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ON (σ, τ)-DERIVATIONS OF PRIME RINGS

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ABSTRACT. Let R be a prime ring with characteristics not 2 and $\sigma, \tau, \alpha, \beta$ be automorphisms of R. Suppose that d_1 is a (σ, τ)-derivation and d_2 is a (α, β)-derivation on R such that $d_2\alpha = \alpha d_2, d_2\beta = \beta d_2$. In this note it is shown that; (1) If $d_1d_2(R) = 0$ then $d_1 = 0$ or $d_2 = 0$. (2) If $[d_1(R), d_2(R)] = 0$ then R is commutative. (3) $If(d_1(R), d_2(R)) = 0$ then R is commutative. (4) If $[d_1(R), d_2(R)]_{\sigma,\tau} = 0$ then R is commutative.

1. INTRODUCTION

Throughout, R will be a prime ring with characteristic not 2 and $\sigma, \tau, \alpha, \beta$ be automorphisms of R. For any $x, y \in R$, set $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ and $(x, y) = xy + yx$, $(x, y)_{\sigma,\tau} = x\sigma(y) + \tau(y)x$. An additive mapping $d : R \longrightarrow R$ is called a (σ, τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all x, $y \in R$. We shall use the following identities.

$(A) : [xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$

$(B) : [x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]\sigma(z)$

$(C) : (xy, z)_{\sigma,\tau} = x(y, z)_{\sigma,\tau} - [x, \tau(z)]y = x[y, \sigma(z)] + (x, z)_{\sigma,\tau}y$

Let d be a (α, β)-derivation such that $d\alpha = \alpha d$, $d\beta = \beta d$ and $[d(x),\ d(y)] = 0$ for all $x, y \in R$. In this case the commutativity of R was proved by J. C. Chang in [3, Theorem 2(i)]. We generalize this result in Theorem 2 and Theorem 4.

In [1], Ashraf and Rehman proved that, if d_1 and d_2 are two $(\sigma, \tau)-$derivations of R such that $d_1\sigma = \sigma d_1$, $d_1\tau = \tau d_1$, $d_2\sigma = \sigma d_2$, $d_2\tau = \tau d_2$ and $d_1d_2(R) = 0$ then $d_1 = 0$ or $d_2 = 0$.

In this paper we generalized the preceding result as follows: Let d_1 be a (σ, τ)-derivation and d_2 a (α, β)-derivation such that $d_2\alpha = \alpha d_2, d_2\beta = \beta d_2$. If $d_1d_2(R) =$

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0 then $d_1 = 0$ or $d_2 = 0$. On the other hand, we gave some results about the commutativity of prime rings with derivations.

2. RESULTS

Lemma 1 and Lemma 2 can be found in [2].

Lemma 1. *Let d be a nonzero (σ,τ)-derivation of R and U a right ideal of R. If $d(U) \subset Z$ then R is commutative.*

Lemma 2. *Let d be a (σ,τ)-derivation of R and U, a nonzero ideal of R. If $a\, d(U) = 0$ (or $d(U)a = 0$) then $a = 0$ or $d = 0$.*

We begin with the following lemmas.

Lemma 3. *Let $0 \neq d_1 : R \longrightarrow R$ be a (σ,τ)-derivation and $0 \neq d_2 : R \longrightarrow R$ an (α,β)-derivation. If $d_1d_2(R) = 0$ then,*

(i) $d_2\beta^{-1}\alpha + \alpha\beta^{-1}d_2 = 0 = d_2\alpha^{-1}\beta + \beta\alpha^{-1}d_2$,
(ii) $d_2\alpha^{-1}d_2 = 0 = d_2\beta^{-1}d_2$.

Proof. For any $x, y \in R$ we have

$$\begin{aligned} 0 &= d_1d_2(xy) \\ &= d_1(d_2(x)\alpha(y) + \beta(x)d_2(y)) \\ &= d_1d_2(x)\sigma(\alpha(y)) + \tau d_2(x)d_1(\alpha(y)) \\ &\quad + d_1(\beta(x))\sigma(d_2(y)) + \tau\beta(x)d_1d_2(y) \end{aligned}$$

and so,

$$\tau d_2(x)d_1(\alpha(y)) + d_1(\beta(x))\sigma(d_2(y)) = 0, \text{ for all } x, y \in R. \tag{2.1}$$

Now replacing x by $\beta^{-1}d_2(x)$ in (2.1), we get $(\tau d_2\beta^{-1}d_2(x))(d_1\alpha(y)) = 0$, for all $x, y \in R$. Since α is onto, if we use Lemma 2, the above relation yields $\tau d_2\beta^{-1}d_2(x) = 0$ or $d_1 = 0$. Since τ is an automorphism and $d_1 \neq 0$ we get

$$d_2\beta^{-1}d_2(x) = 0, \text{ for all } x \in R. \tag{2.2}$$

On the other hand, for any $x, y \in R$ we have

$$\begin{aligned} 0 &= d_2\beta^{-1}d_2(xy) \\ &= d_2\beta^{-1}(d_2(x)\alpha(y) + \beta(x)d_2(y)) \\ &= d_2(\beta^{-1}d_2(x)(\beta^{-1}\alpha)(y) + x(\beta^{-1}d_2(y))) \\ &= (d_2\beta^{-1}d_2(x))(\alpha\beta^{-1}\alpha(y)) + d_2(x)(d_2\beta^{-1}\alpha(y)) \\ &\quad + d_2(x)(\alpha\beta^{-1}d_2(y)) + \beta(x)(d_2\beta^{-1}d_2(y)) \\ &= d_2(x)(d_2\beta^{-1}\alpha(y)) + d_2(x)(\alpha\beta^{-1}d_2(y)) \end{aligned}$$

and so

$$d_2(R)(d_2\beta^{-1}\alpha + \alpha\beta^{-1}d_2)(y) = 0, \text{ for all } y \in R. \tag{2.3}$$

From (2.3) and Lemma 2 we have $(d_2\beta^{-1}\alpha + \alpha\beta^{-1}d_2)(y) = 0$, for all $y \in R$. Now replacing y by $\alpha^{-1}d_2(y)$ in the last relation and using (2.2) we get,

$$d_2\alpha^{-1}d_2(y) = 0, \text{ for all } y \in R. \tag{2.4}$$

Using (2.4), for any x, $y \in R$,

$$\begin{aligned} 0 &= d_2\alpha^{-1}d_2(xy) \\ &= d_2(\alpha^{-1}d_2(x)y + (\alpha^{-1}\beta(x))(\alpha^{-1}d_2(y)) \\ &= (\beta\alpha^{-1}d_2(x))d_2(y) + (d_2\alpha^{-1}\beta(x))d_2(y) \end{aligned}$$

is obtained. And so,

$$(\beta\alpha^{-1}d_2 + d_2\alpha^{-1}\beta)(x)d_2(y) = 0,$$

for all $x, y \in R$. Using Lemma 2 in the preceding relation we obtain

$$\beta\alpha^{-1}d_2 + d_2\alpha^{-1}\beta = 0.$$

□

Lemma 4. *Let d_1 be a $(\sigma, 1)$-derivation and d_2 an (α, β)-derivation on R such that $d_2\alpha = \alpha d_2$ and $d_2\beta = \beta d_2$. If $d_1d_2(R) = 0$ then $d_1 = 0$ or $d_2 = 0$.*

Proof. For any $x, y \in R$ we have

$$\begin{aligned} 0 &= d_1d_2(xy) \\ &= d_1(d_2(x)\alpha(y) + \beta(x)d_2(y)) \\ &= d_1d_2(x)(\sigma\alpha(y)) + d_2(x)d_1(\alpha(y)) \\ &\quad + d_1(\beta(x))(\sigma d_2(y)) + \beta(x)d_1d_2(y) \end{aligned}$$

$$= d_2(x)d_1(\alpha(y)) + d_1\beta(x)(\sigma d_2(y)).$$

That is,

$$d_2(x)d_1(\alpha(y)) + d_1\beta(x)(\sigma d_2(y)) = 0, \text{ for all } x, y \in R. \tag{2.5}$$

Replacing y by $d_2(y)$ and using that $d_2\alpha = \alpha d_2$ in (2.5), we get $d_1(R)\sigma d_2^2(R) = 0$. Since R is prime $d_1 = 0$ or $d_2^2(R) = 0$ is obtained by Lemma 2. If $d_2^2(R) = 0$, then for any x, $y \in R$,

$$\begin{aligned} 0 &= d_2^2(xy) \\ &= d_2(d_2(x)\alpha(y) + \beta(x)d_2(y)) \\ &= (\beta d_2(x))d_2(\alpha(y)) + (d_2\beta(x))\alpha(d_2(y)) \end{aligned}$$

is obtained. That is, $(d_2\beta(x))(\alpha d_2(y)) = 0$, for all x, $y \in R$. Since R is prime and β, α onto we obtain $d_2 = 0$ by Lemma 2. □

Lemma 5. *Let d_1 be a $(1,\tau)$−derivation and d_2 an (α,β)−derivation on R such that $d_2\alpha = \alpha d_2$ and $d_2\beta = \beta d_2$. If $d_1d_2(R) = 0$ then $d_1 = 0$ or $d_2 = 0$.*

Proof. Now,

$$\begin{aligned} 0 &= d_1d_2(xy) \\ &= d_1(d_2(x)\alpha(y) + \beta(x)d_2(y)) \\ &= (\tau d_2(x))d_1(\alpha(y)) + d_1(\beta(x))d_2(y), \end{aligned}$$

for all x, $y \in R$. Replacing y by $d_2(y)$ and using that $d_2\alpha = \alpha d_2$ in the above relation we have $d_1(\beta(x))d_2^2(y) = 0$, for all x, $y \in R$. That is, $d_1(R))d_2^2(R) = 0$. If we consider Lemma 2 and the last relation, we obtain $d_1 = 0$ or $d_2^2(R) = 0$. If $d_2^2(R) = 0$, then we proved that $d_2 = 0$ in the proof of Lemma 4. □

Theorem 1. *Let d_1 be a (σ,τ)−derivation and d_2 an (α,β)-derivation on R such that $d_2\alpha = \alpha d_2$, $d_2\beta = \beta d_2$. If $d_1d_2(R) = 0$ then $d_1 = 0$ or $d_2 = 0$.*

Proof. If we use Lemma 3 we obtain, $d_1 = 0$ or $d_2\beta^{-1}d_2 = 0$. Suppose that $d_2\beta^{-1}d_2 = 0$. Since $d_2\beta^{-1} : R \longrightarrow R$ is an $(\alpha\beta^{-1},1)$−derivation, and then $d_2\beta^{-1}d_2 = 0$ implies that $d_2 = 0$ or $d_2\beta^{-1} = 0$, by Lemma 4 and so $d_2 = 0$ is obtained for two case. □

Lemma 6. *Let d_1 be a (σ,τ)−derivation and d_2 an (α,β)-derivation on R such that $d_2\alpha = \alpha d_2$, $d_2\beta = \beta d_2$. If $d_1\sigma^{-1}d_2(R) = 0$ then $d_1 = 0$ or $d_2 = 0$.*

Proof. $d_1\sigma^{-1}: R \longrightarrow R$ is a $(1,\tau\sigma^{-1})$- derivation. Thus, $(d_1\sigma^{-1})d_2(R) = 0$ implies that $d_1\sigma^{-1} = 0$ or $d_2 = 0$ by Lemma 5. That is, $d_1 = 0$ or $d_2 = 0$. □

Theorem 2. *Let $0 \neq d_1 : R \longrightarrow R$ be a (σ,τ)-derivation and $0 \neq d_2 : R \longrightarrow R$ an (α,β)-derivation such that $d_2\alpha = \alpha d_2$, $d_2\beta = \beta d_2$. If $[d_1(R), d_2(R)] = 0$ then R is commutative.*

Proof. For any $x, y, z \in R$ we have

$$\begin{aligned}0 &= [d_1(xz), d_2(y)]\\ &= [d_1(x)\sigma(z) + \tau(x)d_1(z), d_2(y)]\\ &= d_1(x)[\sigma(z), d_2(y)] + [d_1(x), d_2(y)]\sigma(z)\\ &\quad + \tau(x)[d_1(z),\ d_2(y)] + [\tau(x),\ d_2(y)]d_1(z).\end{aligned}$$

That is,

$$d_1(x)[\sigma(z), d_2(y)] + [\tau(x), d_2(y)]d_1(z) = 0, \text{ for all } x,\ y,\ z \in R. \tag{2.6}$$

If we take $\tau^{-1}d_2(y)$ instead of x in (2.6) and use hypothesis, we get

$$d_1\tau^{-1}d_2(y)[\sigma(z), d_2(y)] = 0, \text{ for all } y, z \in R. \tag{2.7}$$

Replacing z by zt, $t \in R$ in (2.7) and using primeness of R we have $[\sigma(z), d_2(y)] = 0$, for all $z \in R$ or $d_1\tau^{-1}d_2(y) = 0$. That is, for any $y \in R$ we have

$$d_2(y) \in Z \text{ or } d_1\tau^{-1}d_2(y) = 0. \tag{2.8}$$

Set $K = \{y \in R \mid d_2(y) \in Z\}$ and $L = \{y \in R \mid d_1\tau^{-1}d_2(y) = 0\}$. K and L are additive groups and $R = K \cup L$. Let us consider the Brauer's trick. If $d_1\tau^{-1}d_2(R) = 0$ then $d_1 = 0$ or $d_2 = 0$ by Lemma 4. Because $d_1\tau^{-1} : R \longrightarrow R$ is a $(\sigma\tau^{-1}, 1)$-derivation. Since $d_1 \neq 0$ and $d_2 \neq 0$ we obtain $d_2(R) \subset Z$, from the (2.8). If $d_2(R) \subset Z$ then R is commutative by Lemma 1. □

Theorem 3. *Let $0 \neq d_1 : R \longrightarrow R$ be a (σ,τ)-derivation and $0 \neq d_2 : R \longrightarrow R$ an (α,β)-derivation such that $d_2\alpha = \alpha d_2$ and $d_2\beta = \beta d_2$. If $(d_1(R),\ d_2(R)) = 0$ then R is commutative.*

Proof. $0 = (d_1(xy),\ d_2(z)) = (d_1(x)\sigma(y) + \tau(x)d_1(y),\ d_2(z))$, for all $x,\ y,\ z \in R$. If we use the identity (C) in the above relation we have

$$0 = d_1(x)[\sigma(y), d_2(z)] + (d_1(x),\ d_2(z))\sigma(y) + \tau(x)(d_1(y), d_2(z)) - [\tau(x), d_2(z)]d_1(y)$$

and so

$$d_1(x)[\sigma(y), d_2(z)] - [\tau(x), d_2(z)]d_1(y) = 0, \text{ for all } x,\ y,\ z \in R. \tag{2.9}$$

Replacing y by $\sigma^{-1}d_2(z)$ in (2.9),

$$[\tau(x), d_2(z)]d_1\sigma^{-1}d_2(z) = 0, \text{ for all } x,\ z \in R \tag{2.10}$$

is obtained. If we take xy, $y \in R$ instead of x in (2.10) and use primeness of R we get $[\tau(x), d_2(z)] = 0$, for all $x \in R$ or $d_1\sigma^{-1}d_2(z) = 0$. That is, for any $z \in R$, $d_2(z) \in Z$ or $d_1\sigma^{-1}d_2(z) = 0$.

Considering Brauer's trick we have $d_2(R) \subset Z$ or $d_1\sigma^{-1}d_2(R) = 0$. If we consider that $d_1 \neq 0$ and $d_2 \neq 0$, then $d_1\sigma^{-1}d_2(R) \neq 0$ is obtained by Lemma 6. Thus we have $d_2(R) \subset Z$. This implies that R is commutative by Lemma 1. □

Theorem 4. *Let d_1 and d_2 be as in the theorem-3. If $[d_1(R), d_2(R)]_{\sigma,\tau} = 0$ then R is commutative.*

Proof. $0 = [d_1(xy),\ d_2(z)]_{\sigma,\tau} = [d_1(x)\sigma(y) + \tau(x)d_1(y),\ d_2(z)]_{\sigma,\tau}$, for all $x,\ y,\ z \in R$, is obtained. If we use identity (A) and hypothesis we have

$$d_1(x)[\sigma(y),\ \sigma d_2(z)] + [\tau(x),\ \tau d_2(z)]d_1(y) = 0, \text{ for all } x,\ y,\ z \in R.$$

Replacing y by $d_2(z)$ in the above relation we get

$$[\tau(x),\ \tau d_2(z)]d_1d_2(z) = 0, \text{ for all } x, z \in R. \tag{2.11}$$

If we take xy instead of x in (2.11) we obtain $[\tau(x),\ \tau d_2(z)] = 0$, for all $x \in R$ or $d_1d_2(z) = 0$. This implies that, for any $z \in R$, $d_2(z) \in Z$ or $d_1d_2(z) = 0$

Considering Brauer's trick we have, $d_2(R) \subset Z$ or $d_1d_2(R) = 0$. If $d_2(R) \subset Z$ then R is commutative by Lemma 2. If $d_1d_2(R) = 0$ then $d_1 = 0$ or $d_2 = 0$ by Theorem 1. Since $d_1 \neq 0$ and $d_2 \neq 0$ we obtain that R is commutative. □

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