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REMARKS ON THE RADIAL SOLUTIONS OF THE SELF-DUAL ABELIAN CHERN-SIMONS MODEL

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ABSTRACT. We consider the nonrelativistic limit for the radial solutions to the self-dual equations in the self-dual Abelian Chern-Simons model. We achieve the limit by fixing the common maximum value of solutions.

1. INTRODUCTION

Let us consider the following equation which arises from the relativistic self-dual Abelian Chern-Simons model in $\mathbb{R}^2$ [7, 10]:

$$\Delta u = \frac{4q^4}{\kappa^2 c^4} e^u (e^u - \sigma^2) + 4\pi \sum_{j=1}^{k} n_j \delta_{p_j}, \tag{1.1}$$
$$u \to -\infty \quad \text{as} \quad |x| \to \infty.$$

Here, $\kappa > 0$ is the Chern-Simons coupling constant, q is the charge of electron, $\sigma > 0$ is the symmetry breaking parameter, and c is the speed of light. The vortex points $p_1, \quad , p_k$ are distinct in $\mathbb{R}^2$, $n_1, \quad , n_k$ are positive integers, and δ_{p_j} denotes the Dirac measure concentrated on the point p_j. The existence and properties of solutions to (1.1) have attracted much attention and some results can be found in [1, 3, 4, 5, 12]. See also [13] for the results on the other boundary conditions.

In this paper, we are interested in the limit $c \to \infty$ for the solutions to (1.1), which is called the nonrelativistic limit. Considering the Lagrangian density of the relativistic Abelian Chern-Simons model, we find the mass of the scalar Higgs field is $m = \hbar q^2 \sigma^2 / \kappa c^3$, where $\hbar$ is the Plank constant (see [6, 7, 10] for more information). We will accompany the limit $c \to \infty$ with m fixed. To this end, we set

$$\kappa c = \text{constant} =: \mu.$$

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Then

$$\kappa = \frac{\mu}{c} \quad \text{and} \quad \sigma = c\sqrt{\frac{m\mu}{\hbar q^2}}, \tag{1.2}$$

which give a difference from the method in [9]. Indeed, in [9], not only m but also κ is kept fixed and only σ varies in the limit $c \to \infty$. Then the matter part of the nonrelativistic Lagrangian contains the constant c, which yields difficulties in proving the nonrelativistic limit by mathematical arguments. To overcome this, we vary both κ and σ as (1.2) in the limit $c \to \infty$ (see [6] for more information).

If we set $v = u + \ln 2m$, then (1.1) becomes

$$\begin{aligned} \Delta v &= \frac{q^4}{m^2\kappa^2c^4}e^v(e^v - 2m\sigma^2) + 4\pi\sum_{j=1}^{k} n_j\delta_{p_j} \\ &= \frac{q^4}{m^2\kappa^2c^4}e^{2v} - \frac{2q^2}{\hbar\mu}e^v + 4\pi\sum_{j=1}^{k} n_j\delta_{p_j}. \end{aligned} \tag{1.3}$$

Letting $c \to \infty$, we may derive formally that v converges to a solution of

$$\begin{aligned} &\Delta w = -\frac{2q^2}{\hbar\mu}e^w + 4\pi\sum_{j=1}^{k} n_j\delta_{p_j}, \\ &w \to -\infty \quad \text{as} \quad |x| \to \infty. \end{aligned} \tag{1.4}$$

This is the well-known Liouville equation with singular sources and appears in the nonrelativistic self-dual Abelian Chern-Simons model [8, 9].

It is known that there are infinitely many solutions of (1.3). See [1, 3, 4, 12] for example. Therefore it is not surprising that there may be a sequence of solutions to (1.3) which blows up as $c \to \infty$ instead of converging to a solution to (1.4). In this point of view, it is important to find a sequence of solutions to (1.3) converging to a solution to (1.4). This means we need a kind of conditions for solutions of (1.3) to make sure the convergence in the limit $c \to \infty$. In [6], the authors consider such a condition for radial solutions when there is only one-vortex point. They fix a shooting constant for radial solutions in the limit $c \to \infty$ to prove the nonrelativistic limit. In this paper, we consider a different condition for radial solutions to guarantee the limit. In fact, we will show that if we choose a sequence of radial solutions to (1.3) with the common maximum value, then it converges to a solution to (1.4). In the next section, we prove this statement and establish the nonrelativistic limit.

2. Main Theorem

In this section, we assume that there is only one-vortex point. In this case, the equations (1.3) and (1.4) are rewritten as

$$\text{(2.1)}\qquad \begin{aligned}&\Delta v = \frac{q^4}{m^2\kappa^2c^4}e^v(e^v - 2m\sigma^2) + 4\pi N\delta_0,\\ &v \to -\infty \quad \text{as } |x| \to \infty,\end{aligned}$$

and

$$\text{(2.2)}\qquad \begin{aligned}&\Delta w = -\frac{2q^2}{\hbar\mu}e^w + 4\pi N\delta_0,\\ &w \to -\infty \quad \text{as } |x| \to \infty.\end{aligned}$$

We are interested in the radial solutions to (2.1) and (2.2) .

To investigate (2.1) and (2.2) further, we first transform the equation (2.2) into

$$\text{(2.3)}\qquad w_{rr} + \frac{1}{r}w_r = -\frac{2q^2}{\hbar\mu}e^w, \quad r = |x| > 0$$

with the constraint

$$\text{(2.4)}\qquad \lim_{r\to 0}\frac{w(r)}{\ln r} = \lim_{r\to 0} r\,w_r(r) = 2N, \quad \lim_{r\to\infty} w(r) = -\infty.$$

It follows from the result of [2, 11] that every radial solution to (2.3) with (2.4) is of the form

$$\text{(2.5)}\qquad w(r) = \ln\frac{8\lambda(N+1)^2r^{2N}}{(\lambda + r^{2N+2})^2} - \ln\frac{2q^2}{\hbar\mu},$$

where λ is any positive constant.

Similarly, for (2.1) we get

$$\text{(2.6)}\qquad v_{rr} + \frac{1}{r}v_r = \frac{q^4}{m^2\kappa^2c^4}e^v(e^v - 2m\sigma^2) =: g(v)$$

with the constraint

$$\text{(2.7)}\qquad \lim_{r\to 0}\frac{v(r)}{\ln r} = \lim_{r\to 0} r\,v_r(r) = 2N, \quad \lim_{r\to\infty} v(r) = -\infty.$$

Concerning to (2.6) and (2.7), we have one parameter family of radial solutions by the result of [12] as follows:

Theorem 1 ([12]). *For any number $\alpha \le \ln m\sigma^2$, there exist a number $r_0 = r_0(\alpha)$ and a solution $v(r) = v(r;\alpha)$ to (2.6) and (2.7) satisfying $v(r_0) = \alpha$, $v_r(r_0) = 0$. Moreover, α is the unique maximum value of $v(r;\alpha)$.*

In [12] Theorem 1 was proved under the conditions $c = q = \sigma = 1$ and $m = 1/2$ for simplicity. In the following, we give a sketch of its proof keeping all the constants for later use.

Sketch of Proof of Theorem 1. Let $\alpha \leq \ln m\sigma^2$ be fixed. By an elementary ODE argument, one can show that for any $r_0 \in \mathbb{R}^+$, there exists a unique global solution $v(r) = v(r; r_0, \alpha)$ to (2.6) with initial data $v(r_0) = \alpha$, $v_r(r_0) = 0$. Moreover, it holds that $v(0) = -\infty = v(\infty)$, $v_r(r) > 0$ for $r < r_0$, and $v(r) \leq \alpha$ for all $r > 0$.

It remains to find a suitable r_0 such that $v(r; r_0, \alpha)$ satisfies (2.7). To this end, let us define $\eta(r_0, \alpha) = \lim_{r\to 0} rv_r(r; r_0, \alpha)$. Multiplying the inequality from (2.6)

$$(rv_r)_r > -\frac{2q^4\sigma^2}{m\kappa^2c^4}re^v \tag{2.8}$$

by $rv_r + 2$ and integrating it over (r, r_0) for $r < r_0$, we obtain

$$0 < rv_r(r; r_0, \alpha) < -2 + 2\sqrt{1 + \frac{q^4\sigma^2}{m\kappa^2c^4}e^\alpha r_0^2} =: R(r_0, \alpha), \quad \forall r < r_0. \tag{2.9}$$

In particular,

$$\eta(r_0, \alpha) < R(r_0, \alpha). \tag{2.10}$$

Integrating (2.9) on (r, r_0), we obtain

$$\alpha + R(\ln r - \ln r_0) < v(r; r_0, \alpha) \leq \alpha \quad \text{for} \quad r < r_0.$$

Since g defined in (2.6) is decreasing if $v \leq \ln(m\sigma^2)$, we are led to

$$\frac{1}{r}(rv_r)_r = g(v) < g(\alpha + R(\ln r - \ln r_0)).$$

Since $\alpha \leq \ln(m\sigma^2)$, the integration of the above inequality over $(0, r_0)$ yields that

$$\begin{aligned}\eta(r_0, \alpha) &> \frac{q^4}{m\kappa^2c^4}e^\alpha r_0^2\Big(\frac{2\sigma^2}{R+2} - \frac{e^\alpha}{m(2R+2)}\Big) \\ &> \frac{q^4\sigma^2e^\alpha r_0^2}{2m\kappa^2c^4}\Big(1 + \frac{q^4\sigma^2}{m\kappa^2c^4}e^\alpha r_0^2\Big)^{-1/2} =: S(r_0, \alpha).\end{aligned} \tag{2.11}$$

Now let T_1 solve $R(T_1, \alpha) = 2N$, namely,

$$T_1 = T_1(\alpha) = \frac{\sqrt{N^2 + 2N}}{e^{\alpha/2}}\frac{\sqrt{m}\kappa c^2}{q^2\sigma}. \tag{2.12}$$

Then by (2.10), $\eta(T_1, \alpha) < 2N$. Similarly, let T_2 be a solution to $S(T_2, \alpha) = 2N$. Then we have

$$T_2 = T_2(\alpha) = \frac{2(2N^2 + N\sqrt{4N^2+1})^{1/2}\sqrt{m}\kappa c^2}{q^2\sigma e^{\alpha/2}}. \tag{2.13}$$

Obviously, $T_1 < T_2$. By virtue of (2.10) and (2.11), we find

$$\eta(T_1, \alpha) < 2N < \eta(T_2, \alpha).$$

Since η is continuous on r_0, there exists $r_0 = r_0(\alpha) \in (T_1, T_2)$ such that $\eta(r_0, \alpha) = 2N$, which completes the proof. □

Remark 1. It is easily shown that if v is a solution to (2.6) and (2.7), then $v < \ln(2m\sigma^2)$ by the maximum principle. Theorem 1 shows the existence of solutions for $\alpha = \sup v \le \ln(m\sigma^2)$. It is still open whether there exists a solution with the property $\ln(m\sigma^2) < \alpha < \ln(2m\sigma^2)$.

Now we proceed in the proof of nonrelativistic limit for the solutions given by Theorem 1. This theorem implies that for each c there are infinitely many solutions to (2.6) and (2.7). Thus, as mentioned at the end of the previous section, if we choose an arbitrary sequence of solutions as $c \to \infty$, then it may diverge. For example, for any sequence $c_n \to \infty$, if we choose a sequence $\alpha_n \to -\infty$ and a sequence v_n of solutions with the maximum value α_n, then $v_n \to -\infty$. Hence, we need an additional condition to make sure the convergence of solutions in the nonrelativistic limit. Although it seems not to be easy to find such a condition for a sequence of solutions to the general equation (1.1), it is not difficult to get a condition for radial solutions of Theorem 1 as we shall see.

From now on, we adopt the relation (1.2) and let $\hbar, q, \mu, m > 0$ be fixed and $\alpha \in \mathbb{R}$ be given. Set

$$\alpha_c = \ln m\sigma^2 = \ln \frac{\mu m^2 c^2}{\hbar q^2}.$$

Since $\alpha_c \to \infty$ as $c \to \infty$, if c is sufficiently large, then there exists a solution to (2.6) and (2.7) satisfying Theorem 1 for α. Let us denote it by $v(r, c)$. Then it follows from Theorem 1 that for a given large $c > 0$ there exists an $r_0 = r_0(c)$ satisfying

$$\begin{cases} v(r_0, c) = \alpha, & v_r(r_0, c) = 0, \\ \max_{r \in \mathbb{R}^+} v(r, c) = \alpha, & \lim_{r \to 0} r v_r(r, c) = 2N. \end{cases} \tag{2.14}$$

Furthermore, $0 < T_1 \le r_0 \le T_2$, where T_1 and T_2 are defined by (2.12) and (2.13). Using (1.2), we can rewrite T_1 and T_2 as

$$\begin{aligned} T_1 &= \frac{\sqrt{N^2 + 2N}\sqrt{\hbar\mu}}{q e^{\alpha/2}}, \\ T_2 &= \frac{2(2N^2 + N\sqrt{4N^2 + 1})^{1/2}\sqrt{\hbar\mu}}{q e^{\alpha/2}}. \end{aligned} \tag{2.15}$$

We observe that T_1 and T_2 are independent of c. The following Theorem completely characterizes the nonrelativistic limit concerning (2.3), (2.4), (2.6), and (2.7).

Theorem 2. *Let $\hbar, q, \mu, m > 0$ be fixed, N be a positive integer, and $\alpha \in \mathbb{R}$ be given. Let $v(r,c)$ be a solution to (2.6) and (2.7) satisfying (2.14) which is constructed by Theorem 1. Then, as $c \to \infty$, $v(r,c)$ converges to $w(r)$ which is a solution to (2.3) and (2.4). The function $w(r)$ is explicitly given by (2.5) with $\lambda = \lambda(\alpha)$ defined by*

$$\lambda = \lambda(\alpha) = N^N (N+2)^{N+2} \Big(\frac{\hbar\mu}{q^2}\Big)^{N+1} e^{-\alpha(N+1)}. \tag{2.16}$$

Moreover, if we set

$$\tilde{v}(r,c) = v(r,c) - 2N\ln r, \quad \tilde{w}(r) = w(r) - 2N\ln r,$$

then for any nonnegative integers k

$$\|\tilde{v} - \tilde{w}\|_{C^k(B_R)} = \|v - w\|_{C^k(B_R)} \to 0 \tag{2.17}$$

as $c \to \infty$, where B_R is the ball of radius R centered at the origin.

Proof. Let c_n be an arbitrary sequence such that $c_n \to \infty$. Set $r_n = r_0(c_n)$ and $v_n(r) = v(r, c_n)$. We split the proof into four steps.

Step 1. Convergence of v_n.

It follows from (2.6) that

$$v_n(r) = \alpha + \int_{r_n}^{r} \frac{1}{s} \int_{r_n}^{s} \frac{q^4}{m^2\kappa^2 c_n^4} \tau e^{v_n}(e^{v_n} - 2m\sigma^2)\, d\tau ds.$$

Since $0 < T_1 < r_n < T_2$ and T_1, T_2 are independent of c_n, we may assume that there exists a subsequence of $\{r_n\}$, still denoted by $\{r_n\}$, satisfying $r_n \to r_*$ for some $r_* \in [T_1, T_2]$. We note that

$$|g(v_n)| \le \frac{q^4}{m^2\kappa^2 c_n^4}(e^{2\alpha} + 2m\sigma^2 e^{\alpha}) = \frac{q^4}{m^2\kappa^2 c_n^4} e^{2\alpha} + \frac{2q^2}{\hbar\mu} e^{\alpha},$$

which means that g is uniformly bounded for all r as $c_n \to \infty$. Therefore for any given $R_2 > R_1 > 0$, if $R_1 < r < R_2$,

$$|v_n(r)| \le |\alpha| + \Big| \int_{r_n}^{r} \frac{1}{s} \int_{r_n}^{s} \tau g(v_n)\, d\tau ds \Big| \le C$$

for some constant C dependent only on R_1 and R_2. Thus we have

$$\sup_{[R_1,R_2]} |v_n(r)| \le C.$$

Since

$$\sup_{\mathbb{R}^2} |\Delta v_n| = \sup_{\mathbb{R}^2} |g(v_n)| \le C,$$

we conclude that $\|v_n\|_{W^{2,p}(B_{R_2}\backslash B_{R_1})} \le C$ for all $p > 1$. This implies that there exist a subsequence, denoted by the same notation, v_n and a function $w \in W^{2,p}(B_{R_2}\backslash B_{R_1})$ such that $v_n \to w \in C^{1,\beta}(B_{R_2}\backslash B_{R_1})$ for any $\beta \in (0,1)$ as $c_n \to \infty$. Moreover, it follows from the bootstrap argument that $v_n \to w$ in $C^k(B_{R_2}\backslash B_{R_1})$ for all nonnegative integers k.

Step 2. Explicit form of w.

Let us show that w is a solution to (2.3). Since

$$g(v_n) = \frac{q^4}{m^2\kappa^2 c_n^4} e^{2v_n} - \frac{2q^2}{\hbar\mu} e^{v_n} \to -\frac{2q^2}{\hbar\mu} e^{w},$$

w satisfies

$$w(r) = \alpha + \int_{r_*}^{r} \frac{1}{s} \int_{r_*}^{s} \tau\Big(\frac{-2q^2}{\hbar\mu}\Big) e^{w(\tau)}\, d\tau ds,$$

which yields

$$w_{rr} + \frac{1}{r} w_r = -\frac{2q^2}{\hbar\mu} e^{w}, \quad r > 0$$

with $w_r(r_*) = 0$, $w(r_*) = \alpha$.

Next, we verify that w satisfies (2.4). Integrating (2.6) on $(0, r_n)$, we get

$$-2N = \int_0^{r_n} \frac{q^4}{m^2\kappa^2 c^4} r e^{v_n} (e^{v_n} - 2m\sigma^2)\, dr.$$

Taking the limit, we obtain

$$2N = \int_0^{r_*} \frac{2q^2}{\hbar\mu} r e^{w}\, dr,$$

which implies $\lim_{r\to 0} r w_r = 2N$. Since $w_r < 0$ for $r > r_*$, there exists

$$\gamma = \inf_{r > r_*} w(r) = \lim_{r\to\infty} w(r) \ge -\infty.$$

If $\gamma > -\infty$, we arrive at a contradiction:

$$\infty > \lim_{r\to\infty} |w(r)| \ge -|\alpha| + \frac{2q^2 e^{\gamma}}{\hbar\mu} \lim_{r\to\infty} \int_{r_*}^{r} \frac{1}{s} \int_{r_*}^{s} \tau d\tau ds = \infty.$$

Hence $w(\infty) = -\infty$ and (2.4) is proved.

As a consequence, w is a radial solution to (2.3) and (2.4). By (2.5), there exists $\lambda > 0$ such that w is of the form

$$w(r) = 2N \ln r - 2\ln(\lambda + r^{2N+2}) + \ln 8\lambda(N+1)^2 - \ln(2q^2/\hbar\mu).$$

It remains to show (2.16). Since $w_r(r_*) = 0$, we have

$$\lambda = \frac{N+2}{N} r_*^{2N+2}. \tag{2.18}$$

In addition, from $w(r_*) = \alpha$, we obtain

$$r_* = q^{-1} e^{-\alpha/2} \sqrt{N(N+2)\hbar\mu}.$$

Now (2.16) is a consequence of substitution of this identity into (2.18).

Step 3. Convergence of $\tilde{v}_n$.

It is not difficult to show that $\tilde{v}_n$ and $\tilde{w}$ are smooth functions on $\mathbb{R}^2$. For instance, see Lemma 3.2 of [12]. We know from Step 1 that for any $0 < R_1 < R_2$

$$\|\tilde{v}_n - \tilde{w}\|_{C^k(B_{R_2} \setminus B_{R_1})} \to 0.$$

We will extend the convergence on any ball B_R for $R > 0$. From $(r(v_n)_r)_r = rg(v_n)$, we get

$$\tilde{v}_n(r) = \alpha - 2N \ln r_n + \int_{r_n}^{r} \frac{1}{s} \int_0^s \tau g(v_n) d\tau ds, \quad r > 0.$$

Since $|g(v_n)| \le C$, we are led that for any $r \le R$

$$\left| \int_{r_n}^{r} \frac{1}{s} \int_0^s \tau g(v_n) d\tau ds \right| \le C_R,$$

which implies that $\sup_{B_R} |\tilde{v}_n| \le C_R$. Since $\sup_{\mathbb{R}^2} |\Delta \tilde{v}_n| \le C$, we conclude that $\|\tilde{v}_n\|_{W^{2,p}(B_R)} \le C_R$ for all $p > 1$. Then as in the Step 1, we can show by the bootstrap argument that $\tilde{v}_n \to \tilde{w}$ in $C^k(B_R)$ for any nonnegative integers k.

Step 4. Convergence of the whole sequence.

Finally, the uniqueness of w implies that the convergence holds true for the whole sequence c_n. Since $\{c_n\}$ was an arbitrary sequence, we conclude that $\tilde{v}(r, c) \to \tilde{w}(r)$ as $c \to \infty$ in $C^k(B_R)$ for any $R > 0$. □

Remark 2. We observe that λ is a decreasing function of α in (2.16), which implies that Theorem 2 completely characterizes the nonrelativistic limit for radial solutions of a one-vortex case. In other words, for each solution $w(r)$ to the nonrelativistic equations (2.3) and (2.4), we can find one parameter family of solutions $v(r, c)$ to the relativistic equations (2.6) and (2.7) such that $v(r, c) \to w(r)$ as $c \to \infty$. Indeed, $w(r)$ is determined by λ via (2.5) and the corresponding $v(r, c)$ converging to $w(r)$ can be realized by the common maximum value α given by (2.16).

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