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WEIGHTED LEBESGUE NORM INEQUALITIES FOR CERTAIN CLASSES OF OPERATORS

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Abstract. We describe the weight functions for which Hardy's inequality of nonincreasing functions is satisfied. Further we characterize the pairs of weight functions (w, v) for which the Laplace transform $\mathcal{L}f(x) = \int_0^\infty e^{-xy} f(y)dy$, with monotone function f, is bounded from the weighted Lebesgue space $L^p(w)$ to $L^q(v)$.

1. Introduction

H.Heinig and V.Stepanov [3] characterized the weight functions w and v for which the inequality

$$\Big(\int_0^\infty f^q(x)w(x)dx\Big)^{1/q} \le C\,\Big(\int_0^\infty f^p(x)v(x)dx\Big)^{1/p}$$

holds for all nonnegative and nondecreasing functions f. And V.Stepanov [7] solved this problem for all nonnegative and nonincreasing functions.

V.Stepanov [6] gave a criterion for the inequality

$$\Big(\int_0^\infty \big(\frac{1}{x}\int_0^x f(y)dy\big)^q w(x)dx\Big)^{1/q} \le C\Big(\int_0^\infty \big(\frac{1}{x}\int_0^x f(y)dy\big)^p v(x)dx\Big)^{1/p}$$

to hold for all nonnegative and nonincreasing functions f in the index range $0 < p \le q < \infty,\ q \ge 1$.

J.Bradley [1] and B.Muckenhoupt [4] described the weight functions w and v for which the Hardy's inequality of the form

$$\Big(\int_0^\infty \big(\int_0^x f(y)dy\big)^q\, w(x)dx\Big)^{1/q} \le C\Big(\int_0^\infty f^p(x)v(x)dx\Big)^{1/p}$$

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holds.

In this paper we give an extensive presentation of results about the weighted norm inequalities for certain classes of operators on monotone functions.

We first establish usable necessary and sufficient conditions for the inequality of the form

$$\Big(\int_0^\infty [Kf(x)]^q\, w(x)dx\Big)^{1/q} \le C\Big(\int_0^\infty [Kf(x)]^p\, v(x)dx\Big)^{1/p},$$

where K is an integral operator with a nonnegative measurable kernel $k(x,y)$ defined by $Kf(x) = \int_0^\infty k(x,y)f(y)dy$,to hold for all nonnegative and monotone functions f in the index range $0 < p \le 1 \le q$.

Next we provide an alternative description of the weight functions for which the Hardy operator $Sf(x) = \int_0^x f(y)dy$, with nonnegative and nonincreasing function f, is bounded between weighted Lebesgue spaces when $0 < p \le 1$, $p \le q$. And then we consider the problem for the reversed Hardy's inequality.

Finally we pass to the discussion of the boundedness of the Laplace transform between weighted Lebesgue spaces. By using proof due to E. Myasnikov, L. Persson and V.Stepanov [5] we find the best constant in the inequality

$$\Big(\int_0^\infty \Big(\int_0^\infty e^{-xy} f(y)dy\Big)^q\, w(x)dx\Big)^{1/q} \le C\, \Big(\int_0^\infty f^p(x)v(x)dx\Big)^{1/p}$$

for all nonnegative and monotone functions f when $0 < p \le q < \infty$, $0 < p \le 1$. In addition we deal with the related problem of the converse inequality provided that $1 \le p \le q < \infty$.

2. Definitions and Notations

In this section we present some of the definitions and notation to be used. Throughout the paper all functions are assumed measurable and nonnegative.

We shall use the notation $f\downarrow$ (respectively $f\uparrow$) to indicate that f is nonincreasing (respectively nondecreasing).

We say that a nonnegative function is a weight if it is locally integrable. If w is a weight, the weighted Lebesgue space $L^p(w)$, $0 < p <$

∞, consists of those measurable functions f for which the quantity $\|f\|_{p,w} = (\int |f(x)|^p\, w(x)dx)^{1/p}$ is finite.

For $f\downarrow$ we define $f^{-1}(t) = \inf\{s : f(s) \le t\}$ and similarly for $f\uparrow$ we define $f^{-1}(t) = \inf\{s : f(s) > t\}$.

The integral operator K is defined by $Kf(x) = \int_0^\infty k(x,y)f(y)dy$, $x > 0$, where $k(x,y)$ is a nonnegative measurable kernel.

The Hardy operator S is defined by $Sf(x) = \int_0^x f(y)dy$.

The Laplace transform of f is defined by $\mathcal{L}f(x) = \int_0^\infty e^{-xy}f(y)dy$.

3. Results

The following two lemmas are needed in proving the theorems given below.

Lemma 1. *Let $-\infty \le a < b \le \infty$ and $f \ge 0$ on (a,b) and let g be continuous on (a,b).*

(i) *Suppose $f\uparrow$ on (a,b) and $g\downarrow$ on (a,b) with $\lim_{x\to b^-} g(x) = 0$. Then for $0 < r \le 1$, we have*

$$\int_a^b f(x)\, d\,[-g(x)] \le (\int_a^b f^r(x)\, d\,[-g^r(x)])^{1/r}.$$

If $1 \le r < \infty$ then the above inequality is reversed.

(ii) *Suppose $f\downarrow$ on (a,b) and $g\uparrow$ on (a,b) with $\lim_{x\to a^+} g(x) = 0$. Then for $0 < r \le 1$, we have*

$$\int_a^b f(x)\, d\, g(x) \le (\int_a^b f^r(x)\, d\,[g^r(x)])^{1/r}.$$

If $1 \le r < \infty$ then the above inequality is reversed.

Proof. (i) Note that

$$f(x)\, d\,[-g(x)] = \frac{1}{r}(f^r(x)g^r(x))^{\frac{1}{r}-1} f^r(x)\, d\,[-g^r(x)].$$

The hypothesis assures us that

$$0 \le f^r(x)\, g^r(x) \le \int_x^b f^r(t)\, d\,[-g^r(t)].$$

Hence if $0 < r \leq 1$ (respectively $1 \leq r < \infty$) then

$$f(x)\,d\,[-g(x)] \leq \frac{1}{r}(\int_x^b f^r(t)\,d[-g^r(t)])^{\frac{1}{r}-1}\,f^r(x)\,d\,[-g^r(x)]$$
$$= -\frac{d}{dx}(\int_x^b f^r(t)\,d\,[-g^r(t)])^{\frac{1}{r}}$$

(respectively $\geq$), and so integrating from a to b yields

$$\int_a^b f(x)\,d\,[-g(x)] \leq (\int_a^b f^r(x)\,d\,[-g^r(x)])^{1/r}$$

(respectively $\geq$).

(ii) Notice that

$$f(x)dg(x) = \frac{1}{r}(f^r(x)g^r(x))^{\frac{1}{r}-1}f^r(x)\,d[g^r(x)].$$

The hypothesis tells us that $0 \leq f^r(x)g^r(x) \leq \int_a^x f^r(t)d[g^r(t)]$. Hence if $0 < r \leq 1$ (respectively $1 \leq r < \infty$) then

$$f(x)\,dg(x) \leq \frac{1}{r}(\int_a^x f^r(t)\,d[g^r(t)]\,)^{\frac{1}{r}-1}f^r(x)\,d\,[g^r(x)]$$
$$= \frac{d}{dx}(\int_a^x f^r(t)d[g^r(t)]\,)^{\frac{1}{r}}$$

(respectively $\geq$), and so integrating from a to b gives

$$\int_a^b f(x)dg(x) \leq (\int_a^b f^r(x)\,d\,[g^r(x)]\,)^{1/r}$$

(respectively $\geq$). □

LEMMA 2. *Let $u \geq 0$ be locally integrable on $(0,\infty)$ and $0 < r < \infty$.*

(i) *If $0 \leq f \uparrow$ on $[0,a)$ and $0 < a \leq \infty$ then*

$$\int_0^a f^r(x)u(x)dx = r\int_0^{f(a)} y^{r-1}(\int_{f^{-1}(y)}^a u(x)dx)\,dy.$$

(ii) *If* $0 \le f \downarrow$ *on* $(0,\infty)$ *then*

$$\int_0^\infty f^r(x)u(x)dx = r\int_0^\infty y^{r-1}(\int_0^{f^{-1}(y)} u(x)dx)dy.$$

(iii) *If* $0 \le \varphi \downarrow$ *on* $(0,\infty)$ *then*

$$\int_0^\infty (\int_{\varphi(y)}^\infty u(x)dx)\varphi^r(y)dy = \int_0^\infty (\int_0^x \varphi^{-1}(s)d(s^r) - x^r\varphi^{-1}(x))u(x)dx.$$

Proof. (i) Change the order of integration and use the fact that $f^{-1}(t) = \inf\{s : f(s) > t\}$ to see that

$$\int_0^a f^r(x)u(x)dx = \int_0^a (r\int_0^{f(x)} y^{r-1}dy)\, u(x)dx$$
$$= r\int_0^{f(a)} y^{r-1}(\int_{f^{-1}(y)}^a u(x)dx)\, dy.$$

(ii) Changing the order of integration and using the fact that $f^{-1}(t) = \inf\{s : f(s) \le t\}$, we have

$$\int_0^\infty f^r(x)u(x)dx = \int_0^\infty (\, r\int_0^{f(x)} y^{r-1}dy)u(x)dx$$
$$= r\int_0^\infty y^{r-1}(\int_0^{f^{-1}(y)} u(x)dx)dy.$$

(iii) Changing the order of integration we obtain

$$\int_0^\infty (\int_{\varphi(y)}^\infty u(x)dx)\varphi^r(y)dy = \int_0^\infty (\int_{\varphi^{-1}(x)}^\infty \varphi^r(y)dy)u(x)dx.$$

Under the change of variable $t^{\frac{1}{r}} = z$ and integration by parts, we produce

$$\int_0^{x^r} \varphi^{-1}(t^{\frac{1}{r}})dt = \int_0^x \varphi^{-1}(z)rz^{r-1}dz = [\varphi^{-1}(z)z^r]_0^x - \int_0^x z^r d[\varphi^{-1}(z)]$$
$$= \varphi^{-1}(x)x^r + \int_x^o z^r d[\varphi^{-1}(z)]$$
$$= \varphi^{-1}(x)x^r + \int_{\varphi^{-1}(x)}^\infty \varphi^r(y)dy.$$

On the other hand

$$\int_0^{x^r} \varphi^{-1}(t^{\frac{1}{r}})dt = \int_0^x \varphi^{-1}(s)d(s^r).$$

Therefore we have

$$\int_{\varphi^{-1}(x)}^{\infty} \varphi^r(y)dy = \int_0^x \varphi^{-1}(s)d(s^r) - \varphi^{-1}(x)x^r.$$

This ensures the desired equality. □

In the following we consider the characterization problem for the weighted inequality $\|Kf\|_{q,w} \leq C\,\|Kf\|_{p,v}$, on monotone functions.

Theorem 1. *Let K be an integral operator with a nonnegative kernel $k(x,y)$ and $0 < p \leq 1 \leq q$. Then*

$$(1) \qquad \Big(\int_0^\infty [Kf(x)]^q w(x)dx\Big)^{1/q} \leq C\Big(\int_0^\infty [Kf(x)]^p v(x)dx\Big)^{1/p}$$

holds for all $0 \leq f \uparrow$ on $[0,\infty)$ if and only if

(2)

$$\Big(\int_0^\infty [\int_\alpha^\infty k(x,y)dy]^q w(x)dx\Big)^{1/q} \leq C\Big(\int_0^\infty [\int_\alpha^\infty k(x,y)dy]^p v(x)dx\Big)^{1/p}$$

holds for all $\alpha > 0$.

Proof. The necessary part follows by substituting $f(y) = \chi_{[\alpha,\infty)}(y)$, $\alpha > 0$, in the inequality (1).

To prove sufficiency, we use lemma 2(i), Minkowski's inequality and hypothesis (2) to deduce that

$$\begin{aligned}
&\Big(\int_0^\infty [\int_0^\infty k(x,y)\,f(y)dy\,]^q\,w(x)\,dx\Big)^{1/q} \\
&\quad = \Big(\int_0^\infty [\int_0^\infty (\int_{f^{-1}(t)}^\infty k(x,y)dy\,)dt]^q\,w(x)dx\Big)^{1/q} \\
&\quad = \Big(\int_0^\infty [\int_0^\infty (\int_{f^{-1}(t)}^\infty k(x,y)dy)w^{1/q}(x)\,dt]^q dx\Big)^{1/q} \\
&\quad \leq \int_0^\infty [\int_0^\infty (\int_{f^{-1}(t)}^\infty k(x,y)dy)^q w(x)dx]^{1/q}\,dt \\
&\quad \leq C\int_0^\infty [\int_0^\infty (\int_{f^{-1}(t)}^\infty k(x,y)dy)^p\,v(x)dx\,]^{1/p}dt = CI.
\end{aligned}$$

Again applying Minkowski's inequality and lemma 2(i), we obtain

$$\begin{aligned} I^p &= \Big(\int_0^\infty \Big[\int_0^\infty \Big(\int_{f^{-1}(t)}^\infty k(x,y)dy\Big)^p v(x)dx\Big]^{1/p} dt\Big)^p \\ &\le \int_0^\infty \Big[\int_0^\infty \Big(\int_{f^{-1}(t)}^\infty k(x,y)dy\Big) v^{1/p}(x)\, dt\Big]^p dx \\ &= \int_0^\infty \Big[\int_0^\infty \Big(\int_{f^{-1}(t)}^\infty k(x,y)dy\Big)dt\Big]^p v(x)dx \\ &= \int_0^\infty \Big[\int_0^\infty k(x,y)f(y)dy\Big]^p v(x)dx. \end{aligned}$$

Consequently we have the desired inequality (1). □

THEOREM 2. *Let K be an integral operator with a nonnegative kernel $k(x,y)$ and $0 < p \le 1 \le q$. Then*

$$\Big(\int_0^\infty [Kf(x)]^q w(x)dx\Big)^{1/q} \le C\Big(\int_0^\infty [Kf(x)]^p\, v(x)dx\Big)^{1/p} \tag{3}$$

holds for all $0 \le f \downarrow$ on $[0,\infty)$ if and only if

$$\Big(\int_0^\infty \Big[\int_0^\alpha k(x,y)dy\Big]^q w(x)dx\Big)^{1/q} \le C\Big(\int_0^\infty \Big[\int_0^\alpha k(x,y)dy\Big]^p v(x)dx\Big)^{1/p} \tag{4}$$

holds for all $\alpha > 0$.

Proof. The necessity is proved by substituting $f(y) = \chi_{[0,\alpha]}(y)$, $\alpha > 0$, in the inequality (3).

To prove sufficiency we apply lemma 2(ii), Minkowski's inequality and hypothesis (4) to derive that

$$\Big(\int_0^\infty [\int_0^\infty k(x,y)f(y)dy]^q\, w(x)dx\Big)^{1/q}$$

$$= \Big(\int_0^\infty [\int_0^\infty (\int_0^{f^{-1}(t)} k(x,y)dy)\, dt\,]^q w(x)dx\Big)^{1/q}$$

$$= \Big(\int_0^\infty [\int_0^\infty (\int_0^{f^{-1}(t)} k(x,y)dy) w^{1/q}(x)dt]^q dx\Big)^{1/q}$$

$$\leq \int_0^\infty [\int_0^\infty (\int_0^{f^{-1}(t)} k(x,y)dy)^q w(x)dx]^{1/q} dt$$

$$\leq C \int_0^\infty [\int_0^\infty (\int_0^{f^{-1}(t)} k(x,y)dy)^p v(x)dx]^{1/p} dt = CJ.$$

Again using Minkowski's inequality and lemma 2(ii), we have

$$J^p = (\int_0^\infty [\int_0^\infty (\int_0^{f^{-1}(t)} k(x,y)dy)^p\, v(x)dx\,]^{1/p}\, dt\,)^p$$

$$\leq \int_0^\infty [\int_0^\infty (\int_0^{f^{-1}(t)} k(x,y)dy\,) v^{1/p}(x)\, dt\,]^p\, dx$$

$$= \int_0^\infty [\int_0^\infty (\int_0^{f^{-1}(t)} k(x,y)dy\,)dt\,]^p\, v(x)dx$$

$$= \int_0^\infty [\int_0^\infty k(x,y)f(y)dy\,]^p\, v(x)dx.$$

Hence we end up with the desired inequality (3). □

Corollary 1. *Let $0 < p \leq 1 \leq q$. Then*

$$(\int_0^\infty [\int_0^x f(y)dy]^q w(x)dx)^{1/q} \leq C(\int_0^\infty [\int_0^x f(y)dy]^p v(x)dx)^{1/p}$$

holds for all $0 \leq f \downarrow$ if and only if

$$\Big(\int_0^\infty (\min\{x,\alpha\})^q w(x)dx\Big)^{1/q} \leq C\Big(\int_0^\infty (\min\{x,\alpha\})^p v(x)dx\Big)^{1/p}$$

holds for every $\alpha > 0$.

Proof. The proof follows immediately from theorem 2 by taking $k(x,y) = \chi_{[0,x]}(y)$. □

Naturally the question arises : Does the above result remain valid even in the index range $1 \le p, q < \infty$? This is answered by the theorem stated below.

THEOREM 3. *Let* $1 \le p < \infty$ *Then*

$$(\int_0^\infty [\int_0^x f(y)dy]^p w(x)dx)^{1/p} \le C(\int_0^\infty [\int_0^x f(y)dy]^p v(x)dx)^{1/p} \tag{5}$$

holds for all $0 \le f \downarrow$ *if and only if*

$$\Big(\int_0^\infty (\min\{x,\alpha\})^p w(x)dx\Big)^{1/p} \le C\Big(\int_0^\infty (\min\{x,\alpha\})^p v(x)dx\Big)^{1/p} \tag{6}$$

holds for every $\alpha > 0$.

Proof. The necessity is proved by substituting $f(y) = \chi_{[0,\alpha]}(y)$,$\alpha > 0$, in the inequality (5).

To prove sufficiency, we let $\alpha = \varphi(y)$, where φ is a nonnegative nonincreasing function on $(0,\infty)$. Applying lemma 2(ii) and (iii) we get

$$\begin{aligned}
&\int_0^\infty (\int_0^\infty (\min\{x,\alpha\})^p w(x)dx)dy \\
&\qquad = \int_0^\infty (\int_0^\infty (\min\{x,\varphi(y)\})^p w(x)dx)dy \\
&\qquad = \int_0^\infty [\int_0^{\varphi(y)} x^p\, w(x)dx + \int_{\varphi(y)}^\infty \varphi^p(y)w(x)dx]\, dy \\
&= \int_0^\infty \varphi^{-1}(x)x^p w(x)dx + \int_0^\infty [\int_0^x \varphi^{-1}(s)d(s^p) - x^p\varphi^{-1}(x)]w(x)dx \\
&\qquad = \int_0^\infty [\int_0^x \varphi^{-1}(s)d(s^p)]w(x)dx.
\end{aligned}$$

Also we see that

$$\int_0^\infty (\int_0^\infty (\min\{x,\alpha\})^p v(x)dx)dy = \int_0^\infty [\int_0^x \varphi^{-1}(s)d(s^p)]v(x)dx.$$

It follows from hypothesis (6) that

$$\int_0^\infty [\int_0^x \varphi^{-1}(s)d(s^p)]w(x)dx \le C^p \int_0^\infty [\int_0^x \varphi^{-1}(s)d(s^p)]v(x)dx.$$

Now taking $\varphi^{-1}(s) = f(s)[s^{-1}\int_0^s f(y)dy]^{p-1}$, we have

$$\int_0^x \varphi^{-1}(s)d(s^p) = \int_0^x f(s)[s^{-1}\int_0^s f(y)dy]^{p-1}d(s^p)$$
$$= \int_0^x pf(s)[\int_0^s f(y)dy]^{p-1}ds = [\int_0^s f(y)dy]^p|_{s=0}^{s=x} = (\int_0^x f(y)dy)^p.$$

As a result we obtain the desired inequality (6). □

In the next theorem we characterize the boundedness of the Hardy operator between weighted Lebesgue spaces.

THEOREM 4. *Let S be the Hardy operator defined by $Sf(x) = \int_0^x f(y)dy$, $0 < p \le 1$ and $p \le q$. Then*

$$(7) \qquad (\int_0^\infty [Sf(x)]^q w(x)dx)^{1/q} \le C(\int_0^\infty f^p(x)v(x)dx)^{1/p}$$

holds for all $0 \le f \downarrow$ if and only if

$$(8) \qquad \Big(\int_0^\infty (\min\{x,\alpha\})^q w(x)dx\Big)^{1/q} \le C\Big(\int_0^\alpha v(x)dx\Big)^{1/p}$$

holds for all $\alpha > 0$.

Proof. The necessary part follows by substituting $f(y) = \chi_{[0,\alpha]}(y)$, $\alpha > 0$, in the inequality (7).

To prove sufficiency, we use lemma 2(ii) to see that

$$(\int_0^\infty [\int_0^x f(y)dy]^q w(x)dx)^{1/q}$$
$$= (\int_0^\infty [\int_0^\infty f(y)\chi_{[0,x]}(y)dy]^q w(x)dx)^{1/q}$$
$$= (\int_0^\infty [\int_0^\infty (\int_0^{f^{-1}(t)} \chi_{[0,x]}(y)dy)dt]^q w(x)dx)^{1/q} = I.$$

Assume first that $q \leq 1$. We invoke lemma 1(ii) to deduce that

$$\int_0^\infty (\int_0^{f^{-1}(t)} \chi_{[0,x]}(y)dy)dt \leq (\int_0^\infty (\int_0^{f^{-1}(t)} \chi_{[0,x]}(y)dy)^q d(t^q))^{1/q}$$
$$= (\int_0^\infty q\, t^{q-1} (\int_0^{f^{-1}(t)} \chi_{[0,x]}(y)dy)^q dt)^{1/q}.$$

It follows from hypothesis (8) that

$$I \leq \big(q \int_0^\infty [\int_0^\infty t^{q-1}(\min\{x, f^{-1}(t)\})^q dt] w(x)dx\big)^{1/q}$$
$$= \big(q \int_0^\infty [\int_0^\infty (\min\{x, f^{-1}(t)\})^q w(x)dx] t^{q-1} dt\big)^{1/q}$$
$$\leq C\big(q \int_0^\infty (\int_0^{f^{-1}(t)} v(x)dx)^{q/p}\, t^{q-1}\, dt\big)^{1/q}$$
$$= C\big(\int_0^\infty (\int_0^{f^{-1}(t)} v(x)dx)^{q/p}\, d(t^q)\big)^{1/q}.$$

It takes another appeal to lemma 1(ii) to yields that

$$\int_0^\infty (\int_0^{f^{-1}(t)} v(x)dx)^{q/p}\, d(t^q) \leq \big(\int_0^\infty (\int_0^{f^{-1}(t)} v(x)dx)\, d[t^p]\,\big)^{q/p}.$$

Then in view of lemma 2(ii) we have

$$I \leq C\big(p \int_0^\infty t^{p-1} (\int_0^{f^{-1}(t)} v(x)dx)dt\big)^{1/p} = C(\int_0^\infty f^p(x)v(x)dx)^{1/p},$$

which proves the desired inequality (7).

Now consider the case $q > 1$. Setting $g = f^q$ and using lemma 2(ii) we get

$$\int_0^x f(y)dy = \int_0^\infty g^{1/q}(y)\chi_{[0,x]}(y)dy$$
$$= \frac{1}{q}\int_0^\infty t^{\frac{1}{q}-1} (\int_0^{g^{-1}(t)} \chi_{[0,x]}(y)dy)dt.$$

We take account of Minkowski's inequality and hypothesis (8) to draw that

$$
\begin{aligned}
&\Big(\int_0^\infty \Big[\int_0^x f(y)dy\Big]^q w(x)dx\Big)^{1/q}\\
&\quad = \frac{1}{q}\Big(\int_0^\infty \Big[\int_0^\infty t^{\frac{1}{q}-1}\Big(\int_0^{g^{-1}(t)} \chi_{[0,x]}(y)dy\Big)dt\Big]^q w(x)dx\Big)^{1/q}\\
&\quad = \frac{1}{q}\Big(\int_0^\infty \Big[\int_0^\infty t^{\frac{1}{q}-1}\Big(\int_0^{g^{-1}(t)} \chi_{[0,x]}(y)dy\Big)w^{1/q}(x)dt\Big]^q dx\Big)^{1/q}\\
&\quad \le \frac{1}{q}\int_0^\infty \Big(\int_0^\infty t^{1-q}\Big(\int_0^{g^{-1}(t)} \chi_{[0,x]}(y)dy\Big)^q w(x)dx\Big)^{1/q} dt\\
&\quad = \frac{1}{q}\int_0^\infty t^{\frac{1}{q}-1}\Big(\int_0^\infty (\min\{x, g^{-1}(t)\})^q w(x)dx\Big)^{1/q} dt\\
&\quad \le C\frac{1}{q}\int_0^\infty t^{\frac{1}{q}-1}\Big(\int_0^{g^{-1}(t)} v(x)dx\Big)^{1/p} dt = CJ.
\end{aligned}
$$

By virtue of lemma 1(ii) and lemma 2(ii) we get

$$
\begin{aligned}
J &= \int_0^\infty \Big(\int_0^{g^{-1}(t)} v(x)dx\Big)^{1/p} d(t^{1/q})\\
&\le \Big(\int_0^\infty \Big(\int_0^{g^{-1}(t)} v(x)dx\Big) d[t^{p/q}]\Big)^{1/p}\\
&= \Big(\int_0^\infty \frac{p}{q}\, t^{\frac{p}{q}-1}\Big(\int_0^{g^{-1}(t)} v(x)dx\Big)dt\Big)^{1/p}\\
&= \Big(\int_0^\infty g^{p/q}(x)v(x)dx\Big)^{1/p} = \Big(\int_0^\infty f^p(x)v(x)dx\Big)^{1/p}.
\end{aligned}
$$

Therefore we obtain the required inequality (7). □

THEOREM 5. *Let $0 < p \le 1$ and $p \le q$. Then*

$$\Big(\int_0^a \Big[\int_x^a f(y)dy\Big]^q w(x)dx\Big)^{1/q} \le C\Big(\int_0^a f^p(x)v(x)dx\Big)^{1/p} \tag{9}$$

holds for all $0 \le f \uparrow$ if and only if

$$\Big(\int_0^a [\min\{a-x, a-\alpha\}]^q w(x)dx\Big)^{1/q} \le C\Big(\int_\alpha^a v(x)dx\Big)^{1/p} \tag{10}$$

holds for every $0 < \alpha < a$.

Proof. The necessity is proved by substituting $f(y) = \chi_{[\alpha,a]}(y)$, $0 < \alpha < a$, in the inequality (9).

To prove sufficiency, we invoke lemma 2(i) to infer that

$$\Big(\int_0^a \Big[\int_x^a f(y)dy\Big]^q w(x)dx\Big)^{1/q}$$
$$= \Big(\int_0^a \Big[\int_0^{f(a)} \Big(\int_{f^{-1}(t)}^a \chi_{[x,a]}(y)dy\Big)dt\Big]^q w(x)dx\Big)^{1/q} = I.$$

Assume first that $q \le 1$. An appeal to lemma 1(ii) yields that

$$\int_0^{f(a)} \Big(\int_{f^{-1}(t)}^a \chi_{[x,a]}(y)dy\Big)dt \le \Big(\int_0^{f(a)} \Big(\int_{f^{-1}(t)}^a \chi_{[x,a]}(y)dy\Big)^q d[t^q]\Big)^{1/q}$$
$$= \Big(\int_0^{f(a)} q\, t^{q-1}\Big(\int_{f^{-1}(t)}^a \chi_{[x,a]}(y)dy\Big)^q dt\Big)^{1/q}.$$

Using hypothesis (10), lemma 1(ii) and lemma 2(i) we draw

$$I \le \Big(q\int_0^a \Big[\int_0^{f(a)} t^{q-1}\Big(\int_{f^{-1}(t)}^a \chi_{[x,a]}(y)dy\Big)^q dt\Big] w(x)dx\Big)^{1/q}$$
$$= \Big(q\int_0^{f(a)} t^{q-1}\Big(\int_0^a [\min\{a-x, a-f^{-1}(t)\}]^q w(x)dx\Big)dt\Big)^{1/q}$$
$$\le \Big(q\int_0^{f(a)} t^{q-1}C^q\Big(\int_{f^{-1}(t)}^a v(x)dx\Big)^{q/p}dt\Big)^{1/q}$$
$$= C\Big(\int_0^{f(a)} \Big(\int_{f^{-1}(t)}^a v(x)dx\Big)^{q/p} d[t^q]\Big)^{1/q}$$
$$\le C\Big(\int_0^{f(a)} \Big(\int_{f^{-1}(t)}^a v(x)dx\Big) d[t^p]\Big)^{1/p}$$
$$= C\Big(p\int_0^{f(a)} t^{p-1}\Big(\int_{f^{-1}(t)}^a v(x)dx\Big)dt\Big)^{1/p} = C\int_0^a f^p(x)v(x)dx)^{1/p}.$$

This gives us the desired inequality (9).

Next we deal with the case $q > 1$. Putting $g = f^q$ and using lemma 2(i) we obtain

$$\int_x^a f(y)dy = \int_0^a g^{1/q}(y)\chi_{[x,a]}(y)dy$$
$$= \frac{1}{q}\int_0^{g(a)} t^{\frac{1}{q}-1}(\int_{g^{-1}(t)}^a \chi_{[x,a]}(y)dy)dt.$$

By virtue of Minkowski's inequality and hypothesis (10), we get

$$(\int_0^a [\int_x^a f(y)dy]^q w(x)dx)^{1/q}$$
$$= \frac{1}{q}\Big(\int_0^a [\int_0^{g(a)} t^{\frac{1}{q}-1}(\int_{g^{-1}(t)}^a \chi_{[x,a]}(y)dy)dt]^q w(x)dx\Big)^{1/q}$$
$$= \frac{1}{q}\Big(\int_0^a [\int_0^{g(a)} t^{\frac{1}{q}-1}(\int_{g^{-1}(t)}^a \chi_{[x,a]}(y)dy)w^{1/q}(x)dt]^q dx\Big)^{1/q}$$
$$\leq \frac{1}{q}\int_0^{g(a)} (\int_0^a t^{1-q}(\int_{g^{-1}(t)}^a \chi_{[x,a]}(y)dy)^q w(x)dx)^{1/q}dt$$
$$= \frac{1}{q}\int_0^{g(a)} t^{\frac{1}{q}-1}[\int_0^a (\min\{a-x, a-g^{-1}(t)\})^q w(x)dx]^{1/q}dt$$
$$\leq C\frac{1}{q}\int_0^{g(a)} t^{\frac{1}{q}-1}(\int_{g^{-1}(t)}^a v(x)dx)^{1/p}dt = CJ.$$

Lemma 1(ii) and lemma 2(i) step in to ensure that

$$J = \int_0^{g(a)} (\int_{g^{-1}(t)}^a v(x)dx)^{1/p}\, d[t^{1/q}]$$
$$\leq \Big(\int_0^{g(a)} (\int_{g^{-1}(t)}^a v(x)dx)\, d[t^{p/q}]\Big)^{1/p}$$
$$= \Big(\frac{p}{q}\int_0^{g(a)} t^{\frac{p}{q}-1}(\int_{g^{-1}(t)}^a v(x)dx)\, dt\Big)^{1/p}$$
$$= (\int_0^a g^{\frac{p}{q}}(x)v(x)dx)^{1/p} = (\int_0^a f^p(x)v(x)dx)^{1/p}.$$

Hence we obtain the desired inequality (9). □

Next we treat the related problem of the converse Hardy's inequality.

THEOREM 6. *Let S be the Hardy operator defined by $Sf(x) = \int_0^x f(y)dy$,and $1 \leq p \leq q$. Then*

$$(11) \qquad (\int_0^\infty f^q(x)w(x)dx)^{1/q} \leq C(\int_0^\infty [Sf(x)]^p v(x)dx)^{1/p}$$

holds for all $0 \leq f \downarrow$ if and only if

$$(12) \qquad \Big(\int_0^\alpha w(x)dx\Big)^{1/q} \leq C\Big(\int_0^\infty [\min\{x,\alpha\}]^p v(x)dx\Big)^{1/p}$$

holds for all $\alpha > 0$.

Proof. The necessity is proved by substituting $f(x) = \chi_{[0,\alpha]}(x)$, $\alpha > 0$, in the inequality (11).

To prove sufficiency we let $g = f^q$. We invoke lemma 2(ii), lemma 1(ii) and hypothesis (12) to infer that

$$\int_0^\infty f^q(x)w(x)dx = \int_0^\infty g(x)w(x)dx$$
$$= \int_0^\infty (\int_0^{g^{-1}(t)} w(x)dx)dt \leq \Big(\int_0^\infty (\int_0^{g^{-1}(t)} w(x)dx)^{\frac{p}{q}}\, d[t^{p/q}]\Big)^{q/p}$$
$$\leq C^q \Big(\int_0^\infty (\int_0^\infty [\min\{x, g^{-1}(t)\}]^p v(x)dx)\, d[t^{p/q}]\Big)^{q/p}$$
$$= C^q \Big(\int_0^\infty \int_0^\infty (\int_0^{g^{-1}(t)} \chi_{[0,x]}(y)dy\,)^p\, d[t^{p/q}]\, v(x)dx\Big)^{q/p}.$$

By virtue of lemma 1(ii) and lemma 2(ii), we have

$$\int_0^\infty (\int_0^{g^{-1}(t)} \chi_{[0,x]}(y)dy\,)^p\, d[t^{p/q}]$$
$$\leq \Big(\int_0^\infty (\int_0^{g^{-1}(t)} \chi_{[0,x]}(y)dy)\, d[t^{1/q}]\Big)^p$$
$$= \Big(\frac{1}{q}\int_0^\infty t^{\frac{1}{q}-1} (\int_0^{g^{-1}(t)} \chi_{[0,x]}(y)dy\,)dt\Big)^p$$
$$= (\int_0^\infty g^{1/q}(y)\chi_{[0,x]}(y)dy\,)^p = (\int_0^x f(y)dy)^p.$$

Combining the above inequalities we get

$$\left(\int_0^\infty f^q(x)w(x)dx\right)^{p/q} \le C^p \int_0^\infty \left(\int_0^x f(y)dy\right)^p v(x)dx$$

and so the inequality (11) holds. □

THEOREM 7. *Let $1 \le p \le q$. Then*

$$\left(\int_0^a f^q(x)w(x)dx\right)^{1/q} \le C\left(\int_0^a \left[\int_x^a f(y)dy\right]^p v(x)dx\right)^{1/p} \tag{13}$$

holds for all $0 \le f \uparrow$ if and only if

$$\left(\int_\alpha^a w(x)dx\right)^{1/q} \le C\left(\int_0^a [\min\{a-x, a-\alpha\}]^p v(x)dx\right)^{1/p} \tag{14}$$

holds for every $0 < \alpha < a$.

Proof. The necessary part follows by substituting $f(x) = \chi_{[\alpha,a]}(x)$, $0 < \alpha < a$, in the inequality (13).

To prove sufficiency we put $g = f^q$. We apply lemma 2(i), lemma 1(ii) and hypothesis (14) to deduce that

$$\begin{aligned}
&\int_0^a f^q(x)w(x)dx = \int_0^a g(x)w(x)dx \\
&= \int_0^{g(a)} \left(\int_{g^{-1}(t)}^a w(x)dx\right)dt \le \Big(\int_0^{g(a)} \left(\int_{g^{-1}(t)}^a w(x)dx\right)^{\frac{p}{q}} d[t^{p/q}]\Big)^{q/p} \\
&\quad \le C^q\Big(\int_0^{g(a)} \left(\int_0^a [\min\{a-x, a-g^{-1}(t)\}]^p v(x)dx\right) d[t^{p/q}]\Big)^{q/p} \\
&\qquad = C^q\Big(\int_0^a \int_0^{g(a)} \left(\int_{g^{-1}(t)}^a \chi_{[x,a]}(y)dy\right)^p d[t^{p/q}] v(x)dx\Big)^{q/p}.
\end{aligned}$$

We make use of lemma 1(ii) and lemma 2(i) to obtain that

$$\int_0^{g(a)} \Big(\int_{g^{-1}(t)}^a \chi_{[x,a]}(y)dy\Big)^p d[t^{p/q}]$$
$$\leq \Big(\int_0^{g(a)} \Big(\int_{g^{-1}(t)}^a \chi_{[x,a]}(y)dy\Big) d[t^{1/q}]\Big)^p$$
$$\Big(\frac{1}{q}\int_0^{g(a)} t^{\frac{1}{q}-1}\Big(\int_{g^{-1}(t)}^a \chi_{[x,a]}(y)dy\Big)dt\Big)^p$$
$$= \Big(\int_0^a g^{1/q}(y)\chi_{[x,a]}(y)dy\Big)^p = \Big(\int_x^a f(y)dy\Big)^p.$$

Combining the above inequalities we have

$$\Big(\int_0^a f^q(x)w(x)dx\Big)^{p/q} \leq C^p \int_0^a \Big(\int_x^a f(y)dy\Big)^p v(x)dx$$

and hence the inequality (13) holds. □

In the theorem stated below we find usable necessary and sufficient conditions for the boundedness of the Laplace transform between weighted Lebesgue spaces.

THEOREM 8. *Let $\mathcal{L}$ be the Laplace transform defined by $\mathcal{L}f(x) = \int_0^\infty e^{-xy}f(y)dy$, $0 < p \leq q < \infty$ and $0 < p \leq 1$. Then*

(i) *The inequality*

$$\Big(\int_0^\infty [\mathcal{L}f(x)]^q w(x)dx\Big)^{1/q} \leq C\Big(\int_0^\infty f^p(x)v(x)dx\Big)^{1/p} \tag{15}$$

holds for all $0 \leq f \uparrow$ if and only if

$$D = \sup_{t>0}\Big(\int_0^\infty e^{-qxt}x^{-q}w(x)dx\Big)^{1/q}\Big(\int_t^\infty v(x)dx\Big)^{-1/p} < \infty. \tag{16}$$

(ii) *The inequality*

$$\Big(\int_0^\infty [\mathcal{L}f(x)]^q w(x)dx\Big)^{1/q} \leq C\Big(\int_0^\infty f^p(x)v(x)dx\Big)^{1/p} \tag{17}$$

holds for all $0 \leq f \downarrow$ if and only if

$$E = \sup_{t>0}\Big(\int_0^\infty (1-e^{-xt})^q x^{-q}w(x)dx\Big)^{1/q}\Big(\int_0^t v(x)dx\Big)^{-1/p} < \infty. \tag{18}$$

Proof. (i) The necessity is proved by substituting $f(x) = \chi_{[t,\infty)}(x)$, $t > 0$, in the inequality (15).

To prove sufficiency we take a nondecreasing function $f(x)$ in the form $f(x) = \int_0^x h$, where $h \geq 0$, $\operatorname{supp} h \subset (0, \infty)$ and $(\int_0^\infty f^p(x)v(x)dx)^{1/p} < \infty$. Using the identity

$$\int_0^y h(s)ds = \Big(\int_0^y p\,\Big(\int_0^t h(s)ds)^{p-1}h(t)dt\Big)^{1/p},$$

we get

$$\begin{aligned}\mathcal{L}f(x) &= \int_0^\infty e^{-xy}(\int_0^y h(s)ds)dy \\ &= p^{1/p}\int_0^\infty e^{-xy}\Big(\int_0^y (\int_0^t h(s)ds)^{p-1}h(t)dt\Big)^{1/p}dy = p^{1/p}\,I.\end{aligned}$$

An appeal to Minkowski's inequality ensures that

$$\begin{aligned}I^p &= \Big(\int_0^\infty [\int_0^y (e^{-xy})^p(\int_0^t h(s)ds)^{p-1}h(t)dt]^{1/p}dy\Big)^p \\ &\leq \int_0^\infty [\int_t^\infty e^{-xy}(\int_0^t h(s)ds)^{\frac{p-1}{p}}h^{1/p}(t)dy]^p dt \\ &= \int_0^\infty (\int_0^t h(s)ds)^{p-1}(\int_t^\infty e^{-xy}dy)^p h(t)dt.\end{aligned}$$

Hence

$$\begin{aligned}&\Big(\int_0^\infty [\mathcal{L}f(x)]^q w(x)dx\Big)^{\frac{1}{q}} \\ &\leq p^{\frac{1}{p}}\Big(\int_0^\infty [\int_0^\infty (\int_0^t h(s)ds)^{p-1}(\int_t^\infty e^{-xy}dy)^p w^{\frac{p}{q}}(x)h(t)dt]^{\frac{q}{p}}dx\Big)^{\frac{1}{q}} \\ &= p^{\frac{1}{p}}J.\end{aligned}$$

It takes another appeal to Minkowski's inequality to reveal that

$$\begin{aligned}J^p &= \Big(\int_0^\infty [\int_0^\infty (\int_0^t h(s)ds)^{p-1}(\int_t^\infty e^{-xy}dy)^p w^{\frac{p}{q}}(x)h(t)dt]^{\frac{q}{p}}dx\Big)^{\frac{p}{q}} \\ &\leq \int_0^\infty [\int_0^\infty (\int_0^t h(s)ds)^{\frac{(p-1)q}{p}}(\int_t^\infty e^{-xy}dy)^q w(x)h^{q/p}(t)dx]^{p/q}dt \\ &= \int_0^\infty (\int_0^t h(s)ds)^{p-1}[\int_0^\infty e^{-qxt}x^{-q}w(x)dx]^{p/q}h(t)dt.\end{aligned}$$

Combining the above inequalities with the hypothesis (16) we arrive at

$$\begin{aligned}(\int_0^\infty &[\mathcal{L}f(x)]^q w(x)dx)^{1/q}\\ &\le D\,p^{1/p}\big(\int_0^\infty (\int_0^t h(s)ds)^{p-1}(\int_t^\infty v(x)dx)h(t)dt\big)^{1/p}\\ &= D\big(\int_0^\infty \int_0^x p\,(\int_0^t h(s)ds)^{p-1}h(t)dtv(x)dx\big)^{1/p}\\ &= D\big(\int_0^\infty (\int_0^x h(s)ds)^p v(x)dx\big)^{1/p} = D(\int_0^\infty f^p(x)v(x)dx)^{1/p}.\end{aligned}$$

(ii) The necessary part follows by substituting $f(x) = \chi_{[0,t]}(x)$,$t > 0$, in the inequality (17).

To prove sufficiency we take a nondecreasing function $f(x)$ in the form $f(x) = \int_x^\infty h$, where $h \ge 0$, $\operatorname{supp} h \subset (0,\infty)$ and

$$\left(\int_0^\infty f^p(x)v(x)dx\right)^{1/p} < \infty.$$

The identity

$$\int_y^\infty h(s)ds = (\int_y^\infty p\,(\int_t^\infty h(s)ds)^{p-1}h(t)dt\big)^{1/p}$$

leads us to have that

$$\begin{aligned}\mathcal{L}f(x) &= \int_0^\infty e^{-xy}(\int_y^\infty h(s)ds)dy\\ &= p^{1/p}\int_0^\infty e^{-xy}\big(\int_y^\infty (\int_t^\infty h(s)ds)^{p-1}h(t)dt\big)^{1/p}dy = p^{1/p}I.\end{aligned}$$

We apply Minkowski's inequality to produce

$$\begin{aligned}I^p &= \big(\int_0^\infty [\int_y^\infty (e^{-xy})^p(\int_t^\infty h(s)ds)^{p-1}h(t)dt]^{1/p}dy\big)^p\\ &\le \int_0^\infty [\int_0^t e^{-xy}(\int_t^\infty h(s)ds)^{\frac{p-1}{p}}h^{1/p}(t)dy]^p dt\\ &= \int_0^\infty (\int_t^\infty h(s)ds)^{p-1}(\int_0^t e^{-xy}dy)^p h(t)dt.\end{aligned}$$

Therefore

$$
\begin{aligned}
&(\int_0^\infty [\mathcal{L}f(x)]^q w(x)dx)^{1/q} \\
&\le p^{\frac{1}{p}}\Big(\int_0^\infty [\int_0^\infty (\int_t^\infty h(s)ds)^{p-1}(\int_0^t e^{-xy}dy)^p w^{\frac{p}{q}}(x)h(t)dt]^{\frac{q}{p}}dx\Big)^{\frac{1}{q}} \\
&= p^{\frac{1}{p}} J.
\end{aligned}
$$

By another application of Minkowski's inequality we obtain

$$
\begin{aligned}
J^p &= \Big(\int_0^\infty [\int_0^\infty (\int_t^\infty h(s)ds)^{p-1}(\int_0^t e^{-xy}dy)^p w^{\frac{p}{q}}(x)h(t)dt]^{\frac{q}{p}}dx\Big)^{\frac{p}{q}} \\
&\le \int_0^\infty [\int_0^\infty (\int_t^\infty h(s)ds)^{\frac{(p-1)q}{p}}(\int_0^t e^{-xy}dy)^q w(x)h^{q/p}(t)dx]^{p/q}dt \\
&\quad = \int_0^\infty (\int_t^\infty h(s)ds)^{p-1}[\int_0^\infty (1-e^{-xt})^q x^{-q} w(x)dx]^{p/q}h(t)dt.
\end{aligned}
$$

Combining the above inequalities with the hypothesis (18) we get

$$
\begin{aligned}
&(\int_0^\infty [\mathcal{L}f(x)]^q w(x)dx)^{1/q} \\
&\qquad \le E p^{1/p}\Big(\int_0^\infty (\int_t^\infty h(s)ds)^{p-1}(\int_0^t v(x)dx)h(t)dt\Big)^{1/p} \\
&\qquad = E\Big(\int_0^\infty \int_x^\infty p\,(\int_t^\infty h(s)ds)^{p-1}h(t)dt\, v(x)dx\Big)^{1/p} \\
&\quad = E\Big(\int_0^\infty (\int_x^\infty h(s)ds)^p v(x)dx\Big)^{1/p} = E(\int_0^\infty f^p(x)v(x)dx)^{1/p}.
\end{aligned}
$$

This completes the proof. □

Finally we solve the similar problem for the reversed inequality.

THEOREM 9. *Let $\mathcal{L}$ be the Laplace transform defined by $\mathcal{L}f(x) = \int_0^\infty e^{-xy}f(y)dy$, and $1 \le p \le q < \infty$. Then*

(i) *The inequality*

$$
(\int_0^\infty f^q(x)w(x)dx)^{1/q} \le C(\int_0^\infty [\mathcal{L}f(x)]^p v(x)dx)^{1/p} \tag{19}
$$

holds for all $0 \le f \uparrow$ *if and only if*

$$D = \sup_{t>o} \Big(\int_t^\infty w(x)dx\Big)^{1/q} \Big(\int_0^\infty e^{-pxt} x^{-p} v(x)dx\Big)^{-1/p} < \infty. \tag{20}$$

(ii) *The inequality*

$$\Big(\int_0^\infty f^q(x)w(x)dx\Big)^{1/q} \le C\Big(\int_0^\infty [\mathcal{L}f(x)]^p v(x)dx\Big)^{1/p} \tag{21}$$

holds for all $0 \le f \downarrow$ *if and only if*

$$E = \sup_{t>o} \Big(\int_0^t w(x)dx\Big)^{1/q} \Big(\int_0^\infty (1-e^{-xt})^p x^{-p} v(x)dx\Big)^{-1/p} < \infty. \tag{22}$$

Proof. (i) The necessity is proved by substituting $f(x) = \chi_{[t,\infty)}(x)$, $t > 0$, in the inequality (19).

To prove sufficiency we take a nondecreasing function $f(x)$ in the form $f(x) = \int_0^x h$, where $h \ge 0$, $\operatorname{supp} h \subset (0,\infty)$ and

$$\Big(\int_0^\infty [\mathcal{L}f(x)]^p v(x)dx\Big)^{1/p} < \infty.$$

Note that

$$\Big(\int_0^\infty f^q(s)w(s)ds\Big)^{1/q} = \Big(\int_0^\infty \Big[\int_0^s d(f^p(t))\Big]^{q/p} w(s)ds\Big)^{1/q} = I.$$

Using first Minkowski's inequality and then hypothesis (20) we get

$$\begin{aligned} I^p &= \Big(\int_0^\infty \Big[\int_0^s w^{p/q}(s)d(f^p(t))\Big]^{q/p} ds\Big)^{p/q} \\ &\le \int_0^\infty \Big(\int_t^\infty w(s)ds\Big)^{p/q} d(f^p(t)) \\ &\le D^p \int_0^\infty \Big[\int_0^\infty \Big(\int_t^\infty e^{-xy}dy\Big)^p v(x)dx\Big] d(f^p(t)) = D^p J. \end{aligned}$$

An integration by parts shows that

$$J = [f^p(t)\int_0^\infty (\int_t^\infty e^{-xy}dy)^p\, v(x)dx\,]_0^\infty$$
$$- \int_0^\infty f^p(t)\, d\,[\int_0^\infty (\int_t^\infty e^{-xy}dy)^p v(x)dx]$$
$$= p\int_0^\infty v(x)dx \int_0^\infty (\int_t^\infty e^{-xy}dy)^{p-1} e^{-xt} f^p(t)dt.$$

By another use of Minkowski's inequality we get

$$\Big(\int_0^\infty (\int_t^\infty e^{-xy}dy)^{p-1} e^{-xt} f^p(t)dt\Big)^{1/p}$$
$$= \Big(\int_0^\infty [\int_0^t h(s)(\int_t^\infty e^{-xy}dy)^{\frac{p-1}{p}} (e^{-xt})^{\frac{1}{p}}\, ds]^p dt\Big)^{1/p}$$
$$\leq \int_0^\infty [\int_s^\infty h^p(s)(\int_t^\infty e^{-xy}dy)^{p-1} e^{-xt} dt]^{1/p} ds$$
$$= \int_0^\infty h(s)p^{-1/p}(\int_s^\infty e^{-xy}dy)ds$$
$$= p^{-1/p}\int_0^\infty e^{-xy}\int_0^y h(s)dsdy$$
$$= p^{-1/p}\int_0^\infty e^{-xy} f(y)dy = p^{-1/p}\mathcal{L}f(x).$$

Therefore we have

$$J = \int_0^\infty [\mathcal{L}f(x)]^p v(x)dx$$

and so

$$(\int_0^\infty f^q(s)w(s)ds)^{1/q} \leq D(\int_0^\infty [\mathcal{L}f(x)]^p v(x)dx)^{1/p}.$$

(ii) The necessary part follows by substituting $f(x) = \chi_{[0,t]}(x), t > 0$ in the inequality (21).

To prove sufficiency we take a nondecreasing function $f(x)$ in the form $f(x) = \int_x^\infty h$, where $h \geq 0$, $\operatorname{supp} h \subset (0, \infty)$ and

$$(\int_0^\infty [\mathcal{L}f(x)]^p v(x)dx)^{1/p} < \infty.$$

Notice that

$$\Big(\int_0^\infty f^q(s)w(s)ds\Big)^{1/q} = \Big(\int_0^\infty \Big[\int_s^\infty d(-f^p(t))\Big]^{q/p} w(s)ds\Big)^{1/q} = I.$$

An appeal to Minkowski's inequality in combination with hypothesis (22) reveals that

$$\begin{aligned} I^p = \Big(\int_0^\infty \Big[\int_s^\infty & w^{p/q}(s)d(-f^p(t))\Big]^{q/p} ds\Big)^{p/q} \\ &\le \int_0^\infty \Big(\int_0^t w(s)ds\Big)^{p/q} d(-f^p(t)) \\ &\le E^p \int_0^\infty \Big[\int_0^\infty \Big(\int_0^t e^{-xy}dy\Big)^p v(x)dx\Big]\, d(-f^p(t)) = E^p J. \end{aligned}$$

An integration by parts assures us that

$$\begin{aligned} J = \Big[-f^p(t) \int_0^\infty & \Big(\int_0^t e^{-xy}dy\Big)^p v(x)dx\,\Big]_0^\infty \\ &+ \int_0^\infty f^p(t)\, d\Big[\int_0^\infty \Big(\int_0^t e^{-xy}dy\Big)^p v(x)dx\Big] \\ &= p\int_0^\infty v(x)dx \int_0^\infty \Big(\int_0^t e^{-xy}dy\Big)^{p-1} e^{-xt} f^p(t)dt. \end{aligned}$$

It takes another appeal to Minkowski's inequality to yield that

$$\begin{aligned} \Big(\int_0^\infty & \Big(\int_0^t e^{-xy}dy\Big)^{p-1} e^{-xt} f^p(t)dt\Big)^{1/p} \\ &= \Big(\int_0^\infty \Big[\int_t^\infty h(s)\Big(\int_0^t e^{-xy}dy\Big)^{\frac{p-1}{p}} (e^{-xt})^{\frac{1}{p}} ds\,\Big]^p dt\Big)^{1/p} \\ &\le \int_0^\infty \Big[\int_0^s h^p(s)\Big(\int_0^t e^{-xy}dy\Big)^{p-1} e^{-xt}dt\,\Big]^{1/p} ds \\ &= \int_0^\infty h(s)p^{-1/p}\Big(\int_0^s e^{-xy}dy\Big)ds \\ &= p^{-1/p}\int_0^\infty e^{-xy}\int_y^\infty h(s)ds\, dy \\ &= p^{-1/p}\int_0^\infty e^{-xy}f(y)dy = p^{-1/p}\mathcal{L}f(x). \end{aligned}$$

Thus we get

$$J = \int_0^\infty [\mathcal{L}f(x)]^p v(x)dx$$

and hence

$$\left(\int_0^\infty f^q(s)w(s)ds\right)^{1/q} \le E\left(\int_0^\infty [\mathcal{L}f(x)]^p v(x)dx\right)^{1/p}.$$

This ends the proof. □

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