An Economic-Statistical Design of Moving Average Control Charts

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Abstract

Control charts are important tools of statistical quality control. In 1956, Duncan first proposed the economic design of $\bar{x}$-control charts to control normal process means and insure that the economic design control chart actually has a lower cost, compared with a Shewhart control chart. An moving average (MA) control chart is more effective than a Shewhart control chart in detecting small process shifts and is considered by some to be simpler to implement than the CUSUM. An economic design of MA control chart has also been proposed in 2005. The weaknesses to only the economic design are poor statistics because it dose not consider type I or type II errors and average time to signal when selecting design parameters for control chart. This paper provides a construction of an economic-statistical model to determine the optimal parameters of an MA control chart to improve economic design. A numerical example is employed to demonstrate the model's working and its sensitivity analysis is also provided.

Key Words: MA Control Chart, Time to Signal, Economic-statistics Design

1. Introduction

A control chart is one of the most important tools of statistical quality control. It is used to monitor a process in a state of statistical control and applied to reduce the variability of a process. The parameters that are selected for control charts design are the sample size *n*, the sampling interval *h*, and the width of control limits *k* (as a multiple of standard deviation). Duncan (1956) first proposed the economic design of $\bar{x}$ control charts to control normal process means and insure that the economic design control chart actually has a lower cost, compared with a Shewhart control chart. The cost items in Duncan's model consists of the sampling and testing, increasing from out-of-control process, false alarm and the searching and repairing the assignable cause. From then on, this research method has been widely used in subsequent studies on the subject. For example, Duncan (1971) extended his research

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from single to multiple assignable causes. A joint economic design of $\bar{x}$ and R control charts for two assignable causes was proposed by Saniga (1977). In order to reduce the model's complication, a general method for determining the economic design of control charts was proposed by Lorenzen and Vance (1986). This method can be applied regardless of the statistic used. Collani (1986) proposed a different procedure to determine the economic design of $\bar{x}$ control charts. Banerjee and Rahim (1988) extended Duncan's model (1956) from exponential to the Weibull distribution.

Montgomery (2001) pointed out that an MA control chart is more effective than a Shewhart control chart in detecting small process shifts and is considered by some to be simpler to implement than the CUSUM. An MA economic model were proposed by Chen and Yu (2003), Yu and Chen (2005). Another extension of MA economic model using Taguchi loss function was provided by Yu (2005). Economic design does not consider statistical properties such as the probability of type I and type II errors when selecting the parameters for the control chart. Statistically designed control charts have desirable statistical properties, but the operating cost can be high.

In order to improve economic design, Saniga (1989) presents a method to economically design control charts with two bounds of the probability on type I and type II errors. This study will extend Yu and Chen's (2005) model from economic to economic-statistical model. A numerical example is also provided to illustrate the model's working and to demonstrate its utility.

2. Economic Statistical Design of Control Charts

An economic statistical design of a control chart can be defined as that the economic-loss function is minimized subject to a constrained maximum value of probability of type I and type II errors, also the maximum value of the out-of-control ATS (Average Time to Signal). To select statistical constraints, control charts are then designed to have small ATS values in signaling when the process is out of control (Saniga, 1989, Yang, 1998). Let Y be the set of design parameters for the MA control charts and L be expected hourly loss of the process operation. Then, the economic statistical model of the control charts can be formulated as:

$$\min mize\ L(Y)$$

$$\text{subject to } \begin{cases} \alpha \le \alpha_u \\ \beta \le \beta_u \\ ATS \le ATS_u \end{cases} \tag{1}$$

where α_u, β_u and ATS_u are the desired bounds on the probability of type I error, type II error and ATS, respectively. The solution of this model is an improvement to the economic design because the properties of statistics and minimization of loss cost has been considered. A solution without the constraints is the optimum economic design of the control charts (Yang, 1998).

3. Definition and Assumptions

The features of the model considered in this article are as follows (Yu and Chen, 2005):

(1) The failure rate for an assignable cause is the exponential distribution.

(2) The distribution of x and $\bar{x}$ is normal.

(3) The process will produce a shift $\delta\sigma$, if it has been distributed by the assignable cause.

(4) The process is either in-control or out-of-control and in-control at the beginning.

(5) Production is continuous during the search and repair.

(6) The detected probability when assignable cause occurs is greater than $1\text{-}\beta_u$.

(7) The type I error of the control chart is less than a given value α_u.

On the basis of the above assumptions, the analysis model is constructed based on the Lorenzen and Vance's (1986) method also McWilliam's (1994) searching technique is employed and modified to find the optimum design parameters of *n, h* and *k* to minimize the loss-cost per unit time. At this time, there are two elements to be found, they are the expected cycle time and the expected loss in this cycle time.

4. Expected Cycle Length

Some notations used in this research are defined as follows:

α: the type I error, $\alpha = 2\int \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$

T_1: the time when the process operates in control and $T_1 = 1/\lambda$ where λ is the failure rate of the exponential distribution for an assignable cause-per-unit-time.

$p_{j,i}$: detected probability in subsequent *i* moving subgroup when assignable cause shifts to $\mu_0 + \delta\sigma$ between the *j*th and (*j*+*1*)th sample and it can be expressed as.

$$P_{j,i} = \begin{cases} 1-\Phi\left(k-\frac{i\delta}{\sqrt{i+j}}\right)+\Phi\left(-k\frac{i\delta}{\sqrt{i+j}}\right) & if(j+i) < n \\ 1-\Phi\left(k-\frac{i\delta}{\sqrt{n}}\right)+\Phi\left(-k\frac{i\delta}{\sqrt{n}}\right) & if(j+i) \geq n \text{ and } i < n \\ 1-\Phi(k-\sqrt{n}\,\delta)+\Phi(-k-\sqrt{n}\,\delta) & if(j+i) \geq n \text{ and } i \geq n \end{cases} \quad (2)$$

where $\Phi(x)$ is a cumulative density function (CDF) of the standard normal distribution and k is the coefficient of the control limit. Also $P_{j,i} = P_{j,n} = P$, $Q_{j,i} = 1 - P = Q$, in the third situation.

Then the time interval between the assignable cause occurrence and detection can be shown as the following equation (Yu and Chen, 2005):

$$T_2 = h^*\left((1-e^{-\lambda h})\sum_{s=0}^{n-2} e^{-s\lambda h} E_s + e^{-(n-1)\lambda h} E_{n-1}\right) - \frac{e^{\lambda h} - (1+\lambda h)}{\lambda(e^{\lambda h}-1)} \tag{3}$$

where E_s is the expected number of samplings when an assignable cause occurs between the sth and $(s+1)$th samples and E_s can be shown as:

$$E_s = \left\{P_{s,1} + \sum_{i=2}^{n-1} iP_{s,i}\left[\prod_{d=1}^{i-1} Q_{s,d}\right] + Q_{s,r}\left[n + \frac{Q}{P}\right]\right\} \quad if \ s \leq (n-1) \tag{4}$$

$$= E_{n-1} \quad if \ s \geq n$$

where $Q_{s,r} = \prod_{d=1}^{n-1} Q_{s,d}$.

At this time, the expected cycle time can be expressed as:

$$T = T_1 + T_2 + e + D \tag{5}$$

where e is a constant time for sampling and testing, D is the average repairing time

5. The Derivation of Loss Cost

The loss cost of control chart in this model during a cycle consists of the expected additional loss-per-unit time from an out-of-control state, the expected cost per unit time to identify the false alarms per cycle, the loss-cost-per-unit time to repair an out-of-state process and the expected sampling and maintaining control chart cost. It can be expressed as the following equation:

$$L = \left[(T_2 + e + D)^* M + V^* \frac{\alpha\, e^{-\lambda h}}{1 - e^{-\lambda h}} + W\right] / T + (b + c)/h \tag{6}$$

where M is a penalty cost due to poor quality when an assignable cause occurs, V is the average cost per false alarm, W is the repairing cost when assignable cause occurs, b is the cost of taking a sample and c is the variable cost per item for sampling, testing and plotting.

The goal of the economic-statistical design of MA control charts is to find the design parameters, n, h and k, to minimize the loss-cost function in equation (6). Since L is a very complicated function of the decision variables, n, h and k. The Grid search (McWilliams, 1994) technique is employed and modified for the calculation in this research.

6. Numerical Example

The following example illustrates the use of the model. If the rate of occurrence of assignable causes (λ) is given as 0.01 and the process produces a shift of 2 σ when assignable cause occurs, the parameters of cost and time are selected on the basis of the example from Duncan (1956), i.e. $M=100$, $e=0.05$, $D=2$, $V=50$, $W=25$, $b=0.5$, $c=0.1$. Suppose the statistical constraints in Type I error probability (α) and Type II error probability (β) and average Time to Signal (ATS) are $\alpha_u=0.0052$, $\beta_u=0.05$ and $ATS_{u=4}$. To run the optimization program, a comparison between the pure economic design (ED) and economic statistical design (ESD) are shown in table 1. Table 1 reveals that the sampling sizes are 3 and 5 for pure economic and economic statistical design, respectively, and the loss cost increases about 7.39% ((5.2569−4.8952)/4.8952) from pure economic design to economic statistical design. And the economic statistical design has smaller control limits and a longer sampling interval. The type I error probability (α) and ATS are all in the desired limits, and the test power (1-β) is improved from 0.6482 to 0.9523 (46.98%).

Table 1. MA designs for $\lambda=0.01$, $\delta=2$, $M=100$, $e=0.05$, $D=2$, $V=50$, $W=25$, $b=0.5$, $c=0.1$

Designs	n	H	K	a	β	ATS	Loss cost
ED	3	0.5328	3.0836	0.002	0.3518	0.8220	4.8952
ESD*	5	0.5750	2.8049	0.0050	0.0477	0.6038	5.2569

* $\alpha \leq 0.0052$, $\beta \leq 0.05$, $ATS \leq 4$

ED: pure economic design

ESD: economic statistical design

7. Sensitivity Analysis

This section discusses the robustness of the model to verify the inaccuracies in the parameters. Suppose that the user incorrectly estimates any of the values by up to 50% and the values of the parameters given in the example are assumed to be the basic case. At this time, only one parameter among δ, λ, M, e, D, V, W, b, and c is changed at the rate 0.5, 0.75, 0.9, 1.1, 1.25, 1.5 of original value. Then for each of the 6×9 cases run, the optimal values of n, h and k are determined, the results of which are shown in table 2. For example, when it is estimated that $\delta = 1$, instead of the original values of 2, the model calls for design parameters $n = 21$, $h = 0.3063$ and $k = 2.90$. The loss cost will be 7.4438 instead of 5.2569 per unit time, or an increase of 41.60% ((7.4438 - 5.2569)/5.2569 = 41.60%) over the true basic case optimum. When the shift (δ) varies from 0.5, 0.75, 0.9, 1.1, 1.25, to 1.5 times the original value, the estimated minimum cost will be 141.60% to 90.68% of the original and h will change in the same direction in which δ changes.

Such an analysis also gives an indication of the sensitivity to each of the input parameters. Other interesting observations are the following:

(1) When the rate of occurrence of assignable cause (λ) increases, there is no significant changes to the value of k and the loss cost increases. Also the higher the rate at which assignable cause occurs, the shorter the sampling interval. The loss cost changes from 62.69% to 132.61% of the original.

(2) The increasing cost (M) when the process is out-of-control has the same effect on h as λ. The smaller the M, the larger the h. The loss cost changes from 63.91% to 132.12% of the original.

(3) The time for locating the assignable cause (e) also has a negligible effect on h and k, whereby the loss cost changes from 99.56 to 100.44% of the original.

(4) The repair time (D) has only a negligible effect on h or k, whereby the loss cost changes from 82.25% to 117.41%.

(5) There are no significant impacts to the values of h and k with the average cost per false alarm (V) when the process is in-control. The loss cost will increase from 95.78% to 102.70% when V increases.

(6) The location and repair cost (W) have only a negligible effect on h or k, whereby the loss cost changes from 97.71% to 102.29% of the original.

(7) The values of h will change in the same direction as a fixed or variable sampling cost per unit sampling (b or c). The loss cost will decrease from 89.49% to 107.82% when b increases.

(8) It is necessary to pay more attention to obtaining the parameters of the magnitude of shift (δ), the rate of occurrence of assignable causes (λ), the repair time (D) and the increasing cost (M) when the process is out-of-control.

Detailed changing rates of loss cost are shown in table 2.

Table 2. Loss cost in sensitivity analysis

δ	λ	M	e	D	V	W	b	c	n	h	K	Loss cost	% of original
2	0.01	100	0.05	2.0	50	25	0.5	0.1	5	0.57505	2.8049	5.2569	100
1.00	0.01	100	0.05	2.0	50	25	0.5	0.1	21	0.3063	2.9000	7.4438	141.60
1.50	0.01	100	0.05	2.0	50	25	0.5	0.1	10	0.3875	3.0000	6.0360	114.82
1.80	0.01	100	0.05	2.0	50	25	0.5	0.1	7	0.4563	3.0000	5.5117	104.85
2.20	0.01	100	0.05	2.0	50	25	0.5	0.1	5	0.5875	2.8049	5.1358	97.70
2.50	0.01	100	0.05	2.0	50	25	0.5	0.1	5	0.6250	2.8049	4.9793	94.72
3.00	0.01	100	0.05	2.0	50	25	0.5	0.1	5	0.6750	2.8049	4.7672	90.68
2	0.0051	100	0.05	2.0	50	25	0.5	0.1	5	0.7875	2.8049	3.2953	62.69
2	0.0085	100	0.05	2.0	50	25	0.5	0.1	5	0.6516	2.8049	4.3195	82.17
2	0.009	100	0.05	2.0	50	25	0.5	0.1	5	0.5938	2.8049	4.8899	93.02
2	0.011	100	0.05	2.0	50	25	0.5	0.1	5	0.5375	2.8045	5.6143	106.80
2	0.013	100	0.05	2.0	50	25	0.5	0.1	5	0.51259	2.8049	6.1355	116.71
2	0.015	100	0.05	2.0	50	25	0.5	0.1	5	0.4750	2.8049	6.9711	132.61
2	0.01	50	0.05	2.0	50	25	0.5	0.1	5	0.8125	2.8049	3.3595	63.91
2	0.01	75	0.05	2.0	50	25	0.5	0.1	5	0.6625	2.8049	4.3463	82.68
2	0.01	90	0.05	2.0	50	25	0.5	0.1	5	0.5984	2.8049	4.8994	93.20
2	0.01	110	0.05	2.0	50	25	0.5	0.1	5	0.5375	2.8049	5.6063	106.65
2	0.01	125	0.05	2.0	50	25	0.5	0.1	5	0.5125	2.8049	6.1186	116.40
2	0.01	150	0.05	2.0	50	25	0.5	0.1	5	0.4625	2.8049	6.9456	132.12
2	0.01	100	0.025	2.0	50	25	0.5	0.1	5	0.5750	2.8049	5.2338	99.56
2	0.01	100	0.038	2.0	50	25	0.5	0.1	5	0.5750	2.8049	5.2453	99.78
2	0.01	100	0.045	2.0	50	25	0.5	0.1	5	0.5750	2.8049	5.2523	99.91
2	0.01	100	0.055	2.0	50	25	0.5	0.1	5	0.5750	2.8049	5.2615	100.09
2	0.01	100	0.063	2.0	50	25	0.5	0.1	5	0.5750	2.8049	5.2684	100.22
2	0.01	100	0.075	2.0	50	25	0.5	0.1	5	0.5750	2.8049	5.2799	100.44
2	0.01	100	0.05	1	50	25	0.5	0.1	5	0.5750	2.8049	4.3240	82.25
2	0.01	100	0.05	1.5	50	25	0.5	0.1	5	0.5750	2.8049	4.7927	91.17
2	0.01	100	0.05	1.8	50	25	0.5	0.1	5	0.5750	2.8030	5.0718	96.48
2	0.01	100	0.05	2.2	50	25	0.5	0.1	5	0.5750	2.8049	5.4413	103.51
2	0.01	100	0.05	2.5	50	25	0.5	0.1	5	0.5750	2.8049	5.7166	108.74
2	0.01	100	0.05	3	50	25	0.5	0.1	5	0.5750	2.8049	6.1719	117.41
2	0.01	100	0.05	2.0	25	25	0.5	0.1	5	0.5250	2.8049	5.0348	95.78
2	0.01	100	0.05	2.0	37.5	25	0.5	0.1	5	0.5438	2.8049	5.1480	97.93
2	0.01	100	0.05	2.0	45	25	0.5	0.1	5	0.5625	2.8049	5.2137	99.18
2	0.01	100	0.05	2.0	55	25	0.5	0.1	6	0.4938	3.0000	5.2956	100.74
2	0.01	100	0.05	2.0	62.5	25	0.5	0.1	6	0.5008	3.0000	5.3347	101.48
2	0.01	100	0.05	2.0	75	25	0.5	0.1	6	0.5125	3.0000	5.3988	102.70
2	0.01	100	0.05	2.0	50	12.50	0.5	0.1	5	0.5750	2.8049	5.1363	97.71
2	0.01	100	0.05	2.0	50	18.75	0.5	0.1	5	0.5750	2.8049	5.1966	98.85
2	0.01	100	0.05	2.0	50	22.50	0.5	0.1	5	0.5750	2.8049	5.2328	99.54
2	0.01	100	0.05	2.0	50	27.50	0.5	0.1	5	0.5750	2.8049	5.2810	100.46
2	0.01	100	0.05	2.0	50	31.25	0.5	0.1	5	0.5750	2.8049	5.3172	101.15
2	0.01	100	0.05	2.0	50	37.50	0.5	0.1	5	0.5750	2.8049	5.3775	102.29
2	0.01	100	0.05	2.0	50	25	0.25	0.1	6	0.39388	3.0000	4.7044	89.49
2	0.01	100	0.05	2.0	50	25	0.38	0.1	6	0.4438	3.0000	5.0017	95.15
2	0.01	100	0.05	2.0	50	25	0.45	0.1	6	0.4750	3.0000	5.1652	98.26
2	0.01	100	0.05	2.0	50	25	0.55	0.1	5	0.5750	2.8049	5.3439	101.65
2	0.01	100	0.05	2.0	50	25	0.63	0.1	5	0.6125	2.8049	5.4692	104.04
2	0.01	100	0.05	2.0	50	25	0.75	0.1	5	0.6469	2.8049	5.6681	107.82
2	0.01	100	0.05	2.0	50	25	0.5	0.05	6	0.4750	2.3000	5.1652	99.26
2	0.01	100	0.05	2.0	50	25	0.5	0.08	5	0.5625	2.8049	5.2123	99.15
2	0.01	100	0.05	2.0	50	25	0.5	0.09	5	0.5750	2.8049	5.2395	99.67
2	0.01	100	0.05	2.0	50	25	0.5	0.11	5	0.5750	2.8049	5.2743	100.33
2	0.01	100	0.05	2.0	50	25	0.5	0.13	5	0.5750	2.8049	5.3003	100.83
2	0.01	100	0.05	2.0	50	25	0.5	0.15	5	0.5750	2.8049	5.3439	101.65

7. Conclusions

There are certain weakness to only the economic design because it dose not consider statistics properties, such as type I or type II errors and average time to signal when selecting design parameters for control charts. This paper has shown a construction of an economic-statistical model to determine the optimal parameters of an MA control chart to improve economic design. A numerical example is also provided to verify the effectiveness of the proposed model as compared with the original pure economic model and it indicates that statistical performance of control charts can be improved significantly with a slight increase in the cost by using an economic-statistical design. A sensitivity analysis has shown that the parameters of magnitude of shift (δ), increasing cost (M) when the process is out-of-control, and the rate of occurrence of assignable causes (λ) and the repair time (D) should receive more attention when the data for loss-cost calculation are estimated.

References

1. Banerjee, P. K. and Rahim, M. A.(1988), Economic design of $\bar{x}$ control charts under Weibull shock models. *Technometrics*, Vol. 30, No. 4, pp. 407-414.
2. Chen, Y. S. and Yu, F. J.(2003), Determination of optimal design parameters of moving average control charts. *International Journal of advanced Manufactuing Technology*, No. 21, pp. 397-402.
3. Collani, E. V.(1986), A simple procedure to determine the economic design of an $\bar{x}$ control chart. *Journal of .Quality Technology*, Vol. 18, No. 3, pp. 145-151.
4. Duncan, A. J.(1956), The economic design of $\bar{x}$ charts used to maintain current control of a process, *Journal of the American. Statistical. Association.* No. 51, pp. 228-242.
5. Duncan, A. J.(1971), The economic design of $\bar{x}$ charts when there is a multiplicity of assignable causes. *Journal of the American. Statistical. Association*, No. 66, pp. 107-121.
6. Lorenzen, T. J. and Vance, L. C.(1986), The economic design of control charts: a unified approach, *Technometrics*, Vol. 28, No. 1, pp. 3-10.
7. McWilliams, T. P.(1994), Economic, statistical, and economic-statistical $\bar{x}$ chart design, *Journal of Quality Technology*, No. 26, pp. 227-238.
8. Montgomery, D. C.(2001), *Introduction to Statistical Quality Control*, John Wiley and Sons, New York.
9. Saniga, E. M.(1977), Joint economically optimal design of $\bar{x}$ and R control charts, *Management Science*, Vol. 24, No. 4, pp. 420-431.

10. Saniga, E. M.(1989), Economic statistical control chart designs with an application to $\overline{X}$ and R charts, *Technometrics*, No. 31, pp. 313-320.
11. Yang, S. F.(1998), Economic statistical design of S control charts using Taguchi loss function, *International Journal of Quality & Reliability Management*, Vol. 15, No. 3, pp. 259-272.
12. Yu, F.-J. and Chen, Y. S.(2005), Economic design of moving average control charts, *Quality Engineering*, Vol. 17, No. 3. pp. 391-397.
13. Yu, F.-J.(2005), Economic design of moving average control charts using Taguchi loss function, *The Proceedings of the 11^{th} International Conference on Industrial Engineering and Engineering Management*, Northeastern University, Shenyang, China, pp. 243-246.