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The Understanding of Improper Integration – A Case Study[1]

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Although improper integrals constitute a concept of great utility for Mathematics students, it appears that students are unable to assimilate this concept within the wider system of concepts they learn in their first year of Mathematics studies. In this paper we describe a competence model used in a study about the kind of understanding students possess about improper integral calculus when two registers of representation come into play. Competence will be considered as the coherent articulation of different semiotic registers.

After analysing the results of a questionnaire, six students were selected to be interviewed on the basis of their overall results and the significance of their answers. For the interview, five original questions from the questionnaire were used together with a new question. In this article we will analyse, from our theoretical point of view, the work carried out by one student who was interviewed to show how our competence model works and we will discuss this formal competence model used.

Keywords: improper integral, competence model, register of representation, articulation.
ZDM Classification: I55
MSC2000 Classification: 97B40

1. INTRODUCTION

At our university, the improper integral concept is introduced during the first year of the university Mathematics degree course, in the subject called "Mathematical Analysis

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II." This is an important concept, as it is used thereafter to calculate integral transforms and in the calculation of Fourier series, most fundamentally, as well as in other applications, especially in Physics.

In our official syllabuses, Integral Calculus is taught after Differential Calculus, and the contents of improper integration immediately after the contents of definite and indefinite integration. However, experience shows us that students have great difficulty understanding these concepts and relating them well to their previous mathematical knowledge (such as sequences, series and definite integrals); moreover, some paradoxes in the graphic register make them hesitate and feel uncertain[2]. We believe that students learn the tools and concepts needed for this field in a way which is decontextualized and unrelated to their previous learning.

With this in mind, we undertook to discover how First Year Mathematics students learn the contents related to improper integration, as well as to try to identify any obstacles and difficulties that might arise while they are learning these concepts.

While organising this study, one of the main difficulties that we found was the lack of published works focusing on the teaching and learning of the improper integral. Although this concept appears implicitly in some works, it is usually discussed in relation to the definite integral.

Orton (1983) points out that the introduction of integration to students is darkened by algebraic manipulations. In his research he asks students to calculate the integral of the function $1/x^2$ within the interval [-1, 2] and he provided them with a graph of the function. This question brought about special difficulties for the student participants, although they just had elementary knowledge about integration. Calvo (1997) points out that, when illustrating the definition of definite integral, a curve without any pathology is usually presented, within a positive interval, using a reasonable number of rectangles ... but there is no insistence on which of these elements are essential and which ones are not. Related to this, we have found that our students do not know which the necessary conditions to define the Riemann integral are.

Moreover, identifying the integral as an area may provoke wrong meaning ascriptions, in a way that students may not later distinguish the meaning of the integral when the function is positive and when it is negative. On the other hand, Schneider (1991) identifies some errors made by the students when calculating areas and volumes, which

[2] For example, we have observed that many students 'accept' that the integral $\int_1^\infty 1/x\,dx$ diverges while $\int_1^\infty 1/x\,dx$ converges. However, when they interpret that the first one is an area and the second one the volume produced by the same figure, a conflict arises. More details about ruptures between figural and conceptual aspects of mathematical concepts are analysed in Bartolini and Mariotti (1999).

show the presence of some misconceptions. In her work, she identifies the *heterogeneity of dimensions* obstacle, to which we will refer later, that induces students to mix quantities of different dimensions (volumes with areas and areas with lines); the evidences shown indicate that this obstacle has an epistemological character.

Rasslan and Tall (2002) designed a questionnaire to explore the cognitive schemas evocated by the students of the definite integral concept. For this purpose, they analysed whether the student participants, students were able to define the definite integral concept[3]; they state that only seven students out of 41 gave a definition of the definite integral concept[4]. In their questionnaire they also introduce two improper integrals, but the students do not identify them since they do not know the related theoretical aspects. Finally, we mention the work of Camacho and Aguirre (2001), who propose some improper integrals to a group of students and teachers; they observe that both have great difficulties to calculate their values and many get lost in algebraic manipulations. Camacho and Aguirre also observe that both teachers and students tend to substitute the extreme values of the integral and that they do not calculate integrals by means of the limit processes, but rather generalise the Barrow rule and directly substitute the integral's extremes in the variable, even when one of the extremes is infinity.

Among other things, we decided to use non-algorithmic and non-routine type questions (Monaghan et al. 1999; González-Martín & Camacho 2004) to check students' levels of understanding, where understanding is seen from the point of view of the semiotic systems of representation (Duval 1993). We also wished to find out which system of representation students feel most comfortable with and whether they make any geometrical interpretation of the results they obtain. The following are guiding questions that helped us to structure this study:

- What intuitive approaches do students use?
- What kind of articulations do they perform, if any, between different systems of representation in questions about improper integrals?
- Do they carry out any process of transference (Hitt 2000, 2001) from other concepts they have learned prior to the ones under study and, if so, which ones?

Bearing these ideas in mind, a theoretical framework was established in order to evaluate students' levels of understanding when handling the algebraic (or formal) and graphic representation systems. With the help of this theoretical framework, we developed a competence model that would allow us to classify levels of understanding

[3] This question coincides with our first question, both in the questionnaire and the interview protocol.

[4] Although, in our opinion, their answers are doubtfully correct. Four students define it as '*the area between the graph and the OX axis from $x = a$ and $x = b$*', which does not take into account the change of sign of the function, and three students define it as '$\int_a^b f(x)dx = F(b) - F(a)$', purely operative definition that obviates some conditions.

when only two mathematical representation systems come into play.

2. THEORETICAL FRAMEWORK

In accordance with Duval (1993), we consider that the distinction between an object and its representation is fundamental in Mathematical understanding. The use of different semiotic representations of a mathematical object is absolutely necessary to achieve this aim from the theoretical point of view we adopt. One definition of what we understand as semiotic representations is given by Duval (1993):

Mental representations cover the set of images and, globally, the conceptions that an individual may have about an object, about a situation and about whatever is associated to them.

Semiotic representations are productions constituted by the use of signs which belong to a system of representation, which has its own meaning and functioning constraints.

A geometric figure, a statement in natural language, an algebraic formula, and a graph, are semiotic representations that belong to different semiotic systems.

Duval goes further on this idea to characterise a representation register as follows:

A semiotic system may be a representation register if it allows three cognitive activities related to semiosis[5]:

- The formation of an identifiable representation as a representation of a given register.
- The treatment of a representation, which is the transformation of the representation within the same regis ter where it has been created. The treatment is an internal transformation of a register.
- The conversion of a representation, which is the transformation of the representation to another one from other register which preserves the totality or a part of the meaning of the initial representation. The conversion is an external transformation to the initial register.

Duval (1993) stated that, as each representation is partial with regard to what it represents, we must consider as absolutely necessary the interaction between different representations for the formation of the concept. It can be said, so, that a concept is built up by tasks which involve the use of different representation registers and promote the coherent articulation between representations. From this point of view, an individual's knowledge about a concept is stable if s/he is able to articulate different representations of the concept free of contradictions.

From this perspective, we are interested in identifying some errors and difficulties that naturally arise when making conversions between the two chosen registers (algebraic and

[5] It is defined as the apprehension or the production of a semiotic representation.

graphic) or when carrying out operations in one of them. We also aim to study relevant conceptual and procedural cognitive aspects.

Socas (1997) distinguishes five basic elements as producers of difficulties in the Mathematics curriculum: the necessary abilities to develop mathematical capabilities which define a student's competence in Mathematics, the necessity of previous contents, the required abstraction level and the logical nature of school Mathematics. These they may be organised as follows:

- Difficulties associated with the complexity of the objects of Mathematics.
- Difficulties associated with the processes of mathematical thinking.
- Difficulties associated with the processes of teaching developed for the learning of Mathematics.
- Difficulties associated with the processes of the students' cognitive development.
- Difficulties associated with affective and emotional attitudes towards Mathematics.

The two first difficulties are related to the discipline itself, the third to the teaching processes, and the fourth to the students' cognitive processes and the fifth to the affective field.

These difficulties are related and form networks where they are strengthened, taking shape in practice as obstacles and becoming apparent as errors. Socas (1997) also gives some characteristics of the obstacles:

- An obstacle is an acquired knowledge, not a lack of it.
- It has an efficiency domain and is used to produced answers adapted to a certain context.
- When this knowledge is used outside its context, it generates inadequate, even incorrect, answers.
- It is resistant and will be more resistant the more acquired it is or the more it has shown its efficiency in the former validity domain.
- Even its inaccuracy is noted, it carries on becoming apparent sporadically.

Socas (2001) developed a study similar to ours taking into account the algebraic language in Secondary School. Adapting parts of his proposal to our work, the bases for our model are the following: Stages of cognitive development of representation systems (Socas 1997), Duval's theory of semiotic representation registers and cognitive functioning of thought (Duval 1993; Hitt 2000) and difficulties and errors in learning (Socas 2001).

Several researchers, such as Hitt (2000), consider that transformation within a representation register and conversion between registers are not the only important tasks, but that it is also important to confront examples and counterexamples. On the other hand Duval's work does not mention the role of representations in problem solving. With these theoretical aspects, the notion of transference gains great importance as it proves to be essential in the field of problem solving.

Competence will be understood to be the coherent articulation of different semiotic registers: To be competent in Mathematics is to be able to articulate coherently the different representations of a mathematical concept when having to solve "non-routine" problems. According to us, the evaluation of our students' knowledge means that this should be analysed on the basis of activities that aim to clarify possible connections (articulations) made by them while constructing a given concept.

To design our competence model, we decided to adapt different stages of development noted in cognitive representation systems in the case of algebraic language (Socas 2001; Hiebert & Carpenter 1992) to our own concept when using two registers of representation. These stages are the semiotic (1), the structural (2) and the autonomous (3) and they originally refer to the relationships between old and new knowledge in the algebraic register of representation. In our case, we will use them as a guide to refer the relationships between graphic and algebraic representations in the construction of the concept of improper integral. At each stage two categories of behaviour can be distinguished (A and B). This way, we generate the following categories:

Category 1A:	The student's ideas about the improper integral are imprecise and s/he mixes incoherently different semiotic representations.
Category 1B:	The student recognizes the elements of a semiotic representation system in relation to the improper integral.
Category 2A:	The student knows one semiotic representation system and carries out transformations within that representation system.
Category 2B:	The student correctly carries out activities to make conversions from one semiotic representation system to another; there is a system to these conversion activities, which the student controls to facilitate conversions from one system to another.
Category 3A:	The student articulates two semiotic representation systems. S/he can take either of them to refer correctly to the improper integral object independently from the other. The student autonomously handles the two semiotic representation systems.
Category 3B:	The student coherently articulates different semiotic representation systems and exercises control over the semiotic representations s/he uses. S/he is aware of the improper integral object as structure and can control coherent and incoherent aspects of this object.

In this study we only examine the following registers of semiotic representation: Representations in Habitual Language (H)[6], Representations in Formal (algebraic)

[6] As Radford (2002) states, it is also important to pay attention to the interface between the spoken and the seen or perceive.

Language (F), and Representations in Graphical Visual Language (G). For each action, we indicate with a subscript the register (or the interval between two registers) in which we consider the action to occur. Thus the actions identified are as follows:

- Recognition of the elements of a semiotic representation system: $R_{H,}$, R_{F}, R_G
- Internal transformations in a semiotic representation system: T_H, T_F, T_G
- Conversions (external transformations) between semiotic representation systems: $C_{H\to F}$, $C_{H\to G}$, $C_{F\to G}$, $C_{G\to F}$
- Coordination between different semiotic representation systems: $C_{F\leftrightarrow G}$, $C_{G\leftrightarrow F}$, $C_{H\leftrightarrow F}$, $C_{H\leftrightarrow G}$
- Production of semiotic representations to solve a task: PS_F, PS_G

This notation will be used to describe the demands made on the students interviewed when answering the questions set. Also, this code will be adapted to place students' actions in one of the six behaviour categories described above.

3. METHODOLOGY

We developed an initial questionnaire (Q1) consisting of nine questions that covered various aspects of improper integrals both from an algebraic and a graphic point of view (González-Martín & Camacho 2002). This Q1 was administered to thirty-one, First-Year Mathematics students: thirteen male students and eighteen female students.

After analysing the data obtained from the Q1, we wrote the interview script and selected six students to interview (fifteen days after the questionnaire was administered) on the basis of the overall results of their questionnaires and the significance of their answers. The six students covered almost a fifth of the total population and their levels of performance on the questionnaire varied from the lowest range to the highest. For the interview, five original questions from the questionnaire were used together with a new question. The aim of adding one new question was to use it as a control question; we wanted to see how the students faced a new question and to hear their reasoning. We wanted to avoid the risk they would have studied the answers to all the questions of the questionnaire. The new question involved relations between series and improper integrals.

The type of interview held might be called a task-based interview, in the sense given to this term by Goldin (2000). The interviews can also be called semi-structured (as the only scripts were the six questions asked) and focalised (as all the questions covered the same theme). The interviews were both video- and audio-taped. At the same time as making an exhaustive analysis of the interviews, the course contents taught by the subject teacher were checked.

4. RESULTS ANALYSIS

To illustrate how the competence model works, we will present our analysis of data obtained from the interview with Student MA4. We will describe three of the questions asked during the interview and the actions undertaken by the student while solving them.

We should point out that the interview lasted approximately 50 minutes and the student interviewed was studying Mathematics at this level for the first time.

Item 1. How would you explain to a classmate the meaning of $\int_a^b f(x)dx$?

We attempted to check the students' understanding of the definite integral, as the improper integral is obtained when one of the two following conditions fails:

1) Finite interval;
2) Function bounded within the interval.

With this question we wanted to know whether these conditions are clear to the students, or well assimilated, the concept from which it is supposed to be constructed that of improper integral. In the analysis of the questionnaire we observed that almost all the students defined it as an area (29 out of the 31 students). Considering that the graphic register would be very present in the other questions to be posed, we were interested in knowing whether the students are coherent with the registers they use to define the integral and to work with it.

We also observed in the questionnaires that only four students mentioned that the definite integral is an area only if the function is positive within the interval of integration and that four other students referred to the interval of integration as being finite. With this question we intended to find out whether the students did not mention this as a matter of a lapse, or whether they thought it always represents an area. Finally, we though that the students would feel more comfortable if we began the interviews with a question they considered beforehand as "easy."

The actions we considered a priori, according to our competence model, are the following ones:

Table 1. Actions of Item 1

R_H, R_F
$C_{H\to F}$, $C_{F\to G}$
$C_{F\leftrightarrow G}$, $C_{G\leftrightarrow F}$
PS_G

Student MA4's answer was:

S: *Let's see* (he writes on the board: $\int_a^b f(x)dx$). *Well, the integral Is ... the ... let's see how I can explain ...* (he continues looking at the board). *Er ... It'd be ... Well... The integral of a function is to see ... er... the surface*[7] *the function covers between* (he starts drawing a curve on coordinated axes; essentially, it is the same as the one drawn by previous subjects interviewed. See below) *... which is the function* (he points at the curve) *and axis OX, for example...*

I: *Good.*

S: *... or the axis OY, depending.*

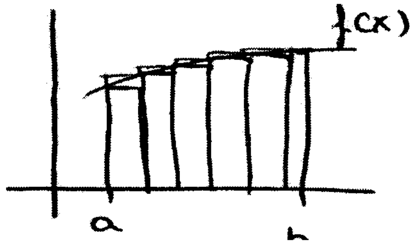

Figure 1. Student MA4's drawing

In the Q1, this student did not draw any graph to explain this definition. We can see that he makes no distinction for the sign of the function. Consider his answer to the next question:

I: *And does it always represent the area of the surface or are there cases where it doesn't...?*

S: *Er... well... yes, exactly. Sometimes you can also represent... volumes putting... multiplying by* π *and squaring the... function* (he changes the text until it is written: $\pi\int_a^b f(x)^2 dx$). *Then you can... the functions supposed to be turning* (he goes over to the graph he drew and makes a gesture as if turning around the axis OX), *and also represents volumes...*

We note that his definition does not include a consideration of the sign of the function. When asked about the conditions that the function needs to fulfil, he at first believed that it is continuity, but then corrects himself and says that he cannot remember. We can deduce that his idea of definite integral is not absolutely clear, as he is unable to put conditions on the function; the student is not fully coordinating the formal register with the graphic ($C_{F\leftrightarrow G}$):

I: *Do you think we have to impose any condition to the function or something like that, or...? For every function?*

S: *Well, yes, that... the function should be continuous* (he goes over again a piece of the curve) *because... well... no, it hasn't always to be* (he stares at the graph)... *eh...* (pause, whispers something to himself; long pause). *So I don't remember the conditions.*

I: *Aha.*

[7] This student uses the term 'surface' in the sense of 'plain surface' to refer the term 'area'.

S: *I know there are functions... it had to be continuous sometimes, or derivable, but... I don't remember the exact cases...*

Similarly, he does not seem to carry out actions from the graphic register. We also believe that his conversion from the formal to the graphic register is incomplete ($C_{F\to G}$). The actions undertaken by this student are schematised in Table 4.

Item 2. We know that f(x) is a continuous and strictly positive function. Suppose that we have calculated $\int_1^{10} f(x)\ dx$, and that this gives us a finite quantity. We now calculate $\int_1^{20} f(x)\ dx$ and this also gives us a finite quantity which is greater than the first one. If we continue calculate $\int_1^{30} f(x)\ dx$, $\int_1^{40} f(x)\ dx$, $\int_1^{50} f(x)\ dx$, ..., $\int_1^{10n} f(x)\ dx$, ..., what do you think will happen? Might this sequence ever converge? What type of f(x) functions can bring about a converging sequence?

This is the only new question introduced in the protocol with the aim of providing data from which we can analyse the students' reasoning faced with a new situation, which, in principle, the student did not foresee. In this case, we are trying to observe what type of transferences students can undertake from their previous knowledge about sequences and series. We also wish to check whether they achieve coordination between the algebraic and graphic registers imagining infinite shapes that might enclose a finite area.

The actions we expected in this question are shown below:

Table 2. Actions of Item 2

R_H, R_F
T_F
$C_{H\to G}$, $C_{F\to G}$, $C_{G\to F}$
$C_{F\leftrightarrow G}$, $C_{G\leftrightarrow F}$
PS_G

Student MA4 directly tackles the text in an intuitive way, since he tells us he does not know many of the theorems and does not have a good grasp of the formal tools:

S: *Well... er... No, it doesn't have to be infinite. You can't say why it might tend to infinity or not. Because there are integrals that... keep growing but... on the... the value is bounded... it gets closer and closer to a value*[8].

I: *Good.*

S: *It doesn't tend to infinite, but to a value; for example π, or a function I can't... er... I can't remember now which one was... Let's see, which tended... which... which tended, tended... was greater and greater, but it didn't... didn't go beyond 3.14.*

[8] In the case of this student, other excerpts make us believe that he thinks that the limit may be reached. We have seen in other students they use the term *tend to* in the sense of '*approaches, but without reaching*'. For more information about the different conceptions of students about these terms, see Monaghan (1991).

Although he does not appear to be very sure, he states that the sequence does not have to always diverge. In this case, it appears that he is transferring his knowledge of improper integrals over to this situation with sequences (action T_F). However, what is unclear is in which register he operates in order to state that we cannot say anything.

When asked what type of functions can give rise to a convergent sequence, he says he does not know the equations but rather the graphs and draws (PS_G) a bell-shaped function and another of the $1/x^k$ type. When asked if he can think of any necessary condition to assure convergence, in spite of mentioning that his two functions tend asymptotically to zero, he is unable to verbalize any condition[9]:

I: So, er... in general... does it occur to you any... any characterization of the functions? You've drawn two... Do you think they have any characteristic in common?

S: (looks at both graphs) *Er...* (Pause) *Well, they both tend to infinity... that... let me see... They're continuous in such a certain point, but no...* (Points to the first one) *This one is continuous in all? But this one isn't* (points to the second one). ... *They both...* (Nods) *they both tend to zero when going to infinity and they're functions that... the surface when you do the integral so... it's a finite value.*

I: *So, in general, in order to what you have said, does it occur to you any... indispensable condition to this integral to be finite?*

S: (stares at the blackboard) *First, to be integrable* (smiles). ... *Here they both are integrable ... But now I don't know...*

We can note how the student has a clear idea about the form of the functions although he cannot manage to verbalize it (he does not do $C_{G\to F}$). Though he cannot remember any formula or equation, he does have an idea about their shape. He can produce answers in the graphic register (PS_G), but without a complete meaning for him.

Also, when asked to characterize his functions, he begins to use "formal" tools (continuity, derivability), and abandons the first result arrived at intuitively.

We believe, therefore, that he has undertaken conversions from the habitual and formal languages to the graphic register ($C_{H\to G}$, $C_{F\to G}$), but was incapable of formally expressing the graphic result. So, although we think he has to some extent coordinated his formal and graphic representations ($C_{F\leftrightarrow G}$), he has not done so from the graphic to the formal ($C_{G\leftrightarrow F}$). The student's actions are tabulated at the end of the section.

Item 6. We know that $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$ and $\sum_{n=1}^{\infty}\frac{1}{2}=\frac{\pi^2}{6}$. In view of these results, what can you say about the value of $\int_1^{\infty}\frac{1}{x}dx$ and $\int_1^{\infty}\frac{1}{x^2}dx$? Use the graph provided

[9] We should point out that it is not necessary for the function to tend to zero to have a convergent integral, under the conditions given in the text. However, at this level, it seems a natural answer.

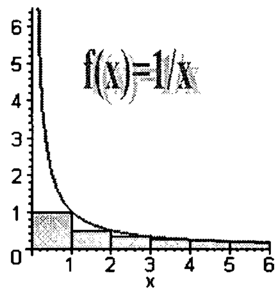

Figure 2. $\sum_{n=1}^{\infty}\frac{1}{n}$

This question was answered in the written test in the expected way, that is, establishing an explicit relation between the integrals, the series and using the graph provided, only by one student out of the 31 who filled in the questionnaire (Student MA5). This was one of the reasons to include it in the protocol, as we wanted to find out why some students had not even tackled it (even some students with a high performance in the whole test). We also wondered whether with some clue the students would establish the needed relations.

In this question, an explicit relation between series and integrals is approached. Again, it is necessary to use the combination of the graphic and formal registers to obtain all the richness of this relation. Apparently, the graph had been rejected by the great majority of the participant students. We also wondered why one might prefer to work exclusively in the algebraic register a question which explicitly asks one to use a graph. One of the doubts we had was whether this was due to non-transference of knowledge about series or to a non-adequate transference of knowledge about integration to the graphic register.

The expected actions are the following:

Table 3. Actions of Item 6

R_G, R_F
T_F, T_G
$C_{F\to G}, C_{G\to F}$
$C_{F\leftrightarrow G}, C_{G\leftrightarrow F}$
PS_F, PS_G

Student MA4's first answer was:

S: (he keeps reading the text) *Er... it's... Let's see, integrals have to do with the sum... where... what the integral does is add. It adds rectangles... Infinite rectangles.*

I: *Good.*

S: *Well... That's why they coincide, don't they?* (he speaks to himself while reading the paper). *Let's see how it'd be... That is...* (he looks at the paper again). *Well, yes. Er...* (he keeps looking at the paper; there's a long pause).

The transference process (TF) is not altogether correct, as he seems to believe the result of the additions will match the value of the integrals. From another fragment of the interview, we can see that he cannot interpret the meaning of the rectangles that appear in the graph either. Our opinion, then, is that he is not coordinating the graphic representation with the algebraic information he is provided with (CG↔F).

I: *Do you know what it represents... the graph you have?*
S: (continues looking at the paper) *What it represents?*
I: *Yes ... Do you think the graph has any relation with the heading of the question or...?*
S: *Yes ... Well, what it represents is... the surface of the graph, but it happens that the rectangles...*
I: *Yes?*
S: *You're taking... the base equal to one, the unit.*
I: *Aha.*
S: *What the integral does is taking the base and makes it smaller every time, making it tend to infinity. To ... to zero.*
I: *Aha.*
S: *So, there are every time more ... more rectangles. There are infinity rectangles.*

When asked to calculate the area of each rectangle, he realizes they represent the sum and his interpretation is that the integral is the limit when a tends to zero of a/n. We believe he was unable to explain the relationship between the additions and the integrals.

5. CONCLUSIONS

As it has been already said, we have illustrated how our competence model can be used, so we do not present an exhaustive analysis of every item of the interview. For further details of this analysis and the whole interviews, see González-Martín (2002). The actions undertaken by the student throughout the interview are shown in the Table 4 below, where we have highlighted in bold the items analysed in this paper:

Once the six questions asked during the interview were analysed, we concluded that the student's level of coordination between the algebraic and graphic registers could be placed in category 2B, since we have seen the student holds up in the formal register and carries out conversion activities from this register to the graphic one. We do not think he could be placed in the third category, because the articulations he develops are not always complete or correct (see the third column of Table 4).

The main difficulties we detected are due to the lack of meaning or knowledge of previous concepts (we have seen his definition of integral does not take into account the changes in the sign of the function; he also presents a lack of knowledge about concepts related to sequences and convergence).

Table 4. Summary

Item	Actions he performs	Actions he does not perform	Incomplete Actions
First	R_H, R_F $C_{H\to F}$	$C_{G\leftrightarrow F}$	$C_{F\to G}$ $C_{F\leftrightarrow G}$ PS_G
Second	R_H, R_F T_F $C_{H\to F}, C_{F\to G}$	$C_{G\to F}$ $C_{G\leftrightarrow F}$	$C_{F\leftrightarrow G}$ PS_G
Third	R_F T_F $C_{F\leftrightarrow G}$		
Fourth	R_F	$C_{G\leftrightarrow F}$ PS_G	T_F $C_{F\leftrightarrow G}$
Fifth	R_H, R_F T_F $C_{H\to F}, C_{H\to G}, C_{F\to G}$ $C_{F\leftrightarrow G}$ PS_F		R_G $C_{G\to F}$ $C_{G\leftrightarrow F}$
Sixth		T_F, T_G $C_{G\leftrightarrow F}$ PS_F, PS_G	R_G, R_F $C_{F\to G}, C_{G\to F}$ $C_{F\leftrightarrow G}$

We can also see that, although he has managed to make representations in the graphic register to some extent, going from graphic to the formal remains more difficult than going from the formal to the graphic. The student compensates for the lack of meaning or the paradoxes evident in some graphic representations by relying on algebraic calculations. This is undoubtedly the system of reference he feels most secure in, in spite of his own limitations.

We believe that in this study the use of representation systems, including the role of transference and error, is of value to describe how Mathematics students coordinate the information given algebraically with graphic representations. The competence model we present in this study appears to be a useful instrument to describe the actions students undertake and to study what type of understanding they possess about the concept of improper integral when algebraic and graphic registers come into play.

As we have studied the case of just two registers of representation, the categories of our model may be particularised to this situation. This way, we should have a competence model to study the coordination between both the graphic and algebraic registers in our students. It is important to select conversion and coordination activities where analysing the comprehension level would be easier. Questions where only one register is used each time will be also included in later works.

In this particular case, teaching using more graphic elements and reinforcing previous knowledge might help this student overcome the difficulties we have seen.

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