho^*, \lambda) \notin \delta$. Then $(\lambda, \rho^*) \notin \delta$ implies $c_\delta(\rho^*) \leq \lambda^*$. Thus,

$$c_\delta^{-1}(\lambda) = \bigwedge\{\mu \in L^X \mid c_\delta(\mu^*) \leq \lambda^*\}$$

$$\leq \bigwedge\{\rho \in L^X \mid (\rho^*, \lambda) \notin \delta\} = c_\delta(\lambda).$$

Let $c_\delta(\mu^*) \leq \lambda^*$. Since $c_\delta(\mu^*) = \bigwedge\{\rho_i \in L^X \mid (\rho_i^*, \mu^*) \notin \delta\}$, by (P), $(\bigvee \rho_i^*, \mu^*) \notin \delta$. Since $\lambda \leq (c_\delta(\mu^*))^* = \bigvee \rho_i^*$, we have $(\bigvee \rho_i^*, \mu^*) \notin \delta$ implies $(\lambda, \mu^*) \notin \delta$. So, $(\mu^*, \lambda) \notin \delta$. Hence $c_\delta^{-1} \geq c_\delta$.

(3) For all $\lambda \in L^X$, by Lemma 2.2(5),

$$\begin{aligned} c_\delta(\lambda^*) &= \bigwedge\{\rho \in L^X \mid (\rho^*, \lambda^*) \notin \delta\} \\ &= \left(\bigvee\{\rho^* \in L^X \mid (\rho^*, \lambda^*) \notin \delta\}\right)^* \\ &= (\mathbf{I}_\delta(\lambda))^*. \end{aligned}$$

(4) Since c_δ is a join-preserving map from (C6), it has a right adjoint $c_\delta^{\leftarrow}(\lambda) = \bigvee\{\rho \in L^X \mid c_\delta(\rho) \leq \lambda\}$. We only show that $c_\delta^{\leftarrow} = \mathbf{I}_\delta$.

Suppose there exists $\lambda \in L^X$ such that $c_\delta^{\leftarrow}(\lambda) \not\leq \mathbf{I}_\delta(\lambda)$. By the definition of $c_\delta^{\leftarrow}$, there exists $\rho \in L^X$ with $c_\delta(\rho) = \bigwedge\{\mu_i \mid (\mu_i^*, \rho) \notin \delta\} \leq \lambda$ such that $\rho \not\leq \mathbf{I}_\delta(\lambda)$. Since δ is principle, $(c_\delta(\rho)^*, \rho) \notin \delta$. Since $c_\delta(\lambda)^* \geq \lambda^*$, by (P2), $(\lambda^*, \rho) \notin \delta$. So, $\mathbf{I}_\delta(\lambda) \geq \rho$. It is a contradiction. Hence $c_\delta^{\leftarrow} \leq \mathbf{I}_\delta$.

Suppose there exists $\lambda \in L^X$ such that $c_\delta^{\leftarrow}(\lambda) \not\geq \mathbf{I}_\delta(\lambda)$. By the definition of $\mathbf{I}_\delta$, there exists $\rho \in L^X$ with $(\rho, \lambda^*) \notin \delta$ such that $c_\delta^{\leftarrow}(\lambda) \not\geq \rho$. Since $(\rho, \lambda^*) \notin \delta$, $c_\delta(\rho) \leq \lambda$. So, $c_\delta^{\leftarrow}(\lambda) \geq \rho$. It is a contradiction. Hence $c_\delta^{\leftarrow} \geq \mathbf{I}_\delta$.

Let $c_\delta(\lambda) \leq \mu$. By the definition of $c_\delta^{\leftarrow} = \mathbf{I}_\delta$, $\lambda \leq \mathbf{I}_\delta(\mu)$.

Let $\lambda \leq \mathbf{I}_\delta(\mu) = \bigvee\{\rho_i \in L^X \mid c_\delta(\rho_i) \leq \mu\}$. Since $\lambda \leq \bigvee \rho_i$, we have $c_\delta(\lambda) \leq c_\delta(\bigvee \rho_i) = \bigvee c_\delta(\rho_i) \leq \mu$.

Furthermore, $c_\delta(\lambda) \leq c_\delta(\lambda)$ iff $\lambda \leq \mathbf{I}_\delta(c_\delta(\lambda))$. Also, $\mathbf{I}_\delta(\mu) \leq \mathbf{I}_\delta(\mu)$ iff $c_\delta(\mathbf{I}_\delta(\mu)) \leq \mu$.

□

Theorem 3.8. Let δ be a principle, enriched $(L, \wedge)$-proximity on X. We define a subset $\mathbf{U}_\delta$ of $\Omega(X)$ as follows:

$$\mathbf{U}_\delta = \{\phi \in \Omega(X) \mid c_\delta \leq \phi\}.$$

Then (1) $\mathbf{U}_\delta$ is an Hutton $(L, \triangle)$-uniformity induced by δ where

$$\phi_1 \triangle \phi_2(\lambda) = \bigwedge\{\phi_1(\lambda_1) \wedge \phi_2(\lambda_2) \mid \lambda = \lambda_1 \wedge \lambda_2\}.$$

(2) $\delta_{\mathbf{U}_\delta} = \delta$

Proof. First, we show that $c_\delta \triangle c_\delta = c_\delta$.

Suppose there exists $c_\delta \triangle c_\delta(\lambda) \not\geq c_\delta(\lambda)$. Then there exist $\lambda_1, \lambda_2 \in L^X$ with $\lambda = \lambda_1 \wedge \lambda_2$ such that $c_\delta(\lambda_1) \wedge c_\delta(\lambda_2) \not\geq c_\delta(\lambda)$. By the definition of c_δ, there exist $\rho_i \in L^X$ with $(\rho_i^*, \lambda_i) \notin \delta$ for $i = 1, 2$ such that $\rho_1 \wedge \rho_2 \not\geq c_\delta(\lambda)$. On the other hand, since $(\rho_i^*, \lambda_i) \notin \delta$ for $i = 1, 2$, $(\rho_1^* \vee \rho_2^*, \lambda_1 \wedge \lambda_2) \notin \delta$. So, $\rho_1 \wedge \rho_2 \geq c_\delta(\lambda)$. It is a contradiction. Hence $c_\delta \triangle c_\delta \geq c_\delta$. Since $c_\delta \triangle c_\delta(\lambda \wedge 1_X) \leq c_\delta(\lambda) \wedge c_\delta(1_X) = c_\delta(\lambda)$, we have $c_\delta \triangle c_\delta \leq c_\delta$.

(U1) Obvious. (U2) Let $\phi_i \in \mathbf{U}_\delta$ for $i = 1, 2$. Then $c_\delta \leq \phi_i$. Since $c_\delta = c_\delta \triangle c_\delta \leq \phi_1 \triangle \phi_2$, $\phi_1 \triangle \phi_2 \in \mathbf{U}_\delta$

(U3) For $\phi \in \mathbf{U}_\delta$ with $c_\delta \leq \phi$, there exists $c_\delta \in \mathbf{U}_\delta$ such that $c_\delta \circ c_\delta = c_\delta \leq \phi$.

(U4) For $\phi \in \mathbf{U}_\delta$ with $c_\delta \leq \phi$, since $c_\delta^{-1} = c_\delta$, $c_\delta^{-1} \leq \phi^{-1}$ implies $\phi^{-1} \in \mathbf{U}_\delta$.

(2) Let $(\lambda, \rho) \notin \delta_{\mathbf{U}_\delta}$. Then there exists $\phi \in \mathbf{U}_\delta$ such that $\phi(\lambda) \leq \rho^*$. Since $\phi \in \mathbf{U}_\delta$, there exists $c_\delta \leq \phi$ such that $c_\delta(\lambda) \leq \phi(\lambda) \leq \rho^*$. Since $c_\delta(\lambda) = \bigwedge\{\rho_i \mid (\rho_i^*, \lambda) \notin \delta\} \leq \rho^*$, then $\rho \leq \bigvee \rho_i^*$ and $(\bigvee \rho_i^*, \lambda) \notin \delta$. Hence $(\rho, \lambda) \notin \delta$.

Let $(\rho, \lambda) \notin \delta$. Then $c_\delta(\lambda) \leq \rho^*$. Since $c_\delta \in \mathbf{U}_\delta$, $(\lambda, \rho) \notin \delta_{\mathbf{U}_\delta}$.

□

From Theorems 2.6, 2.8 and 3.8, we can obtain the following corollary.

Corollary 3.9. Let δ be a principle, enriched $(L, \wedge)$-proximity on X. We define a subset $\mathbf{D}_\delta$ of $E(X \times X)$ as follows:

$$\mathbf{D}_\delta = \{u \in E(X \times X) \mid \Lambda(c_\delta) \leq u\}.$$

Then $\mathbf{D}_\delta$ is an $(L, \wedge)$-uniformity such that $\delta_{\mathbf{D}_\delta} = \delta$.

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