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# 계층적 스케줄 방식을 고려한 다계층 우선순위 대기행렬시스템의 성능척도 분석

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# Performance Analysis of a Multiclass Priority Queue with Layered Scheduling Algorithm

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여러 형태의 고객이 외부로부터 포아송과정에 따라 각 대기행렬에 도착하고 정해진 서비스규칙에 따라 해당 서비스를 받은 후 마코비안 확률분포에 따라 시스템을 떠나거나 다른 형태의 고객으로 시스템을 다시 돌아 올 수도 있는 $M/G/1$ 대기행렬시스템을 고려한다. 본 연구에서는 기존의 연구 모형을 확장하여 계층적 서비스 규칙을 갖는 우선순위 대기행렬모형을 제시하고 이에 대한 시스템 성능척도를 보다 체계적으로 구할 수 있는 방법을 소개한다. 이를 위하여 먼저 대기행렬시스템의 거동을 나타내는 시스템 상태를 정의하고, 바쁜기간과 서비스기간 분석을 통하여 시스템 상태의 선형 함수로 평균체제시간을 구할 수 있음을 보인다.

**Keywords** : Feedback Queue, Markovian, Mean Sojourn Time

## 1. Introduction

Queueing models with feedback are useful for analyzing manufacturing systems, communication systems, etc. Recently, workgroup networking demands higher bandwidths as users increasingly share and access data across the network. More powerful work stations promote multiple classes of high-bandwidth networked applications using imaging, graphics, and multimedia. Switches are tools for increasing bandwidth, controlling traffic, and dispelling congestion. A switch fabric is a method used to actually route a packet from one port to another port [1, 8].

Also, the shared bus in which an internal high speed backplane is used to interconnect switch ports. These switches usually support packet priority. When we design efficient packet transmissions under the prospective diversification of network service requirements, we should investigate the effects of their scheduling algorithms including priorities and service orders of several type of packets on their system performances [1].

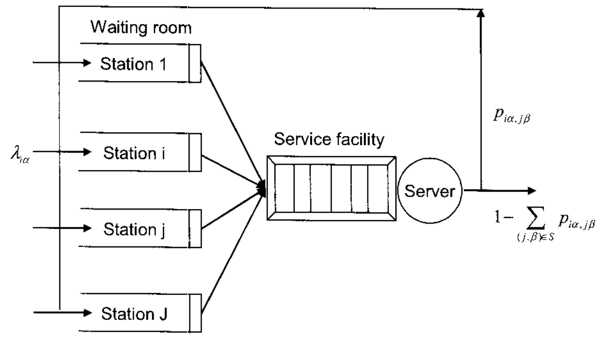
In this paper, we investigate a Markovian feedback queueing system with layered composit scheduling algorithms, and apply our unified solution method developed in [3, 4, 6]. In Markovian feedback systems, each arriving customer joins one of the stations, and each customer who completes its service either return to one of the stations or departs from the system according to a given Markovian probability.

We first define the stochastic process that represents an evolution of the system, and define the expected sojourn times of each customer conditioned on the system state at its arrival epochs. We show that they satisfy the feedback equations (3.4), (3.8), (3.12). Then we obtain the mean sojourn times spent during every service stage of each customer by using busy and service periods. The expressions of the performance measures are obtained by solving the feedback equations. These expressions have the linear structure which is essentially common to all of the queueing systems investigated in [3-6].

## 2. Model Description

A single server serves $J$ groups of customers at $J$ stations. Customers belonging to a group $i$, called $i$-customers, stay at station $i(1 \le i \le J)$. Group $i$ consists of $L_i$-classes of customers. Customers belonging to class α in group $i$, called $(i, \alpha)$-customers, arrive from outside the system according to a Poisson process with rate $\lambda_{i\alpha}$. Let $S=\{(i,\alpha): 1 \le i \le J; 1 \le \alpha \le L_i\}$.

The system is separated into two parts which are called the service facility and the waiting rooms of the stations. Each customer arriving at each station from outside the system or by feedback enters in the service facility when the gate of the station is opened. Otherwise, it enters in the waiting room. The server selects one of the stations at a time, and open its gate in order to admit some customers at the station to its queue in the service facility. Then the server serves the customers in the facility until the server empties it. Because the gates of the stations that are not selected by the server are closed, all customer at such station must wait for service in the waiting room. Once a customer begins a service, its service is not interrupted by other customer. Each time interval from when the server selects a station until the first time the server empties the service facility is called a service period.




<Figure 1> Multiclass queueing model with feedback

Customers are served according to a predetermined scheduling algorithm defined below.

(1) The server selects the stations in the priority order in which station $i$ has priority over station $j$ if $i < j$.
(2) The service order of customers for each group in the service facility is either FCFS or priority. If the server selects a group with FCFS, it serves all customers in the service facility in a first-come- first-served base.

If the server selects a group with priority, it serves all customers in the service facility in the fixed priority base, where class α customers in the group have priority over class $\beta$ customers if $\alpha < \beta$. All customers in each class are served in a first-come-first-served base.

Service times $S_{i\alpha}$ of $(i, \alpha)$-customers are independent and identically distributed with mean $E[S_{i\alpha}]$. After completing a service, $(i, \alpha)$-customer either returns to the system as a $(j, \beta)$-customer with probability $p_{i\alpha,j\beta}$, or departs from the system with probability $1-\sum_{1 \le j \le J}\sum_{1 \le \beta \le L_j} p_{i\alpha,j\beta}$. The feedback probability matrix is given by $\boldsymbol{P}=(p_{i\alpha,j\beta})$. Since we assume that $\boldsymbol{P}^n \to O$ as $n \to \infty$, where $O$ is a zero matrix. Hence all arriving customers eventually leave the system. Also, The arrival processes, service times, and feedback probability processes are assumed to be independent each other.

We define the stochastic process as follow :

$$Q=\left\{ \begin{array}{l} Y(t)=(X(t),\Gamma(t),\kappa(t),a(t),r(t),g(t),n(t),L(t)) \\ ;t \ge 0 \end{array} \right\}$$

that represents an evolution of the system. $(\kappa(t),a(t))$ denotes the station-class pair of a customer being served at time $t$. $r(t)$ denotes the remaining service time of a customer being served at time $t$. The number of$(i,\alpha)$-customers in the service facility(not being served) at time $t$ is denoted by $g_{i\alpha}(t)$, and the number of $(i, \alpha)$-customers in the waiting room at time $t$ is denoted by $n_{i\alpha}(t)$. Let $g_i(t)=(g_{i\alpha}(t);, 1 \le \alpha \le L_i)$ $n_i(t)=(n_{i\alpha}(t); 1 \le \alpha \le L_i)$, $g(t)=(g_1(t),\cdots,g_J(t))$ and $n(t)=(n_1(t),\cdots,n_J(t))$.

The sample pass of these processes are assumed to be left-continuous with right hand limits.

The $e$th customer arrives from outside the system at time $\sigma_0^e$ and is denoted by $c^e(1 \le e \le J)$. The customer$c^e$ arrives from outside the system according to a Poisson process with rate $\lambda$, and then it becomes an $(i, \alpha)$-customer with probability $\lambda_{i\alpha}/\lambda$. Let $M^e$ be the total number of service stages of customer $c^e$ from its arrival from outside the system at time $\sigma_0^e$ until its departure from the system. Let $\sigma_k^e$ be the time just when it arrives at one of the stations(by feedback) or departs from the system after completing its $k$th service stage($1 \le k \le M^e$). We specify information of the system at time $t$ : $L(t) = \{j_m(t), \beta_m(t), s_m(t); m = 1,2,\cdots\}$ where $(j_m(t),\beta_m(t))$ is a station-class pair and $s_m(t)$ is a status of customer who has arrived the $m$th earliest of all customers in the system at time $t$. Let us consider transition epochs of these process consisting of customer arrival epochs and service completion epochs. Then let $(X(t),\Gamma(t))$ denote a station-class pair of an arriving customer at the last transition epoch before or on $t(t \ge 0)$ ; if it is not a customer arrival epoch, then $(X(t), \Gamma(t)) = (0,0)$. $X(t)$ and $L(t)$ are right continuous with left-hand limits.

Possible values of $Y(t)(t \ge 0)$ are called states. and the state space of $Q$ is denoted by $\Omega$.

Let $\overline{T}_{i\alpha,j}$ be the total expected amount of service times a customer receives from when it becomes an $(i, \alpha)$ -customer until its departure from the system or its next entrance one of the stations 1 through $j$. Let

$$\rho_j^+ = \begin{cases} 0, & j=0 \\ \sum_{i=1}^{j}\sum_{\alpha=1}^{L_i} \lambda_{i\alpha} \overline{T}_{k\gamma,j}, & j=1,\cdots,J \end{cases} \quad \cdots\cdots (2.1)$$

We assume that $\rho_J^+ < 1$.

## 3. Performance Measures

We define three types of the system performance measures for a customer $c^e$. First type of the performance measures is related to the waiting times of the customer in the waiting room. The function $C^e_{W,i\alpha}(t) = 1$, if the $c^e$ stays in the waiting room as an $(i, \alpha)$-customer at time $t$ or $C^e_{W,i\alpha}(t) = 0$ otherwise. Then the $c^e$'s waiting time spent in the waiting room as an $(i,\alpha)$-customer is defined by

$$W^e_{i\alpha} = \int_0^\infty C^e_{W,i\alpha}(t)dt \quad \cdots\cdots (3.1)$$

For $l = 0, 1, 2,\cdots$, we define its expected waiting times conditioned on the state of the system($Y$) at time $\sigma^e_l$ which spent by the $c^e$ in the waiting room as an $(i, \alpha)$-customer after time $\sigma^e_l$ :

$$W^e_{i\alpha}(Y, e, l) = E[\int_{\sigma^e_l}^\infty C^e_{W,i\alpha}(t)dt| Y(\sigma^e_l) = Y] \quad \cdots\cdots (3.2)$$

$$W^I_{i\alpha}(Y, e, l) = E[\int_{\sigma^e_l}^{\sigma^e_{l+1}} C^e_{W,i\alpha}(t)dt| Y(\sigma^e_l) = Y] \quad \cdots\cdots (3.3)$$

$W_{i\alpha}(Y, e, l)$ is the overall expected waiting time after time $\sigma^e_l$ where as $W^I_{i\alpha}(Y, e, l)$ is the expected waiting time during a service stage in $[\sigma^e_l, \sigma^e_{l+1})$. The the feedback equation holds.

$$W_{i\alpha}(Y, e, l) = W^I_{i\alpha}(Y, e, l) + E[W_{i\alpha}(Y(\sigma^e_{l+1}), e, l+1| Y(\sigma^e_l) = Y] \quad \cdots\cdots (3.4)$$

for $l = 0, 1, 2,\cdots$.

Second type of performance measures is related to pieces of the $c^e$'s waiting times in the waiting room. Let

$$H^e_{i\alpha}(k) = \int_0^\infty C^e_{W,i\alpha}(t)1\{\kappa(t) = k\}dt, 1 \le \kappa \le J \quad \cdots\cdots (3.5)$$

where 1A=1 if event $A$ occurs, or 1A=0 otherwise. $H^e_{i\alpha}(k)$ is the $c^e$'s waiting time spent in the waiting room as an $(i, \alpha)$-customer while the system is in period $k$. For $l = 0, 1, 2,\cdots$, we define two expected waiting times conditioned on the state of the system($Y$) at time $\sigma^e_l$ which spent by the $c^e$ in the waiting room as an $(i, \alpha)$-customer after time $\sigma^e_l$ while the system is in period $k$.

$$H^e_{i\alpha}(Y, e, l, k) = E[\int_{\sigma^e_l}^\infty C^e_{W,i\alpha}(t)1\{\kappa(t) = k\}dt| Y(\sigma^e_l) = Y] \quad \cdots\cdots (3.6)$$

$$H^I_{i\alpha}(Y, e, l, k) = E[\int_{\sigma^e_l}^{\sigma^e_{l+1}} C^e_{W,i\alpha}(t)1\{\kappa(t) = k\}dt| Y(\sigma^e_l) = Y] \quad \cdots\cdots (3.7)$$

$H_{i\alpha}(Y, e, l, k)$ is the overall expected waiting time after time $\sigma^e_l$ where as $H^I_{i\alpha}(Y, e, l, k)$ is the expected waiting time during a service stage in $[\sigma^e_l, \sigma^e_{l+1})$. Then the feedback equation holds.

$$H_{i\alpha}(Y, e, l, k) = H^I_{i\alpha}(Y, e, l, k) + E[W_{i\alpha}(Y(\sigma^e_{l+1}), e, l+1, k| Y(\sigma^e_l) = Y] \quad \cdots\cdots (3.8)$$

for $l = 0, 1, 2,\cdots$ and $1 \le k \le J$.

Third type of performance measures is related to the sojourn times(i.e.,. sum of the waiting time and service time) of the $c^e$ in the service facility. We define for any $t > 0$ and $(i, \alpha) \in S$, the function $C^e_{F,i\alpha}(t) = 1$, if the $c^e$ stays in the service facility or receives a service as an $(i, \alpha)$ -customer at time $t$ or $C^e_{F,i\alpha}(t) = 0$ otherwise.

Then the $c^e$'s sojourn time spent in the service facility as an $(i, \alpha)$-customer is defined by

$$F_{i\alpha}^{e} = \int_{0}^{\infty} C_{F,i\alpha}^{e}(t)dt \quad \cdots (3.9)$$

For $l = 0, 1, 2, \cdots$, we define its expected sojourn times conditioned on the state of the system( $Y$ ) at time $\sigma_l^e$, which spent by the $c^e$ in the service facility as an $(i, \alpha)$-customer after time $\sigma_l^e$ :

$$F_{i\alpha}^{e}(Y, e, l) = E[\int_{\sigma_l^e}^{\infty} C_{F,i\alpha}^{e}(t)dt | Y(\sigma_l^e) = Y], \quad \cdots (3.10)$$

$$F_{i\alpha}^{I}(Y, e, l) = E[\int_{\sigma_l^e}^{\sigma_{l+1}^e} C_{F,i\alpha}^{e}(t)dt | Y(\sigma_l^e) = Y], \quad \cdots (3.11)$$

$F_{i\alpha}(Y, e, l)$ is the overall expected sojourn time after time $\sigma_l^e$ where as $F_{i\alpha}^{I}(Y, e, l)$ is the expected sojourn time during a service stage in $[\sigma_l^e, \sigma_{l+1}^e)$. Then the feedback equation holds.

$$F_{i\alpha}(Y, e, l) = F_{i\alpha}^{I}(Y, e, l) + E[F_{i\alpha}(Y(\sigma_{l+1}^e), e, l+1 | Y(\sigma_l^e) = Y] \quad \cdots (3.12)$$

for $l = 0, 1, 2, \cdots$.

## 4. Busy and Service Periods Analysis

Now let us consider an $(i, \alpha)$-customer staying at station $i$. Let $T_{i\alpha}^{\delta}$ be the total amount of service times the customers receives until the first time it departs from the set of classes $(i, 1), \cdots, (i, \delta) (\in S)$ at station $i$ after at least receiving its initial service as an $(i, \alpha)$-customer. Let $\overline{T}_{i\alpha}^{\delta}$ be its expected value and $\overline{T}_{i\alpha}^{\delta}(r)$ be its expected value conditioned on its initial remaining service time $r$ as an $(i, \alpha)$-customer. Then, for $(i, \alpha) \in S$ and $\delta = 0, 1, \cdots, L_i$,

$$\overline{T}_{i\alpha}^{\delta} = E[S_{i\alpha}] + \sum_{\beta=1}^{\delta} p_{i\alpha, i\beta} \overline{T}_{i\beta}^{\delta} \quad \cdots (4.1)$$

$$\overline{T}_{i\alpha}^{\delta}(r) = r + \sum_{\beta=1}^{\delta} p_{i\alpha, i\beta} \overline{T}_{i\beta}^{\delta} \quad \cdots (4.2)$$

We define

$$\rho_{i\delta}^{+} = \sum_{\alpha=1}^{\delta} \lambda_{i\alpha} \overline{T}_{i\alpha}^{\delta} \qquad \delta = 0, 1, 2, \cdots, L_i \quad \cdots (4.3)$$

where the empty sum which arises when $\delta = 0$ is equal to 0. For convenience, we define

$$1_{i\alpha}(j, \beta) = \begin{cases} 1, & (j, \beta) = (i, \alpha) \\ 0, & otherwise \end{cases} \quad \cdots (4.4)$$

$$1(r) = \begin{cases} 1, & r > 0 \\ 0, & r = 0 \end{cases} \quad \cdots (4.5)$$

For any transition epoch $\tau$, let $D(\tau)$ and $n_{i\alpha}^{C}(\tau)$ be the remaining length of the current service period (or idle period) $\kappa(\tau)$ and the number of $(i, \alpha)$-customers at the period completion epoch, respectively. Then we define the conditional expected values.

$$D(Y) = E[D(\tau) | Y(\tau) = Y] \quad \cdots (4.6)$$

$$\overline{n}_{i\alpha}^{C}(Y) = E[n_{i\alpha}^{C}(\tau) | Y(\tau) = Y] \quad \cdots (4.7)$$

For any $(k, \gamma) \in S$, let $Y_{k\gamma} \in \Omega$ be the state of the system in which only a $(k, \gamma)$-customer ready to start a service is in the system in a service period $k$. Let $Y_{k\gamma}^{r} \in \Omega$ be the state of the system in which only a $(k, \gamma)$-customer having a remaining service time $r$ is already in service in this state. We define $\overline{n}_{i\alpha}^{C}(k, \gamma) = \overline{n}_{i\alpha}^{C}(Y_{k\gamma})$ and $\overline{n}_{i\alpha}^{C}(k, \gamma, r) = \overline{n}_{i\alpha}^{C}(Y_{k\gamma}^{r})$ which are the expected numbers of $(i, \alpha)$-customers at the completion epoch of service period starting with a $(k, \gamma)$-customer. Then we have

$$\overline{n}_{i\alpha}^{C}(k, \gamma) = \begin{cases} \lambda_{i\alpha} r + p_{k\gamma, i\alpha} + \sum_{\delta=1}^{L_k} (\lambda_{k\delta} r + p_{k\gamma, k\delta}) \overline{n}_{i\alpha}^{C}(k, \delta), & i \neq k \\ 0, & i = k \end{cases} \quad \cdots (4.8)$$

$$\overline{n}_{i\alpha}^{C}(k, \gamma, r) = \begin{cases} \lambda_{i\alpha} r + p_{k\gamma, i\alpha} + \sum_{\delta=1}^{L_k} (\lambda_{k\delta} r + p_{k\gamma, k\delta}) \overline{n}_{i\alpha}^{C}(k, \delta), & i \neq k \\ 0, & i = k \end{cases} \quad \cdots (4.9)$$

For any state $Y = (j, \beta, \kappa_0, a_0, r, \mathbf{g}, \mathbf{n}, L) \in \Omega$ at time $\tau$ ($\mathbf{g} = (g_{i\alpha} : (i, \alpha) \in S)$, $\mathbf{n} = (n_{i\alpha} : (i, \alpha) \in S)$). we have

$$D(Y) = \begin{cases} \dfrac{1(r)\overline{T}_{\kappa_0 a_0}^{L_{\kappa_0}}(r) + \sum_{\gamma=1}^{L_{\kappa_0}} (g_{\kappa_0\gamma} + 1_{\kappa_0\gamma}(j, \beta)) \overline{T}_{\kappa_0\gamma}^{L_{\kappa_0}}}{1 - \rho_{\kappa_0 L_{\kappa_0}}^{+}}, & \kappa_0 > 0 \\ 0, & \kappa_0 = 0 \end{cases} \quad \cdots (4.10)$$

$$\overline{n}_{i\alpha}^{C}(Y) = \begin{cases} n_{i\alpha} + 1_{i\alpha}(j, \beta) + 1(r)\overline{n}_{i\alpha}^{C}(\kappa_0, a_0, r) \\ \quad + \sum_{\gamma=1}^{L_{\kappa_0}} (g_{\kappa_0\gamma} + 1_{\kappa_0\gamma}(j, \beta)) \overline{n}_{i\alpha}^{C}(\kappa_0, \gamma), & i \neq \kappa_0 \\ 0, & i = \kappa_0 \\ 1_{i\alpha}(j, \beta), & \kappa_0 = 0 \end{cases} \quad \cdots (4.11)$$

The customers belonging among groups 1 through $i$ are called $c_i^{+}$-customers. Then we derive the following quantities

$$\overline{N}^{(j\delta)}_{j\alpha,k\gamma}(r)=\lambda_{k\gamma}r+p_{j\alpha,k\gamma}+\sum_{\delta'=1}^{\delta}(\lambda_{j\delta'}r+p_{j\alpha,j\delta'})\overline{N}^{(j\delta)}_{j\delta',k\gamma} \cdots\cdots (5.8)$$

for $0\le\delta\le L_j$ and $(k,\gamma)\in S(\gamma>\delta$, if $k=j)$. Note that the $c^e$'s waiting times related to $F^I_{j\beta}(Y,e,l)$ -(1) from its arrival epoch to the start of its service (for $\kappa_0=j$) and (2) from the visiting instant of station $j$ to the start of its service (for $\kappa_0\neq j$)- are $(j,\beta-1)$-busy periods with initial works of customers with higher priorities than the $c^e$'s staying in the service facility at the beginning of the waiting times. Then we have

$$E[n_{k\gamma}(\sigma^e_{l+1})|Y(\sigma^e_l)=Y] \cdots\cdots (5.9)$$

$$=\begin{cases} n_{k\gamma}+1(r)\overline{N}^{(j\beta-1)}_{ja_0,k\gamma}(r)+\sum_{\alpha=1}^{\beta}g_{j\alpha}\overline{N}^{(j\beta-1)}_{j\alpha,k\gamma} \\ \quad +\lambda_{k\gamma}E[S_{j\beta}], & \kappa_0=j,\ k\neq j \\ \overline{N}^j_{k\gamma}(Y)+\sum_{\alpha=1}^{\beta-1}\overline{N}^j_{j\alpha}(Y)\overline{N}^{(j\beta-1)}_{j\alpha,k\gamma} \\ \quad +n_{j\beta}\overline{N}^{(j\beta-1)}_{j\beta,k\gamma}+\lambda_{k\gamma}E[S_{j\beta}], & \kappa_0\neq j,\ k\neq j \\ 0, & k=j \end{cases}$$

The $(j,\gamma)$-customers in the service facility at the completion epoch of the sojourn time $F^I_{j\beta}(Y,e,l)$ is similar to that for the FCFS groups except that in the case $(j,\gamma)$ -customer($\gamma<\beta$) are cleared from the system when the $c^e$ starts service. Thus we have

$$E[g_{k\gamma}(\sigma^e_{l+1})|Y(\sigma^e_l)=Y] \cdots\cdots (5.10)$$

$$=\begin{cases} 0, & k\neq j \\ \lambda_{j\gamma}E[S_{j\beta}], & k=j,\gamma<\beta \\ 1(r)\overline{N}^{(j\beta-1)}_{ja_0,j\beta}(r)+\sum_{\alpha=1}^{\beta}g_{j\alpha}\overline{N}^{(j\beta-1)}_{j\alpha,j\beta} \\ +\lambda_{j\beta}E[S_{j\beta}], & \kappa_0=j,\ k=j,\ \gamma=\beta \end{cases}$$

$$\begin{cases} g_{j\gamma}+1(r)\overline{N}^{(j\beta-1)}_{ja_0,j\gamma}(r)+\sum_{\alpha=1}^{\beta}g_{j\alpha}\overline{N}^{(j\beta-1)}_{j\alpha,j\gamma}+\lambda_{j\gamma}E[S_{j\beta}], \\ \kappa_0=j, k=j, \\ \overline{N}^j_{j\beta}(Y)-n_{j\beta}-1+\sum_{\alpha=1}^{\beta-1}\overline{N}^j_{j\alpha}(Y)\overline{N}^{(j\beta-1)}_{j\alpha,j\beta} \\ +n_{j\beta}\overline{N}^{(j\beta-1)}_{j\beta,j\beta}+\lambda_{j\beta}E[S_{j\beta}], \quad \kappa_0\neq j, k=j, \gamma=\beta \\ \overline{N}^j_{j\gamma}(Y)+\sum_{\alpha=1}^{\beta-1}\overline{N}^j_{j\alpha}(Y)\overline{N}^{(j\beta-1)}_{j\alpha,j\gamma} \\ +n_{j\beta}\overline{N}^{(j\beta-1)}_{j\beta,j\gamma}+\lambda_{j\gamma}E[S_{j\beta}], \quad \kappa_0=j, k=j, \end{cases}$$

### 5.3 Linear functional expressions of the Quantities

From the analysis of this section, we can see the following important properties :

(1) The component $(j,\beta,\kappa_0,a_0,r,\boldsymbol{g},\boldsymbol{n})$ of state $Y=(j,\beta,\kappa_0,a_0,r,\boldsymbol{g},\boldsymbol{n},L)$ at the $c^{e'}$ arrival epoch $\sigma^e_l$ is sufficient to derive $W^I_{j\beta}(Y,e,l)$, $H^I_{j\beta}(Y,e,l,k)$, $F^I_{j\beta}(Y,e,l)$ and the expected vector of the numbers of customers $E[(\boldsymbol{g}(\sigma^e_{l+1}),\ \boldsymbol{n}(\sigma^e_{l+1})|Y(\sigma^e_l)=Y]$.

(2) These performance measures and conditional expected number of customers at $\sigma^e_{l+1}$ are linear with respect to the component $(\boldsymbol{g},\boldsymbol{n})$.

**Proposition 2.** Let $Y=(j,\beta,\kappa_0,a_0,r,\boldsymbol{g},\boldsymbol{n},L)\in\Omega$, $e=1,2,\cdots$and $l=0,1,2,\cdots$ Then we have

$$W^I_{j\beta}(Y,e,l)=(r,1(r))\boldsymbol{\psi}^{j\beta}(\kappa_0,a_0,0) +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{w}^{j\beta}(\kappa_0,0)+w^{j\beta}(\kappa_0,0) \cdots\cdots (5.11)$$

$$H^I_{j\beta}(Y,e,l,k)=(r,1(r))\boldsymbol{\psi}^{j\beta}(\kappa_0,a_0,k) +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{w}^{j\beta}(\kappa_0,k)+w^{j\beta}(\kappa_0,k), \quad k=1,2,\cdots,J \cdots\cdots (5.12)$$

$$F^I_{j\beta}(Y,e,l)=(r,1(r))\boldsymbol{\eta}^{j\beta}(\kappa_0,a_0) +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{f}^{j\beta}(\kappa_0)+f^{j\beta}(\kappa_0) \cdots\cdots (5.13)$$

For $(i,\alpha)\neq(j,\beta)$

$$W^I_{i\alpha}(Y,e,l)=H^I_{i\alpha}(Y,e,l)=F^I_{i\alpha}(Y,e,l)=0 \cdots (5.14)$$

Also we have

$$E[(\boldsymbol{g}(\sigma^e_{l+1}),\boldsymbol{n}(\sigma^e_{l+1})|Y(\sigma^e_l)=Y] =(r,1(r))\boldsymbol{v}^{j\beta}(\kappa_0,a_0) +(\boldsymbol{g},\boldsymbol{n})\boldsymbol{U}^{j\beta}(\kappa_0)+\boldsymbol{u}^{j\beta}(\kappa_0) \cdots\cdots (5.15)$$

Note the coefficients $\boldsymbol{\psi}^{j\beta}(\kappa_0,a_0,k)$, $\boldsymbol{w}^{j\beta}(\kappa_0,k)$, $w^{j\beta}(\kappa_0,k)$, $\boldsymbol{\eta}^{j\beta}(\kappa_0,a_0)$, $\boldsymbol{f}^{j\beta}(\kappa_0)$, $f^{j\beta}(\kappa_0)$ can be determined from the given system parameters by using the expressions in this section.

## 6. Expressions of the Performance Measures

We obtain the expressions of the performance measures defined in section 2.

Let $\hat{w}_{i\alpha}(j,\beta,k)$, $\hat{f}_{i\alpha}(j,\beta)$ be the solutions of the following equations :

$$\hat{\boldsymbol{w}}_{i\alpha}(j,\beta,k)=p_{j\beta,i\alpha}\boldsymbol{w}^{i\alpha}(j,k)+\sum_{m=1}^{J}\sum_{\delta=1}^{L_m}p_{j\beta,m\delta}\boldsymbol{U}^{m\delta}(j)\hat{\boldsymbol{w}}_{i\alpha}(m,\delta,k) \quad (6.1)$$

$$\hat{\boldsymbol{f}}_{i\alpha}(j,\beta)=p_{j\beta,i\alpha}\boldsymbol{f}^{i\alpha}(j)+\sum_{m=1}^{J}\sum_{\delta=1}^{L_m}p_{j\beta,m\delta}\boldsymbol{U}^{m\delta}(j)\hat{\boldsymbol{f}}_{i\alpha}(m,\delta) \quad (6.2)$$

where $\boldsymbol{w}^{i\alpha}(j,k), \boldsymbol{f}^{i\alpha}(j)$ and $\boldsymbol{U}^{m\delta}(j)$ are given (5.11), (5.12), (5.13) and (5.14). Also let $\hat{w}_{i\alpha}(j,\beta,k)$, $\hat{f}_{i\alpha}(j,\beta)$ be the solutions of the following equations:

$$\hat{w}_{i\alpha}(j,\beta,k)=p_{j\beta,i\alpha}w^{i\alpha}(j,k)+\sum_{m=1}^{J}\sum_{\delta=1}^{L_m}p_{j\beta,m\delta}[\boldsymbol{u}^{m\delta}(j)\hat{\boldsymbol{w}}_{i\alpha}(m,\delta,k)+\hat{w}_{i\alpha}(m,\delta,k)] \quad (6.3)$$

$$\hat{f}_{i\alpha}(j,\beta)=p_{j\beta,i\alpha}f^{i\alpha}(j)+\sum_{m=1}^{J}\sum_{\delta=1}^{L_m}p_{j\beta,m\delta}[\boldsymbol{u}^{m\delta}(j)\hat{\boldsymbol{f}}_{i\alpha}(m,\delta)+\hat{f}_{i\alpha}(m,\delta)] \quad (6.4)$$

where $w^{i\alpha}(j,k), f^{i\alpha}(j)$ and $\boldsymbol{u}^{m\delta}(j)$ are given (5.11), (5.12), (5.13) and (5.14). Let define constants:

$$\boldsymbol{\phi}_{i\alpha}(j,\beta,\kappa_0,a_0,k)=1_{i\alpha}(j,\beta)\boldsymbol{\phi}^{j\beta}(\kappa_0,a_0,k)+\boldsymbol{v}^{j\beta}(\kappa_0,a_0)\hat{\boldsymbol{w}}_{i\alpha}(j,\beta,k)$$

$$\boldsymbol{w}_{i\alpha}(j,\beta,\kappa_0,k)=1_{i\alpha}(j,\beta)\boldsymbol{w}^{j\beta}(\kappa_0,k)+\boldsymbol{U}^{j\beta}(\kappa_0)\hat{\boldsymbol{w}}_{i\alpha}(j,\beta,k)$$

$$w_{i\alpha}(j,\beta,\kappa_0,k)=1_{i\alpha}(j,\beta)w^{j\beta}(\kappa_0,k)+\boldsymbol{u}^{j\beta}(\kappa_0)\hat{\boldsymbol{w}}_{i\alpha}(j,\beta,k)+\hat{w}_{i\alpha}(j,\beta,k)$$

$$\boldsymbol{\eta}_{i\alpha}(j,\beta,\kappa_0,a_0)=1_{i\alpha}(j,\beta)\boldsymbol{\eta}^{j\beta}(\kappa_0,a_0)+\boldsymbol{v}^{j\beta}(\kappa_0,a_0)\hat{\boldsymbol{f}}_{i\alpha}(j,\beta)$$

$$\boldsymbol{f}_{i\alpha}(j,\beta,\kappa_0)=1_{i\alpha}(j,\beta)\boldsymbol{f}^{j\beta}(\kappa_0)+\boldsymbol{U}^{j\beta}(\kappa_0)\hat{\boldsymbol{f}}_{i\alpha}(j,\beta)$$

$$f_{i\alpha}(j,\beta,\kappa_0)=1_{i\alpha}(j,\beta)f^{j\beta}(\kappa_0)+\boldsymbol{u}^{j\beta}(\kappa_0)\hat{\boldsymbol{f}}_{i\alpha}(j,\beta)+\hat{f}_{i\alpha}(j,\beta)$$

for $(\kappa_0,a_0)\in S\cup\{(0,0)\}$, and $0\le k\le J$. Then we summarize these results.

**Proposition 3**: For any $Y=(j,\beta,\kappa_0,a_0,r,\boldsymbol{g},\boldsymbol{n},L), e=1,2,\cdots, l=0,1,2,\cdots,(i,\alpha)\in S$, and $1\le k\le J$, we can obtain the following expressions for the performance measures :

$$\widehat{W}_{i\alpha}(Y,e,l)=(r,1(r))\boldsymbol{\phi}_{i\alpha}(j,\beta,\kappa_0,a_0,0)+(\boldsymbol{g},\boldsymbol{n})\boldsymbol{w}_{i\alpha}(j,\beta,\kappa_0,0)+w_{i\alpha}(j,\beta,\kappa_0,0) \quad (6.5)$$

$$\widehat{H}_{i\alpha}(Y,e,l,k)=(r,1(r))\boldsymbol{\phi}_{i\alpha}(j,\beta,\kappa_0,a_0,k)+(\boldsymbol{g},\boldsymbol{n})\boldsymbol{w}_{i\alpha}(j,\beta,\kappa_0,k)+w_{i\alpha}(j,\beta,\kappa_0,k) \quad (6.6)$$

$$\widehat{F}_{i\alpha}(Y,e,l)=(r,1(r))\boldsymbol{\eta}_{i\alpha}(j,\beta,\kappa_0,a_0)+(\boldsymbol{g},\boldsymbol{n})\boldsymbol{f}_{i\alpha}(j,\beta,\kappa_0)+f_{i\alpha}(j,\beta,\kappa_0) \quad (6.7)$$

**Proposition 4** : Let us consider the system defined in section 2. Then for $(i,\alpha)\in S$ and $k=1,2,\cdots,J$, $\widehat{W}_{i\alpha}(\cdot)$ defined in (6.5) satisfies equation (3.4), $\widehat{H}_{i\alpha}(\cdot,k)$ defined in (6.6) satisfies equation (3.8), $\widehat{F}_{i\alpha}(\cdot)$ defined in (6.7) satisfies equation (3.11).

Then it can be shown that equations (3.4), (3.8) and (3.11) have unique solutions. The proof of Proposition 4 and the uniqueness is similar to that in [3, 4]. Thus $H_{i\alpha}(\cdot,k)=\widehat{H}_{i\alpha}(\cdot,k)$and $F_{i\alpha}(\cdot)=\widehat{F}_{i\alpha}(\cdot)$ and equations (6.5), (6.6) and (6.7) give the linear functional equations of the performance measures defined in section 2.

## 7. Conclusions

We have concerned with a queueing system with multi-layered composite scheduling algorithms. In order to derive the mean sojourn times, we can apply unified solution method [3-6]. The key feature of our method is to derive the linear functional expressions for the performance measures such as (6.5), (6.6) and (6.7). We can obtain the steady state mean sojourn times by simple limiting procedures.

## References


[1] Chen, J. S. and Guerin, R.; "Performance study of an input queueing packet switch with two priority classes," *IEEE Transactions on Communications*, 39 : 117-126, 1991.

[2] Chen, J. S., Guerin, R., and Stern, T. E.; "Markov-modulated flow model for output queues of packet switch," *IEEE Transactions on Communications*, 40, 1908-1110, 1992.

[3] Hong, S. J. and Hirayama, T.; "Sojourn Times in a Multiclass Priority Queue with Random Feedback," *International Journal of Management Science*, 2 : 105-127, 1996.

[4] Hirayama, T.; "Mean Sojourn Times in Multiclass Feedback Queues with Gated Disciplines," *Naval Research Logistics*, 50 : 719-741, 2003.

[5] Hirayama, T., Hong, S. J., and Krunz, M. M.; "A New Approach to Analysis of Polling Systems," *Queueing Systems*, 48 : 135-158, 2004.

[6] Hirayama, T.; "Multiclass Polling Systems with Markovian Feedback : mean sojourn times in gated and exhaustive with local priority and FCFS order," J. *Operations Research. Society of Japan*, 48 : 226-255, 2005.

[7] Paterok, M. and Fisher, O.; "Feedback Queues with Preemption-Distance Priorities," *Performance Evaluation Review*, 17 : 136-145, 1989.

[8] Sisi, M., Levy, H., and Fuhrmann, S. W.; "A Queueing Network with Single Cyclically Roving Server," *Queueing Systems*, 11 : 121-144, 1992.