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BANACH SPACE WITH PROPERTY (β) WHICH CANNOT BE RENORMED TO BE B-CONVEX

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Abstract. In this paper, we study property (β) and B-convexity in reflexive Banach spaces. It is shown that k-uniform convexity implies B-convexity and property (β). We also show that there is a Banach space with property (β) which cannot be equivalently renormed to be B-convex.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X. By B_X and S_X, we denote the closed unit ball and the unit sphere of X, respectively. For any subset A of X by $\mathrm{co}(A)(\overline{\mathrm{co}}(A))$ we denote the convex hull (closed convex hull) of A

$(X, \|\cdot\|)$ is called uniformly convex (UC) if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x,\ y \in B_X$ with $\|x - y\| \geq \epsilon$,

$$\left\| \frac{1}{2}(x+y) \right\| \leq 1 - \delta.$$

A k-uniformly convex space is defined for $k \geq 2$ in an obvious fashion so that a uniformly convex space is just 2-uniformly convex; $(X, \|\cdot\|)$ is k-uniformly convex if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x_1, x_2, \cdots, x_k \in B_X$ with $\|x_i - x_j\| \geq \epsilon$ for $i \neq j$ and $i, j = 1, 2, \cdots, k$,

$$\left\| \frac{1}{k} \sum_{i=1}^{k} x_i \right\| \leq 1 - \delta.$$

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For a sequence (x_n) in X, we let

$$\operatorname{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\}.$$

$(X, \|\cdot\|)$ is said to have the Kadec-Klee property (KK) if for $x_n \in B_X$, $\operatorname{sep}(x_n) > 0$ and $x_n \to x$ weakly, then $\|x\| < 1$. The followings are intermediate notions between (UC) and (KK).

$(X, \|\cdot\|)$ is uniformly Kadec-Klee property (UKK) if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x_n \in B_X$, $\operatorname{sep}(x_n) \geq \epsilon$ and $x_n \to x$ weakly, then $\|x\| \leq 1 - \delta$.

$(X, \|\cdot\|)$ is nearly uniformly convex (NUC) if for all $\epsilon > 0$, there exists $\delta > 0$ if $x_n \in B_X$, $\operatorname{sep}(x_n) \geq \epsilon$, then $\operatorname{co}(x_n) \cap (1-\delta)B_X \neq \emptyset$.

It is easy to see that (UKK) $\Rightarrow$ (KK) and (UC) $\Rightarrow$ (NUC). Huff [3] showed that a space is (NUC) if and only if it is (UKK) and reflexive. We have the following implication.

$$(\mathrm{UC}) \Rightarrow (\mathrm{NUC}) \Rightarrow (\mathrm{UKK}) \Rightarrow (\mathrm{KK})$$

For any subset C, we denote by $\alpha(C)$ its Kuratowski measure of non-compactness, i.e., the infimum of such $\epsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less than ϵ.

For any $x \notin B_X$, the drop determined by x is the set

$$D(x, B_X) = \operatorname{co}\Big(\{x\} \cup B_X\Big)$$

Rolewicz [8] has defined property (β). A Banach space X is said to have property (β) if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\alpha\Big(D(x, B_X) \backslash B_X\Big) < \epsilon$$

whenever $1 < \|x\| < 1 + \delta$.

The following result is found in [5].

A Banach space X has property (β) if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for each element $x \in B_X$ and each sequence $(x_n) \in B_X$ with $\operatorname{sep}(x_n) \geq \epsilon$, there is $k \in \mathbb{N}$ such that

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

By this result we can easily prove that property (β) implies (NUC). We have finally

$$(\text{UC}) \Rightarrow \text{property} \ \ (\beta) \Rightarrow (\text{NUC}) \Rightarrow (\text{UKK}) \Rightarrow (\text{KK})$$

A Banach space is said to have the Banach-Saks property if any bounded sequence in the space admits a subsequence whose arithmetic means converges in norm. S. Kakutani [4] showed that Uniform convexity implies the Banach-Saks property. And T. Nishiura and D. Waterman [7] proved that the Banach-Saks property implies reflexivity in Banach spaces

A Banach space X is called (r, δ)-convex if for any r elements $x_1, x_2 \cdots, x_r$ of X with $\|x_i\| \leq 1$ there is an alternate signs sequence $\epsilon = (\epsilon_i)_{i=1}^r$ of ± 1 such that $\|\epsilon_1 x_1 + \cdots + \epsilon_r x_r\| \leq r(1-\delta)$. A Banach space X which is (r,δ)-convex for some r and some $\epsilon > 0$ is called B-convex.

To end with this introduction, let us mention the following lemma.

Lemma 1 [6]. *Let $(Y, \|\cdot\|)$ be a Banach space with basis $(e_i : i \in I)$ (unconditional if I is noncountable) and such that, for every finite subset J of I,*

$$\text{if } \ 0 \leq |\alpha_j| \leq \beta_j, j \in J, \ \text{then} \ \left\| \sum_{j \in J} \alpha_j e_j \right\| \leq \left\| \sum_{j \in J} \beta_j e_j \right\|.$$

Let $(X_i, i \in I)$ be a family of finite dimensional Banach space. Let

$$Z := \left\{ (x_i)_{i \in I} \in \prod_{i \in I} X_i : \sum_{i \in I} \|x_i\| e_i \in Y \right\}$$

equipped with the norm $\|(x_i)_{i \in I}\| = \left\| \sum_{i \in I} \|x_i\| e_i \right\|$. Then, if $(Y, \|\cdot\|)$ has property (β), $(Z, \|\cdot\|)$ has property (β), too.

2. B-convexity and property (β)

We start with the following results.

THEOREM 2 [2]. *If a Banach space X has property (β), then both X and X^* have the Banach-Saks property.*

THEOREM 3 [1]. *If a Banach space X is reflexive and B-convex, then both X and X^* have the Banach-Saks property.*

By Theorem 2 and Theorem 3, it is natural to consider the relationship between the property (β) in Banach spaces and B-convexity in reflexive Banach spaces. We have another results concerning the properties.

PROPOSITION 4. *If a Banach space X is k-uniformly convex, there exists $\delta > 0$ such that X is (k, δ)-convex.*

Proof. Suppose that X is k-uniformly convex. Then for $\epsilon = 1$, there exists $\delta(1) > 0$ such that if $\|x_i - x_j\| \leq 1$, $i \neq j$, $1 \leq i, j \leq k$ and $x_i \in B_X$, then

$$\left\| \frac{1}{k} \sum_{i=1}^{k} x_i \right\| \leq 1 - \delta(1)$$

Let $\delta = \inf \left\{ \frac{1}{k}, \delta(1) \right\}$. It is suffices to show that for $x_i \in B_X$ and $i = 1, 2, \cdots, k$,

$$\inf_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^{k} \epsilon_i x_i \right\| \leq k(1 - \delta).$$

Suppose that

$$\left\| \sum_{i=1}^{k} \epsilon_i x_i \right\| > k(1 - \delta),$$

for all signs $\epsilon_i = \pm 1$.

For $1 \leq i < j \leq k$,

$$\begin{aligned} \|x_i - x_j\| &\geq \|x_1 + \cdots + x_i + \cdots + x_{j-1} - x_j + \cdots + x_k\| - \sum_{l \neq i,j} \|x_l\| \\ &> k(1 - \delta) - (k - 2) = 2 - k\delta \geq 1 \end{aligned}$$

Then

$$\left\| \frac{1}{k} \sum_{i=1}^{k} x_i \right\| < 1 - \delta(1) \leq 1 - \delta.$$

We get the contradiction. □

Proposition 5. *If X is k-uniformly convex then it has property (β).*

Proof. Suppose that X is k-uniformly convex. Let $\epsilon > 0$. Then there exists $\delta\left(\frac{\epsilon}{3(k-1)}\right) > 0$ such that if $z_i \in B_X$ and $\|z_i - z_j\| \geq \frac{\epsilon}{3(k-1)}$ $i \neq j$ and $1 \leq i, j \leq k$, then $\left\|\frac{1}{k}\sum_{i=1}^{k} z_i\right\| \leq 1 - \delta\left(\frac{\epsilon}{3(k-1)}\right)$.

We show that there exists $\delta > 0$ such that for $x, x_n \in B_X$ with $\mathrm{sep}(x_n) \geq \epsilon$, there exists $m \in \mathbb{N}$ such that $\left\|\frac{x+x_m}{2}\right\| \leq 1 - \delta$.

Take $\delta = \delta\left(\frac{\epsilon}{3(k-1)}\right)$. Let $x \in B_X$ and $x_n \in B_X$ with $\mathrm{sep}(x_n) \geq \epsilon$. Then there exists $m \in \mathbb{N}$ such that $\|x - x_m\| \geq \frac{\epsilon}{3}$. (Indeed, if for all $n \in \mathbb{N}$ $\|x - x_n\| < \frac{\epsilon}{3}$, then $\|x_i - x_j\| \leq \|x_i - x\| + \|x_j - x\| < \frac{2\epsilon}{3}$. This contradicts that $\mathrm{sep}(x_n) \geq \epsilon$.)

Consider $y_1 = x_m$, $y_2 = \frac{1}{k-1}\{x + (k-2)x_m\}$, $y_3 = \frac{1}{k-1}\{2x + (k-3)x_m\}, \cdots$, $y_{k-1} = \frac{1}{k-1}\{(k-2)x + x_m\}$, $y_k = x$. Since for $1 \leq i < j \leq k$

$$\begin{aligned}
\|y_i - y_j\| &= \frac{1}{k-1}\|\{(i-1)x + (k-i)x_m\} - \{(j-1)x + (k-j)x_m\}\| \\
&= \frac{1}{k-1}\|(i-j)x + (j-i)x_m\| = \frac{(j-i)}{(k-1)}\|x_m - x\| \\
&\geq \frac{1}{k-1}\|x_m - x\| \geq \frac{\epsilon}{3(k-1)}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{k} y_i &= \frac{1}{k-1}\sum_{i=1}^{k}\{(i-1)x + (k-i)x_m\} \\
&= \frac{1}{(k-1)}\left\{x\frac{(k-1)k}{2} + x_m\left(k^2 - \frac{k(k+1)}{2}\right)\right\} \\
&= \frac{k}{2}x + \frac{k}{2}x_m = \frac{k(x+x_m)}{2},
\end{aligned}$$

$$\left\|\frac{x+x_m}{2}\right\| = \left\|\frac{1}{k}\sum_{i=1}^{k} y_i\right\| \leq 1 - \delta\left(\frac{\epsilon}{3(k-1)}\right) = 1 - \delta.$$

This completes our proof. □

By Proposition 4 and 5, k-uniformly convexity implies property (β) and B-convexity.

We can get a simple example which is B-convex and does not have property (β), by renorming $(l_2, \|\cdot\|_2)$.

EXAMPLE 6. Define a norm $\|\cdot\|$ in l_2 by

$$\left\|\sum_{n=1}^{\infty} a_n e_n\right\| = \max\left\{|a_1|, \left\|\sum_{n=2}^{\infty} a_n e_n\right\|_2\right\},$$

where (e_n) is a usual unit vector basis of l_2. Then $(l_2, \|\cdot\|)$ and $(l_2, \|\cdot\|_2)$ are isomorphic, since $\|\sum_{n=1}^{\infty} a_n e_n\| \leq \|\sum_{n=1}^{\infty} a_n e_n\|_2 \leq 2\|\sum_{n=1}^{\infty} a_n e_n\|$. Since $(l_2, \|\cdot\|_2)$ is uniformly convex, $(l_2, \|\cdot\|)$ is superreflexive. Since superreflexivity implies B-convexity and reflexivity, $(l_2, \|\cdot\|)$ is reflexive and B-convex.

We show that $(l_2, \|\cdot\|)$ has no property (β). It suffices to show that X is not (KK), since (UC) $\Rightarrow$ property (β) $\Rightarrow$ (NUC) $\Rightarrow$ (UKK) $\Rightarrow$ (KK).

Consider $x_n = e_1 + e_{n+1}$. Then $x_n \to e_1$ weakly (Indeed, for $x^* = \sum_{n=1}^{\infty} \alpha_n e_n \in (l_2, \|\cdot\|)^* = (l_2, \|\cdot\|)$, $x^*(x_n) = a_1 + a_{n+1} \to x^*(e_1) = a_1$) but $x_n \nrightarrow e_1$ in norm, since $\|x_n - e_1\| = \|e_{n+1}\| = 1$. This implies that $(l_2, \|\cdot\|)$ is not (KK).

It is well known fact that B-convexity is isomorphic invariant. We can see that property (β) is not isomorphic invariant by Example 6. Thus it is reasonable to consider the following definition.

DEFINITION 7. A Banach space $(X, \|\cdot\|)$ is called a (β)-space if there is a norm $\|\cdot\|_1$ equivalent to $\|\cdot\|$ such that $(X, \|\cdot\|_1)$ satisfies property (β).

Every superreflexive space is a (β)-space [8] but the converse implication does not hold [6]. It is well known fact that superreflexive space is B-convex and reflexive. Since (β)-space is reflexive, it is an apparent question whether (β)-space is B-convex.

THEOREM 8. *There exists a (β)-space which cannot be equivalently renormed to be B-convex.*

Proof. Since B-convexity is norm isomorphic invariant, it is enough to find non-B-convex space which satisfies (β)-property. Let

$$Z = \left\{ (x_i) \in \prod_{i=1}^{\infty} \mathbb{R}^i : \sum_{i=1}^{\infty} \|x_i\|_\infty e_i \in l_2,\ x_i \in \mathbb{R}^i \right\}$$

equipped with the norm $\|(x_i)\| = \|\sum_{i=1}^{\infty} \|x_i\|_\infty e_i\|_2$ where (e_n) is usual unit vector basis of l_2. Then Z has property (β) by Lemma 1. We prove that Z is not B-convex. It suffices to show that for all $n \in \mathbb{N}$ there exist $x^{(1)},\ x^{(2)}, \cdots, x^{(n)} \in Z$ such that $\|x^{(k)}\| = 1,\ k = 1, 2, \cdots, n$ and $\left\|\sum_{k=1}^{n} \epsilon_k x^{(k)}\right\| = n$, for all $\epsilon_k = \pm 1$.

Define $x^{(k)} = (x_i^{(k)}) \in Z$ with

$$x_i^{(k)} = \begin{cases} \sum_{j=1}^{2^{n-1}} \frac{1}{\sqrt{n}} \epsilon_j^{(k)} e_j^{(i)}, & 2^{n-1} \le i \le 2^{n-1} + n - 1 \\ 0 \in \mathbb{R}^i, & \text{elsewhere}, \end{cases}$$

if $\epsilon_j^{(1)} = 1,\ 1 \le j \le 2^{n-1}$, for $k \ge 2$

$$\epsilon_j^{(k)} = \begin{cases} 1, & (2l-2)2^{n-k} < j \le (2l-1)2^{n-k} \\ -1, & (2l-1)2^{n-k} < j \le 2l \cdot 2^{n-k},\ 1 \le l \le 2^{k-2} \end{cases}$$

and $(e_j^{(i)})_{j=1}^{j=i}$ is usual unit vector basis of $\mathbb{R}^i$.

Then for $k = 1, 2, \cdots, n$,

$$\|x^{(k)}\| = \|(x_i^{(k)})\| = \left(\sum_{i=1}^{\infty} \|x_i^{(k)}\|_\infty^2 \right)^{\frac{1}{2}} = \left(\sum_{i=2^{n-1}}^{2^{n-1}+n-1} \left(\frac{1}{\sqrt{n}} \right)^2 \right)^{\frac{1}{2}} = 1$$

and for $\epsilon_k = \pm 1,\ k = 1, 2, \cdots, n$,

$$\begin{aligned} \|\epsilon_1 x^{(1)} + \cdots + \epsilon_n x^{(n)}\| &= \left\| \sum_{i=1}^{\infty} \|\epsilon_1 x_i^{(1)} + \cdots + \epsilon_n x_i^{(n)}\|_\infty e_i \right\|_2 \\ &= \left\| \sum_{i=2^{n-1}}^{2^{n-1}+n-1} \|\epsilon_1 x_i^{(1)} + \cdots + \epsilon_n x_i^{(n)}\|_\infty e_i \right\|_2 \\ &= \left\| \sum_{i=2^{n-1}}^{2^{n-1}+n-1} \frac{n}{\sqrt{n}} e_i \right\|_2 = n. \end{aligned}$$

This implies that Z is not (n, δ)-convex for all $n \in \mathbb{N},\ \delta > 0$. This completes our proof. $\square$

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