Kangweon-Kyungki Math. Jour. **14** (2006), No. 1, pp. 113–123

BIFURCATION THEORY FOR A CIRCULAR ARCH SUBJECT TO NORMAL PRESSURE

Keumseong Bang*† and JaeGwi Go

Abstract. The arches may buckle in a symmetrical snap-through mode or in an asymmetry bifurcation mode if the load reaches a certain value. Each bifurcation curve develops as pressure increases. The governing equation is derived according to the bending theory. The balance of forces provides a nonlinear equilibrium equation. Bifurcation theory near trivial solution of the equation is developed, and the buckling pressures are investigated for various spring constants and opening angles.

1. Introduction

Assume that the equation

$$F(X,p)=0 \tag{1}$$

has the trivial solution $X=0$ for all pressure p in an open neighborhood of $p_0 \in \mathbb{R}$. If the Frechet derivative $F_X(0,p_0)$ is invertible, then the implicit function theorem guarantees the uniqueness of the trivial solution [5, p310] and, when it is singular, bifurcations usually occur. More precisely, let

$$F : U(0,p_0) \subset H \times \mathbb{R}^1 \to K$$

be a C^2-map on an open neighborhood $U(0,p_0)$ of the point $(0,p_0)$, where H and K are real Banach spaces. The linearized operator $F_X(0,p_0)$: $H \to K$ is assumed to be Fredholm and such that

1. $N(F_X(0,p_0))$= span $\{v\}$

Received May 18, 2006.
2000 Mathematics Subject Classification: 35C23.
Key words and phrases: bifurcation, nonlinear equilibrium equation.
* Corresponding author.
†This work was supported by a research fund of the Catholic University of Korea, 20060065.

2. $N(F_X^*(0, p_0))=$ span $\{v^*\}$

3. $< v^*, F_X(0, p_0)v > \neq 0$ (bifurcation condition).

In [5, p311] it is shown that, under the above assumptions, $(0, p_0)$ is a bifurcation point of the equation (1).

The arches may buckle or deviate from its circular shape if the pressure is larger than a certain critical pressure. The equilibrium equations are derived from a small perturbation from the circular arch, and the buckling pressure is the first eigenvalue of the linearized system. When the pressure is further increased, two types of bifurcation curves, symmetric and anti-symmetric curves develop versus pressure. Such a buckling of a circular arch has been studied by Timoshenko and Gere [4], Tadjbakhsh, I. and Odeh [3], and Pi, Bradford, and Uy [2]. An elastic, thin, and inextensible circular arch is considered in this paper. The arch is under uniformly distributed normal pressure. Bifurcation curve is developed near trivial solution, and buckling pressures are investigated for various spring constants and opening angles.

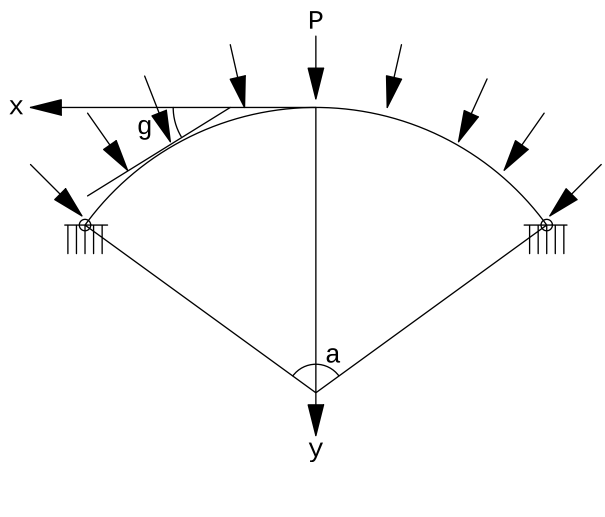

Fig. 1. Normal load uniformly distributed around arch

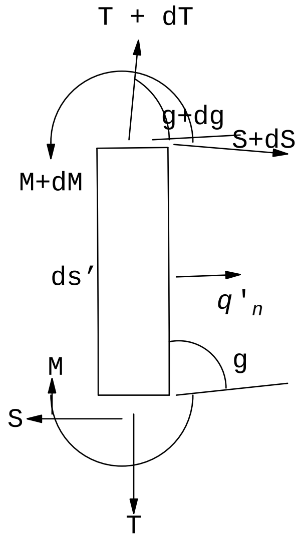

Fig. 2. An elemental length

2. Derivation of equation

The balance (Fig 2) of forces in the normal and tangential directions provides the equations

(2) $$Tdg - dS - q_n^{'} ds^{'} = 0$$

(3) $$dT + Sdg = 0.$$

By the balance of local moment we obtain

(4) $$dM - Sds^{'} = 0.$$

Here T, S, and $q_n^{'}$ are tension, shear, and normal stress on the surface, respectively, and g is the local angle of inclination, and $s^{'}$ the arc length. The Euler-Bernoulli law yields

(5) $$M = EI\frac{dg}{ds'}\,,$$

where EI is flexural rigidity. The combination of equations (2)∼ (5) gives the nonlinear normalized equation

(6) $$g_{ssss}g_s - g_{sss}g_{ss} - q_n g_{ss} + g_{ss}g_s^3 = 0.$$

The arch length and normal stress are normalized by the variables $s = \frac{s^{'}}{R}$ and $q_n = \frac{q_n^{'} R^3}{EI}$, where R is the radius of the circular arch.

The total angle change is resisted by an additional moment at the bases with torsional spring constants. Thus the boundary conditions are

$$\tau^{'}(g(-a)+a) - EI(g_{s'}(-a) - \frac{1}{R}) = 0 \tag{7a}$$

$$\tau^{'}(g(a)-a) + EI(g_{s'}(a) - \frac{1}{R}) = 0, \tag{7b}$$

where τ' is the spring constant and $2a$ is the opening angle. The non-dimensionless forms are

$$(g(0)+a) - \tau(g_s(0) - 2a) = 0 \tag{8a}$$

$$(g(1)-a) + \tau(g_s(1) - 2a) = 0, \tag{8b}$$

where $\tau = EI/\tau^{'}R$. The normalized Cartesian coordinates (x, y) are related to g by

$$x_s = \cos g(s) \qquad y_s = \sin g(s). \tag{9}$$

3. Bifurcation theory near trivial solution

Using the transform $\tilde{g}(s) = g(1-s)$ and then dropping the tilde give the followings:

$$g_{ssss}g_s - g_{sss}g_{ss} - pg_{ss} + g_{ss}g_s^3 = 0$$

$$x_s = \cos g(s) \qquad y_s = \sin g(s)$$

$$(g(0)-a) - \tau(g_s(0) + 2a) = 0$$
$$(g(1)+a) + \tau(g_s(1) + 2a) = 0$$

$$x(0) = y(0) = y(1) = 0 \qquad x(1) = \frac{\sin a}{a}.$$

Now, we let

$$\begin{cases} x_1 = g + 2as - a \\ x_2 = g_s + 2a \\ x_3 = g_{ss} \\ x_4 = g_{sss} \\ x_5 = x(s) - \frac{1}{2a}[\sin a - \sin(a - 2as)] \\ x_6 = y(s) + \frac{1}{2a}[\cos a - \cos(a - 2as)]. \end{cases}$$

Then, the above boundary value problem can be converted into the following form

(11) $$F[X, p] = LX - f(X, p) = 0$$

(12) $$B[X] = B_1X(0) + B_2X(1) = 0,$$

where

(13) $$X = (x_1, x_2, x_3, x_4, x_5, x_6)^T$$

(14) $$L[X] = \frac{d}{ds}X$$

$$f(X, p) = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \dfrac{x_3x_4 + px_3 - x_3(x_2 - 2a)^3}{x_2 - 2a} \\ \cos g - \cos(a - 2as) \\ \sin g - \sin(a - 2as) \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 & -\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us define the domain Banach space to be

$$H = \{X \in (C^1[0,1])^6 : B[X] = 0\},$$

where B is the boundary operator defined by (12). Let the range space be

$$K = \{X \in (C[0,1])^6\}.$$

Note that the equation (11) has the trivial solution X=0 for all p and

$$F_X(0, p_0) = L - \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\gamma^2 & 0 & 0 & 0 \\ -\sin(a-2as) & 0 & 0 & 0 & 0 & 0 \\ \cos(a-2as) & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\gamma^2 = \dfrac{p_0 + 8a^3}{2a}$. Hence $F_X(0,p_0)v = 0$ implies

$$\begin{cases} v_{1,s} = v_2 \\ v_{2,s} = v_3 \\ v_{3,s} = v_4 \\ v_{4,s} = -\gamma^2 v_3 \\ v_{5,s} = -\sin(a-2as)v_1 \\ v_{6,s} = \cos(a-2as)v_1 \end{cases}$$

which, in turn, yields

$$\begin{cases} v_1 = -\frac{A_1}{\gamma^2}\cos\gamma s - \frac{A_2}{\gamma^2}\sin\gamma s - A_3 s + A_4 \\ v_2 = \frac{A_1}{\gamma}\sin\gamma s - \frac{A_2}{\gamma}\cos\gamma s + A_3 \\ v_3 = A_1\cos\gamma s + A_2\sin\gamma s \\ v_4 = -A_1\gamma\sin\gamma s + A_2\gamma\cos\gamma s \\ v_5 = A_1[\dfrac{\sin(a-2as)\sin\gamma s}{\gamma(\gamma^2-4a^2)} - \dfrac{2a\cos(a-2as)\cos\gamma s}{\gamma^2(\gamma^2-4a^2)}] \\ \quad -A_2[\dfrac{\sin(a-2as)\cos\gamma s}{\gamma(\gamma^2-4a^2)} + \dfrac{2a\cos(a-2as)\sin\gamma s}{\gamma^2(\gamma^2-4a^2)}] \\ \quad -A_3[\dfrac{s\cos(a-2as)}{2a} + \dfrac{\sin(a-2as)}{4a^2}] - A_4\dfrac{\cos(a-2as)}{2a} + A_5 \\ v_6 = -A_1[\dfrac{\cos(a-2as)\sin\gamma s}{\gamma(\gamma^2-4a^2)} + \dfrac{2a\sin(a-2as)\cos\gamma s}{\gamma^2(\gamma^2-4a^2)}] \\ \quad +A_2[\dfrac{\cos(a-2as)\cos\gamma s}{\gamma(\gamma^2-4a^2)} - \dfrac{2a\sin(a-2as)\sin\gamma s}{\gamma^2(\gamma^2-4a^2)}] \\ \quad -A_3[\dfrac{s\sin(a-2as)}{2a} - \dfrac{\cos(a-2as)}{4a^2}] - A_4\dfrac{\sin(a-2as)}{2a} + A_6. \end{cases}$$

The boundary conditions are satisfied if

$$\begin{cases} -\dfrac{1}{\gamma^2}A_1 + \dfrac{\tau}{\gamma^2}A_2 - \tau A_3 + A_4 = 0 \\ A_1[-\dfrac{\cos\gamma}{\gamma^2} + \dfrac{\tau\sin\gamma}{\gamma}] - A_2[\dfrac{\sin\gamma}{\gamma^2} + \dfrac{\tau\cos\gamma}{\gamma}] + (1+\tau)A_3 + A_4 = 0 \\ A_1[-\dfrac{2a\cos a}{\gamma^2(\gamma^2-4a^2)}] - A_2[\dfrac{\sin a}{\gamma(\gamma^2-4a^2)}] - A_3\dfrac{\sin a}{4a^2} - A_4\dfrac{\cos a}{2a} + A_5 = 0 \\ A_1[-\dfrac{2a\sin a}{\gamma^2(\gamma^2-4a^2)}] + A_2[\dfrac{\cos a}{\gamma(\gamma^2-4a^2)}] + A_3\dfrac{\cos a}{4a^2} - A_4\dfrac{\sin a}{2a} + A_6 = 0 \\ A_1[-\dfrac{\sin a\sin\gamma}{\gamma(\gamma^2-4a^2)} - \dfrac{2a\cos a\cos\gamma}{\gamma^2(\gamma^2-4a^2)}] + A_2[\dfrac{\sin a\cos\gamma}{\gamma(\gamma^2-4a^2)} - \dfrac{2a\cos a\sin\gamma}{\gamma^2(\gamma^2-4a^2)}] \\ \quad +A_3[-\dfrac{\cos a}{2a} + \dfrac{\sin a}{4a^2}] - A_4\dfrac{\cos a}{2a} + A_5 = 0 \\ A_1[-\dfrac{\cos a\sin\gamma}{\gamma(\gamma^2-4a^2)} + \dfrac{2a\sin a\cos\gamma}{\gamma^2(\gamma^2-4a^2)}] + A_2[\dfrac{\cos a\cos\gamma}{\gamma(\gamma^2-4a^2)} + \dfrac{2a\sin a\sin\gamma}{\gamma^2(\gamma^2-4a^2)}] \\ \quad +A_3[\dfrac{\sin a}{2a} + \dfrac{\cos a}{4a^2}] + A_4\dfrac{\sin a}{2a} + A_6 = 0. \end{cases}$$

Numerical investigations show that this system has one dimensional null set at infinitely many isolated points p_0 and it is known [1, p41] that the null space of the adjoint problem has the same dimension. To make sure that bifurcation happens at those points, it is enough to verify the bifurcation condition 3. This will be done just for one case bellow, but it appears that the bifurcation condition is always satisfied possibly except in some special cases.

The adjoint $F_X^*(0,p_0)$ of $F_X(0,p_0)$ is given [1, p40] by

$$F_X^*(0,p_0)Z = -LZ - f_X^T(0,p_0)Z = 0$$

with the boundary condition

$$PZ(0) + QZ(0) = 0$$

where

$$P = \begin{pmatrix} \tau & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \tau & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

If $F_X^*(0,p_0)v^* = 0$, then we have

$$\begin{cases} v_{1,s}^* = \sin(a-2as)v_5^* - \cos(a-2as)v_6^* \\ v_{2,s}^* = -v_1^* \\ v_{3,s}^* = -v_2^* + \gamma^2 v_4^* \\ v_{4,s}^* = -v_3^* \\ v_{5,s}^* = 0 \\ v_{6,s}^* = 0, \end{cases}$$

and hence,

$$
v^* = \begin{pmatrix}
\frac{k_1}{2a}\cos(a-2as) + \frac{k_2}{2a}\sin(a-2as) + k_3 \\
\frac{k_1}{4a^2}\sin(a-2as) - \frac{k_2}{4a^2}\cos(a-2as) - k_3 s + k_4 \\
\frac{k_1}{2a(\gamma^2-4a^2)}\cos(a-2as) + \frac{k_2}{2a(\gamma^2-4a^2)}\sin(a-2as) \\
+\frac{k_3}{\gamma^2} + k_5\gamma\sin\gamma s - k_6\gamma\cos\gamma s \\
\frac{k_1}{4a^2(\gamma^2-4a^2)}\sin(a-2as) - \frac{k_2}{4a^2(\gamma^2-4a^2)}\cos(a-2as) \\
-\frac{k_3}{\gamma^2}s + \frac{k_4}{\gamma^2} + k_5\cos\gamma s + k_6\sin\gamma s \\
k_1 \\
k_2
\end{pmatrix}.
$$

In order that the boundary conditions of the adjoint problem are satisfied, k_i's, $i = 1, 2, ..., 6$, must satisfy the system:

$$
\begin{cases}
k_1[\frac{2a\tau\cos a + \sin a}{4a^2}] + k_2[\frac{2a\tau\sin a - \cos a}{4a^2}] + \tau k_3 + k_4 = 0 \\
k_1[\frac{2a\tau\cos a + \sin a}{4a^2}] + k_2[\frac{-2a\tau\sin a + \cos a}{4a^2}] + (1+\tau)k_3 - k_4 = 0 \\
k_1[\frac{\cos a}{2a(\gamma^2-4a^2)}] + k_2[\frac{\sin a}{2a(\gamma^2-4a^2)}] + k_3\frac{1}{\gamma^2} - \gamma k_6 = 0 \\
k_1[\frac{\sin a}{4a^2(\gamma^2-4a^2)}] + k_2[\frac{-\cos a}{4a^2(\gamma^2-4a^2)}] + k_4\frac{1}{\gamma^2} + k_5 = 0 \\
k_1[\frac{\cos a}{2a(\gamma^2-4a^2)}] + k_2[\frac{-\sin a}{2a(\gamma^2-4a^2)}] + k_3\frac{1}{\gamma^2} + k_5[\gamma\sin\gamma] - \gamma\cos\gamma k_6 = 0 \\
k_1[\frac{-\sin a}{4a^2(\gamma^2-4a^2)}] + k_2[\frac{-\cos a}{4a^2(\gamma^2-4a^2)}] - \frac{1}{\gamma^2}k_3 + k_4\frac{1}{\gamma^2} + \cos\gamma k_5 + \sin\gamma k_6 = 0.
\end{cases}
$$

This system has nontrivial solution if and only if the system for $A_1, \ldots, A_6$ has nontrivial solution [1, p41].

Using Mathematica we will now verify bifurcation condition for angle $a = \frac{\pi}{2}$ and spring constant $\tau = 0$. $F_X(0, p_0)$ is non-invertible if $\gamma = 9.42478$, in which case we can choose

TABLE 1. Buckling pressures

a/τ	0	.01	.1	1	10	100	∞
$\pi/12$	42.24	40.463	31.9219	22.4848	20.7366	20.5484	20.5273
$\pi/6$	84.208	80.9883	63.4116	44.1913	40.6209	40.2364	40.1933
$\pi/4$	125.695	120.859	94.0955	64.3533	58.8019	58.2037	58.1368
$\pi/3$	166.634	160.18	123.734	82.2333	74.4316	73.5905	73.4964
$5\pi/12$	207.212	199.156	152.341	94.1487	86.6682	85.5374	85.4108
$\pi/2$	248.05	238.449	180.405	108.523	94.6812	93.1862	93.0188

$$\begin{cases} A_1 = 1 & k_1 = 1 \\ A_2 = -3312371257 & k_2 = 165.9995 \\ A_3 = 2.3116318 \times 10^{-11} & k_3 = -\dfrac{2}{\pi^2} \\ A_4 = 0.0112579 & k_4 = -\dfrac{1}{\pi^2} \\ A_5 = -4451197.619 & k_5 = -0.000142586 \\ A_6 = 0.0015886, & k_6 = 0.0707641. \end{cases}$$

The bifurcation coefficient is then

$$< v^*, F_{Xp}(0, p_0)v >= -1.29416 \times 10^7.$$

Therefore, $(0, p_0)$ with $p_0 = 248.05021$ is a bifurcation point in this case.

The buckling pressures for various values of the angle a and the spring constants τ are given in Table 1.

References

[1] P. Baird and J. Eells, A conservation law for harmonic maps, Lecture Notes in Mathematics, 894, Springer(1981), pp. 1-25.

[2] P. Baird and S. Gudmundsson, p-harmonic maps and minimal submanifolds, Math. Ann., 294 (1992), 611-624.

[3] Miklavcic, M. (1998). *Applied Functional Analysis And Partial Differential Equations*, World Scientific.

[4] Pi, Y.L., Bradford, M.A., and Uy, B.(2002) *In-Plane Stability Of Arches*, Inter. J. Solids and Structures, Vol. 39, 2002, pp. 105-125.

[5] Tadjbakhsh, I. AND Odeh, F.(1967) *Equilibrium States Of Elastic Rings*, J. Math. Anal. Appl., Vol. 18, pp59-74.

[6] Timoshenko, S.P. and Gere, J.M.(1961). *Theory Of Elastic Stability*, McGraw-Hill Second Edition.

[7] Zeidler, E. (1995) *Applied Functional Analysis : Main Principle and Their Applications*, Springer-Verlag.

Keumseong Bang
Department of Mathematics
The Catholic University of Korea
Bucheon, 420–743, Korea
E-mail: bang@catholic.ac.kr

JaeGwi Go
Department of Mathematics
Changwon National University
Changwon, 641–773, Korea
E-mail: gojaegwi@msu.edu