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C^1 Continuous Piecewise Rational Re-parameterization

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Abstract – A new method to obtain explicit re-parameterization that preserves the curve degree and parametric domain is presented in this paper. The re-parameterization brings a curve very close to the arc length parameterization under L_2 norm but with less segmentation. The re-parameterization functions we used are C^1 continuous piecewise rational linear functions, which provide more flexibility and can be easily identified by solving a quadratic equation. Based on the outstanding performance of Mobius transformation on modifying pieces with monotonic parametric speed, we first create a partition of the original curve, in which the parametric speed of each segment is of monotonic variation. The values of new parameters corresponding to the subdivision points are specified a priori as the ratio of its cumulative arc length and its total arc length. C^1 continuity conditions are imposed to each segment, thus, with respect to the new parameters, the objective function is linear and admits a closed-form optimization. Illustrative examples are also given to assess the performance of our new method.

Key Words **: Parametric curves, Arc-length parameterization, Mobius transformation, Optimization**

1. Introduction

Parametric representation is one of the most common ways to describe curves in CAD/CAM and related areas. Arc length parameterization is the most natural parameterization for a given curve because of its nice mathematical properties and useful applications. For instance, in computer animation and computer numerical control machining, it is indispensable to have a control of the speed, which can be readily obtained from the arc length parameterization. However, the impossibility [1] of constant parametric speed is a fundamental limitation to polynomial and rational curve parameterizations. Therefore, a number of methods [2-4] have been proposed to obtain approximations of the arc-length functions or inverses of the arc-length functions [5,6].

To obtain re-parameterization of a polynomial curve but still keeps its parameter domain and degree, Mobius transformation is a class of appropriate re-parameterization functions. In [7], Farouki first introduces an optimality criterion to measure the deviation of a curve from its arc length parameterization. He also gives a method to obtain optimal parameterizations using Mobius transformation. Bert Juttler further derived a simplification of the method. Farouki's method is very attractive for its simplicity [8], unfortunately, it is also limited because the class of rational linear function is too small to achieve good approximation to arc-length parameterization. For higher order curves with several undulations in their parametric speeds above and below unity, the method gives only negligible improvements.

Costantini et al. [9] expand the class of re-parameterization functions to the space of piecewise rational linear functions. They show that, for fixed knots, the optimal piecewise rational linear re-parame-terization can be defined by a simple recursion relation, but this representation is only C^0 continuous with respect to new parameters. In most applications, C^1 continuous re-parameterizations with respect to new parameters are preferable. Thus some schemes to achieve C^1 continuous re-parameterizations have also been proposed, however, objective functions of these schemes, which provides three sets of free parameters, are highly nonlinear and do not admit a closed-form optimization. In fact, for a pre-specified error, partitions using fixed knots usually give too many residual segments.

In this paper, we will first analyze some characteristics of Mobius transformation, and their effects on parametric speed. Based on the outstanding performance of Mobius transformation on modifying pieces with monotonic parametric speed, we create a partition of the original curve, in which the parametric speed of each segment is of monotonic variation. The values of new parameters corresponding to the subdivision points are specified a priori as the ratio of its cumulative arc length and its total arc length. C^1 continuity conditions are imposed to each segment, thus, with respect to the new parameters, the objective function is linear and admits a closed-form optimization. Illustrative examples are also given to assess the performance of our new method.

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2. Preliminaries

Let $r(t)$ for $t \in [0,1]$ be a regular parametric curve, and its normalized form be $p(t) = r(t)/S$, where $S = \int_0^1 |r'(t)|dt$ is the total arc length of $r(t)$.

We assume that $p(t)$ is a polynomial Bezier curve of degree n

$$p(t) = \sum_{k=0}^{n} p_k B_k^n(t) \tag{1}$$

where $B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k}$. Our objective is to obtain a re-parameterization $q(u) = p(t(u))$ of $p(t)$, which is an optimal approximation to its arc length parameterization.

As used in [7], the optimal criterion to measure the deviation of a curve from its arc length parameterization can be defined as

$$J = \int_0^1 |p'(t)|^2 dt \tag{2}$$

As $p(t)$ is normalization of $r(t)$, where $p(t)=r(t)/S$, we know that $\int_0^1 |p'(t)|dt = 1$, which gives the obvious result $J \geq 1$. Thus, an optimal parameterization is a representation that exhibits the least value for J among a given class of admissible parameterizations.

Mobius transformation, also known as bilinear transformation or linear fractional transformation, is an important class of elementary mapping. It can be expressed as the ratio of two linear expressions $t = t(u) = \frac{au+b}{cu+d}$, where $ad \neq bc$. The most general form of Mobius transformation, which maps intervals $t \in [0, 1]$ and $u \in [0, 1]$ onto each other, is

$$t = t(u) = \frac{(1-\gamma)u}{\gamma(1-u)+(1-\gamma)u} \tag{3}$$

where $0 < \gamma < 1$. In particular it satisfies $t(0) = 0$, $t(1) = 1$, and $t(1/2) = 1 - \gamma$. The inverse of expression (3) is

$$u = \frac{\gamma t}{(1-\gamma)(1-t)+\gamma t}.$$

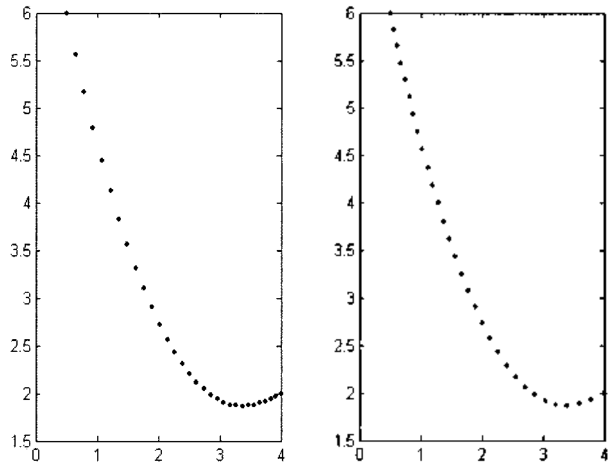
Now let us analyze how Mobius transformation can modify the speed variation. Substitute (3) into (1), we can obtain the re-parameterization of $q(u)$. The derivative of $q(u)$ is $q'(u) = \frac{dt}{du}\frac{dp}{dt}$. From (3) we have

$$\frac{dt}{du} = \frac{\gamma(1-\gamma)}{[\gamma(1-u)+(1-\gamma)u]^2} \tag{4}$$

Expression (4) is a modification factor to the original parametric speed. By a projective transformation of the parameter domain, we can figout that expression (4) has a double-pole at point $u_\infty = \frac{\gamma}{2\gamma-1}$, and it becomes unity at points $u_* = \frac{\gamma \pm \sqrt{\gamma(1-\gamma)}}{2\gamma-1}$, which lies to the left and right of u_∞, respectively. Thus, the parametric speed is sped up over $[0, u_*]$ and slowed down over $[u_*, 1]$, or vise versa. Therefore, we get the conclusion that if the parametric speed with respect to the original parameters is monotonic, expression (4) can improve the parametric flow dramatically to closer constant parametric flow; otherwise, the improvement is negligible.

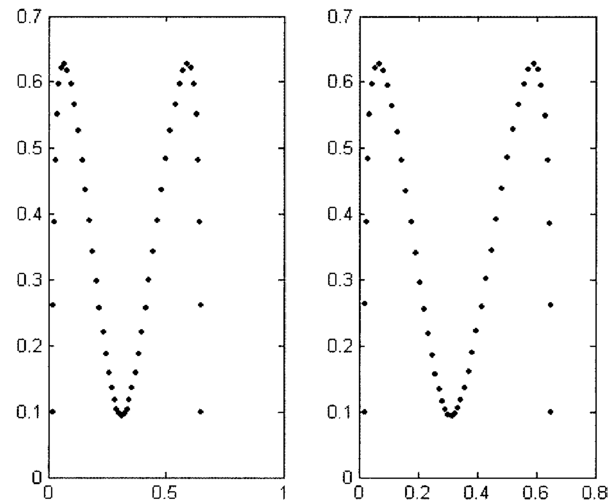
In Figs. 1-2, we give two examples of parametric curve. For the curve in Fig. 1, the original polynomial parameterization has monotonically decreasing parametric speed. For the curve in Fig. 2, the original parameterization has several undulations in its parametric speed.

Now, we improve the parametric flow of these two curves by using Mobius transformation. And in Figs. 3 and 4 we compare the parametric speeds of the original



(1) (2)

Fig. 1. Parametric flow of curve 1.(1) Original parametric flow, (2) Improved parametric flow using Mobius transformation.



(1) (2)

Fig. 2. Parametric flow of curve 2. (1) Original parametric flow, (2) Improved parametric flow using Mobius transformation.

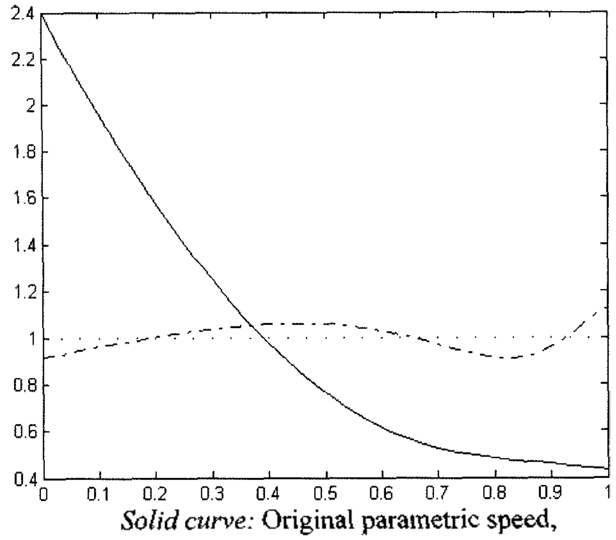


Fig. 3. Parametric speed of curve 1.
Solid curve: Original parametric speed, *Dashed curve:* Improved parametric speed using Mobius transformation.

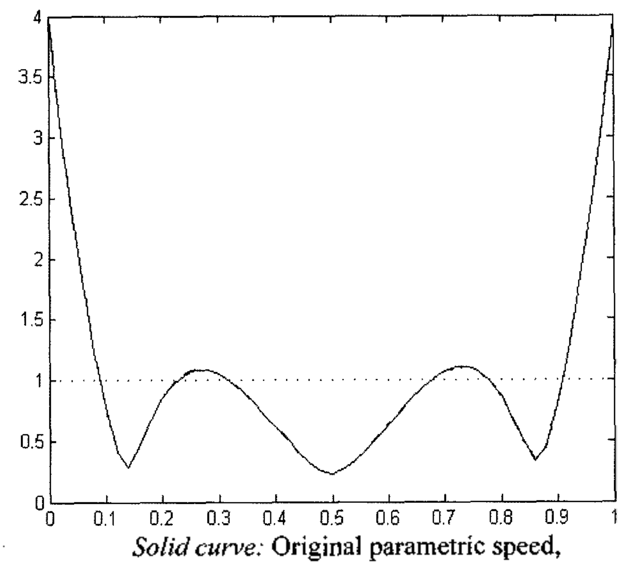


Fig. 4. Parametric speed of curve 2.
Solid curve: Original parametric speed, *Dashed curve:* Improved parametric speed using Mobius transformation.

curve and that of improved curve. It is obvious that the parametric speed of curve 1 becomes flatter after applying re-parameterization, while the speed of curve 2 has no observable changes.

We also listed out the values of functions J of original curve and that of improved curve for curve 1 and curve 2, respectively.

From Figs. 1-4 and Table 1 we can see that, for curve 1 with monotonic parametric speed, the re-parameterization using Mobius transformation gives a satisfactory result. However, for curve 2 with several undulations in its parametric speed, the re-parameterization only shows negligible improvement on the parametric flow of the original curve.

Table 1. Values of J for curve 1 and curve 2

J	Original curve	Improved curve
Curve 1	1.3336	1.0028
Curve 2	1.57686	1.57684

3. C¹ continuous piecewise rational re-parameterization

From section 2 we know, the class of rational linear function is too small to achieve good approximation to arclength parameterization, so re-parameterization functions with more flexibility, such as piecewise rational linear function, are needed. In this section, we will provide a flexible method with closed-form optimization to re-parameterize the polynomial and rational parametric curves.

Similar to (3), we define a piecewise rational linear function as

$$t(u)=t_j+\frac{\Delta t_j(1-\gamma_j)\hat{u}}{\gamma_j(1-\hat{u})+(1-\gamma_j)\hat{u}} \tag{5}$$

where, $\hat{u}=\dfrac{u-u_j}{\Delta u_j}$, $u\in[u_j, u_{j+1}]$.

From (5), we can derive

$$\frac{dt}{du}=\frac{\Delta t_j}{\Delta u_j}\frac{\gamma_j(1-\gamma_j)}{[\gamma_j(1-\hat{u})+(1-\gamma_j)\hat{u}]^2} \tag{6}$$

To obtain C^1 continuous transformation at points u_j ($j = 1, \cdots, N-1$), the following conditions must be satisfied

$$\gamma_{j+1}=\frac{m_{j+1}(1-\gamma_1)}{m_j\gamma_j+m_{j+1}(1-\gamma_1)} \tag{7}$$

where $m_j=\dfrac{\Delta t_j}{\Delta u_j}$ ($j=0, \cdots, N$).

Substituting (5) into (1), we get

$$q(u)=p(t(u))=$$

$$\frac{\sum_{k=0}^{n}p_k\binom{n}{k}[t_j\gamma_j(1-\hat{u})+t_{j+1}(1-\gamma_{j+1})\hat{u}]^k[(1-t_j)\gamma_j(1-\hat{u})+(1-t_{j+1})(1-\gamma_j)\hat{u}]^{n-k}}{\sum_{k=0}^{n}(1-\gamma_j)^k\gamma_j^{n-k}B_k^n(\hat{u})} \tag{8}$$

For the re-parameterization $q(u)=p(t(u))$, the optimal criterion J can be written as

$$J=\int_0^1\left|\frac{dq}{du}\right|^2du=\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}}\left|\frac{dp}{dt}\right|^2\frac{dt}{du}dt \tag{9}$$

To obtain the least value for J, we will discuss how to determine the knots $t_0, \cdots, t_N$, $u_0, \cdots, u_N$, and $\gamma_0, \cdots, \gamma_{N-1}$.

3.1 Determining knots $t_0, \cdots, t_N$ and $u_0, \cdots, u_N$

We begin with choosing the subdivision points using piecewise rational re-parameterization. Without lose of generality, assume $0 = t_0 < t_1 < \cdots < t_N = 1$ and $0 = u_0 < u_1 < \cdots < u_N = 1$, and denote $\Delta t_j = t_{j+1}\Delta - t_j$, $\Delta u_j = u_{j+1} - u_j$, for $j = 0, \cdots, N-1$.

To create a partition to the original curve, in which the parametric speed of each segment is of monotonic variation, we specify t_j ($j = 1, \cdots, N-1$) as roots of the equation $\frac{d\left|p'(t)\right|}{dt} = 0$

As we know that, for arc length parameterization, the points corresponding to equally spaced values of the parameter will be uniformly distributed along the curve. Thus, the corresponding new knots u_j are defined as

$$u_j = \frac{s(t_j)}{s(t_N)},\ (j = 1, \cdots, N-1) \tag{10}$$

where $s(t_j) = \int_0^{t_j}\left|\frac{dp}{d\tau}\right|d\tau$ are the cumulative arc-length functions.

3.2 Determining $\gamma_0, \cdots, \gamma_{N-1}$

Obviously, imposing C^1 continuity conditions to piecewise rational re-parameterization incurs a dependency of parameters $\gamma_1, \cdots, \gamma_{N-1}$ on γ_0. Thus, γ_0 is the only free para-meter that is used to minimize the value of the function in (9).

Denote

$$I_{jk} = \int_{t_j}^{t_{j+1}}\left|\frac{dp}{d\tau}\right|^2 (t-t_j)^k dt, \quad k = 0, 1, 2$$

$$B_j = \frac{I_{j2}}{\Delta t_j^2},$$

$$C_j = \frac{I_{j1}}{\Delta t_j} - B_j,$$

$$D_j = I_{j0} - \frac{I_{j1}}{\Delta t_j} - C_j$$

$$\varphi_0 = \omega_0 = 1,\ \varphi_{k+1} = m_0, \cdots, m_{2k},$$

$$\omega_{k+1} = m_1 \mathrm{L}\ m2_{k+1},\ k = 0, \cdots, \lfloor (N-1)/2 \rfloor$$

$$P = \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \frac{\varphi_{k+1}^2}{\omega_k^2}(B_{2k} + D_{2k+1})$$

$$Q = \sum_{k=0}^{\lfloor (N-1)/2 \rfloor}\left(\frac{\omega_k^2}{\varphi_k^2}D_{2k} + \frac{\omega_{k+1}^2}{\varphi_{k+1}^2}B_{2k+1}\right)$$

$$R = \sum_{k=0}^{\lfloor (N-1)/2 \rfloor}(m_{2k}C_{2k} + m_{2k+1}C_{2k+1})$$

Then, the problem to minimize function J possesses a global minimum γ_0. This solution satisfies the equation

$$\frac{dJ}{d\gamma_0} = \frac{\gamma_0}{(1-\gamma_0)m_0}P + \frac{(1-\gamma_0)m_0}{\gamma_0}Q + 2R = 0 \tag{11}$$

The roots of equation (11) are

$$\gamma_0 = \frac{m_0^2 Q \pm \sqrt{PQ}}{m_0^2 Q - P} \tag{12}$$

If $\gamma_0 \in (0, 1)$, they identify the extrema of J.

3.3 Solutions to the re-parameterization

The above algorithm can be summarized as follows:

Input: degree n and control points $p_0, \ldots, p_n$ of $r(t)$.

Step 1. Compute knots of original parameter at subdivision points $t_0, \ldots, t_N$;

Step 2. Compute knots of new parameter $u_0, \ldots, u_N$ corresponding to the original knots $t_0, \ldots, t_N$;

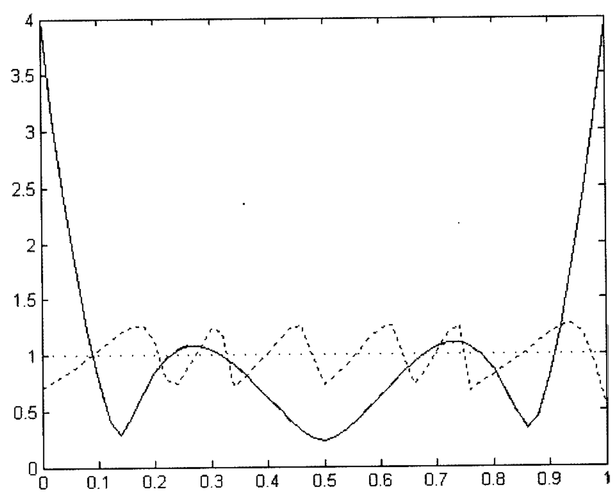
Step 3. Compute the values $\gamma_0, \ldots, \gamma_{N-1}$;

Output: re-parameterized rational representation $q(u)$.

4. Illustrative examples

To assess the performance of our new method, we will give a comparison of our method with rational linear re-parameterization [7] in this section.

Fig. 5 shows the parametric speed of curve 2 in Section 2. We can see that an improved parametric



Solid curve: the parametric speed of original curve,

Fig. 5. Parametric speed of curve 2.
Solid curve: the parametric speed of original curve, *Dot-line curve:* rational linear re-parameterization, *Dashed curve:* improved parametric speed using C^1 continuous piecewise rational linear re-parameterization.

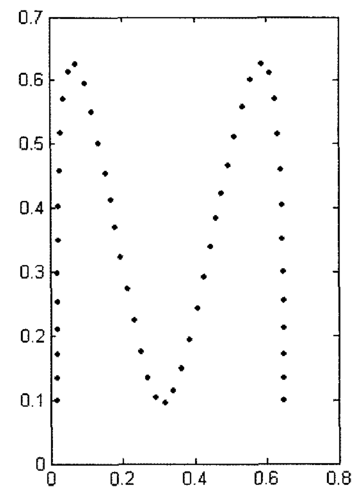


Fig. 6. Parametric flow of curve 2 obtained by new method.

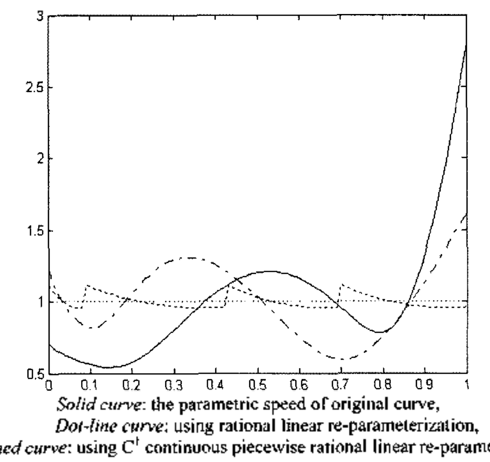


Fig. 7. Parametric speeds of curve 3.
Solid curve: the parametric speed of original curve, *Dot-line curve*: using rational linear re-parameterization, *Dashed curve*: using C^1 continuous piecewise rational linear re-parameterization.

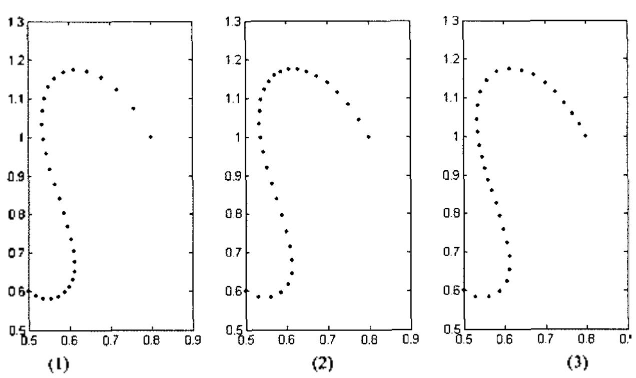


Fig. 8. Parametric flows of curve 3.
(1) The original parametric flow, (2) The parametric flow using rational linear function, (3) The parametric flow using C^1 continuous piecewise rational linear re-parameterization.

speed is given by the new method. Fig. 6 shows the parametric flow of curve 2 obtained by the new method.

Table 2. Values of J for curve 2 and curve 3

	Original	Integral	Piecewise
Curve 2	1.57686	1.57684	1.03153
Curve 3	1.17539	1.06339	1.00263

In Figs. 7 and 8, we give out another example, say 'curve 3'. Fig. 7 shows the parametric speed of original curve, the parametric speed obtained by rational linear re-parameterization, and the parametric speed obtained by the new method. Fig. 8 shows the corresponding parametric flow of curve 3.

In Table 2, we also give values of the J function of original curve, rational linear re-parameterization (integral), and C^1 continuous piecewise rational linear re-parameterization (piecewise) for curve 2 and curve 3.

5. Conclusions

In this paper, we present a C^1 continuous piecewise rational linear re-parameterization, which provides more flexibility than rational linear re-parameterization and can be easily identified by solving a quadratic equation. A partition is done on the original curve, in which the parametric speed of each segment is of monotonic variation. It should be noted that, in order to satisfy a pre-specified error bound, further partitions could be used as needed, which split each new parameter interval into halves. The values of new parameters corresponding to the subdivision points are specified a priori as the ratio of its cumulative arc length and its total arc length. C^1 continuity conditions are imposed to each segment, thus, with respect to the new parameters, the objective function is linear and admits a closed-form optimization. Analysis of examples shows that our method brings a curve very close to the arc-length parameterization under L_2 norm but with fewer segments.

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