Kangweon-Kyungki Math. Jour. **14** (2006), No. 1, pp. 65–70

ON GENERALIZED BOUNDARY CLUSTER SETS

Bo-Hyun Chung

Abstract. In this article, we mention some subsequent developments of the theory of cluster sets, and present a new boundary cluster set for a simply connected domain in the complex plane and its applications.

1. Introduction

Sir E. F. Collingwood wrote [9] concerning the origin of the theory of cluster sets of arbitrary real functions and its development.

The theory of the cluster sets of arbitrary real functions originated with W. H. Young. The story begins with his paper [14], in which he showed that the points of inequality of right and left upper and lower limits of a function of a single variable are enumerable. This was followed by a number of papers, some in collaboration with G. C. Young, of which the most important are [16], in which he proved that for a function of a single variable the points of inequality of right and left cluster sets, although not under that or any other compendious name, are enumerable, with analogous theorems for several variables; and [17] which completes and summarizes his theory. Young considered only real-functions and was evidently unaware of Painleve's definition of a cluster set which had been formulated in 1895 for complex functions.

The work of H. Blumberg ([5]and [4]), who had independently discovered Young's theorem on discontinuities [3], developed Young's theory of arbitrary real function as a good deal further and gave rise to the theorems of Jarrnik [11] and Bagemihl [1] whose well known ambiguous point theorem has important implications for complex function theory.

Received March 13, 2006.
2000 Mathematics Subject Classification: 30D40, 30C85.
Key words and phrases: Boundary behavior, cluster sets, prime ends.

Let f be a mapping from the real line R into the extended real line $\bar{R}$, and let x be a point in R.

The right cluster set $C^+(f, x)$ of f at x is defined to be the set of all points in $w \in \bar{R}$ for which

$$f^{-1}(U) \cap (x, x + r) \neq \emptyset$$

for each $r > 0$ and each neighborhood U of $\bar{R}$.

The left cluster set $C^-(f, x)$ of f at x is defined analogously, and the cluster set of f at x is defined to be the set

$$C(f, x) = C^+(f, x) \cup C^-(f, x)$$

In 1908, W. H. Young [15] proved the following theorem.

Theorem 1.1. (Young's Theorem) *If $f : R \to \bar{R}$ is an arbitrary one or multi-valued function, then*

$$C^+(f, x) = C^-(f, x)$$

for all but countably many points $x \in R$.

Let $f : H \to \bar{R}$ be an arbitrary function on the open upper half plane. The cluster set $C(f, x)$ of f at $x \in R$ is the set of all points $w \in \bar{R}$ for which there exists a sequence of points $z_n \in H$ with $z_n \to x$ and $f(z_n) \to w$. Then the right boundary cluster set $C_{Br}(f, x)$ of f at x is

$$C_{Br}(f, x) = \cap_{\varepsilon > 0} \overline{Y(x, \varepsilon)}$$

where

$$Y(x, \varepsilon) = \cup_{x < y < x+\varepsilon} C(f, y),$$

and the bar denotes closure. The left boundary cluster set $C_{Bl}(f, x)$ is defined similarly.

Collingwood [9] proved the following theorem.

Theorem 1.2. *If $f : H \to \bar{R}$ is an arbitrary function, then*

$$C_{Br}(f, x) = C_{Bl}(f, x) = C(f, x)$$

for all but countably many points $x \in R$.

Let γ be an arc at $x \in R$. The cluster set $C_\gamma(f, x)$ of $f : H \to \bar{R}$ along the arc γ at x is the set of all points $w \in \bar{R}$ for which there exists a sequence of points z_n of γ with $z_n \to x$ and $f(z_n) \to w$.

The function $f : H \to \bar{R}$ is said to be ambiguous at $x \in R$ if there exists two arcs α and β at x with that

$$C_\alpha(f, x) \cap C_\beta(f, x) = \emptyset$$

F. Bagemihl [1] gave the final answer to the general ambiguity problem:

THEOREM 1.3. (Bagemihl [1]) *An arbitrary function $f : H \to \bar{R}$ can have only countably many ambiguous points $x \in R$.*

All the results given above are true for complex valued functions $f : D \to \bar{\mathbf{C}}$, where D is an unit disc and $\bar{\mathbf{C}}$ is the extended complex plane. Bagemihl's theorem can be stated as :

THEOREM 1.4. *If $f(z)$ is a complex-valued function defined on D, then the set of ambiguous points $e^{i\theta}$ on D with the property that there exists two arcs α and β such that*

$$C_\alpha(f, e^{i\theta}) \cap C_\beta(f, e^{i\theta}) = \emptyset$$

is at most countable.

2. A new boundary cluster set and applications

Let $f(z)$ be meromorphic on the unit disc T. Let $t_0 = e^{i\theta_0}$ be a fixed point on Γ the boundary of T and Λ an open arc of Γ containing t_0.

We assumed that E is a set of linear measure zero containing t_0 and contained in Λ. We associate with every $e^{i\theta}$ in $\Lambda - E$ an arbitrary curve Λ_0 in T terminating at $e^{i\theta}$ and the cluster set $C_{\Lambda_0}(f, e^{i\theta})$ of $f(z)$ at $e^{i\theta}$ along Λ_0.

Noshiro [12] defined the boundary cluster set $C^*_{\Gamma-E}(f, t_0)$ of $f(z)$ at t_0 to be

$$C^*_{\Gamma-E}(f, t_0) = \cap_{r>0} M_r,$$

Where M_r is the closure of the union $\cup C_{\Lambda_0}(f, e^{i\theta})$ for all $e^{i\theta}$ in the intersection of $\Lambda - E$ with $\{z : |z - t_0| < r\}$.

We define a new boundary cluster set for a simply connected domain in $\mathbf{C}$. Let $f(z)$ be a meromorphic function in a simply connected domain D, $\bar{E}$ a for conformal null set of prime ends of D such that E the union of impressions of prime ends in E contains t_0.

We associate with every accessible boundary point a with $P(a)$ in $\bar{D} - \bar{E}$ ($\bar{D}$ is the set of all prime ends of D) an arc Λ at $P(a)$ is D

terminating at $z(a)$ and the cluster set $C_\Lambda(f, z(a))$ of $f(z)$ at $z(a)$ along Λ.

DEFINITION 2.1. We define a new boundary cluster set $C^*_{\bar{D}-\bar{E}}(f, t_0)$ of $f(z)$ to be

$$C^*_{\bar{D}-\bar{E}}(f, t_0) = \cap_{r>0} M_r$$

where M_r is the closure of the $\cup C_\Lambda(f, z(a))$ for all accessible points a with $P(a) \in \bar{D} - \bar{E}$ and $z(a)$ in $\{z : |z - t_0| < r\}$.

As an application of the new boundary cluster set we can prove a generalized version of the F. Bagemihl's result [2].

THEOREM 2.2. *Let D be a simply connected domain which is not the whole plane. $\bar{E}$ a conformal null set. Let $f(z)$ be analytic in D. Suppose that at every prime end p of D that does not belong to $\bar{E}$ there is a curve Λ_p such that*

$$\lim_{z \to p}(\sup |f(z)|) \leq m, \ z \in \Lambda_p$$

and such that $f(z)$ does not have the asymptotic value ∞ at any prime end of D. Then f is bounded in D.

The proof for Theorem 2.2 was given in [6].

In our discussion we will need the following theorem.

THEOREM 2.3. (Seidel and Frostman [9] pp 37) *Let $w = f(z)$ be a function of class (U). Suppose that $f(z)$ has a singularity at $x = e^{ie}$. Then every value of $D_w = \{w : |w| < 1\}$ is assumed by $f(z)$ infinitely often in any neighborhood of t except perhaps for a set of values of capacity zero. that is, $D_W - R_D(f, t)$ is at most of capacity zero.*

we proved the following theorem in [6].

THEOREM 2.4. *Let D be a simply connected domain in the z-plane, which is not the whole plane. And let t_0 be a boundary point of D, contained in the union of impressions of prime ends in $\bar{E}$, a D-conformal null set. Let $f(z)$ be single-valued and meromorphic in D. If*

$$\alpha \in C_D(f, z_0) - C^*{}_{\bar{D}-\bar{E}}(f, t_0)$$

is an exceptional value of $f(z)$ is a neighborhood of t_0 then either, α is an asymptotic value of $f(z)$ at z_0, or there is a sequence of points z_n in the boundary of D converging to t_0, such that α is an asymptotic value of $f(z)$ at each z_n.

we present an application of Theorem 2.4.

THEOREM 2.5. *If*

$$S = C_D(f, z_0) - C^*{}_{\bar{D}-\bar{E}}(f, t_0)$$

is not empty, then the range of values $R_D(f, z_0)$ of $f(z)$ covers S, with a possible exceptional set of capacity zero.

Proof. Suppose that

$$(R_D(f, z_0))^c \cap S$$

is of positive capacity, where E^c denotes the complement of E_0. Then there exists a positive number r_0 such that the complement of the set of values taken by $f(z)$ in the intersection of D and

$$D_r = \{z : |z - t_0| < r\}$$

contains a Borel set B of positive capacity in S.

We can select a point α of B such that for all positive number r, $B \cap D_r$ is of positive capacity. Now choose a positive number $r(< r_0)$ so that $\alpha \notin M_r$ and denote by d a positive number less than the distance between α and M_r.

By Theorem 2.2, there exists a point t_0' (distinct from t_0 or may not) arbitrarily near t_0 such that α is an asymptotic values of $f(z)$ at t'_0. If $t'_0 = t_0$, applying Seidel-Frostman's Theorem, we see that $R_D(f, z_0)$ covers the disc

$$D_\alpha = \{w : |w - \alpha| < d\}$$

with a possible exceptional set of capacity zero.

This is a contradiction.

In the case $t_0' \neq t_0$, it follows that t_0' belongs to

$$E = \{z(a) : P(a) \in \bar{E}\}$$

Accordingly, we can define

$$C^*{}_{\bar{D}-\bar{E}}(f, t_0).$$

Then

$$C_D(f, z_0) - C^*{}_{\bar{D}-\bar{E}}(f, t_0) \supset D_\alpha$$

and the above reasoning shows that $R_D(f, z_0)$ covers the disc D_α with a possible exceptional set of capacity zero. Thus we arrive at a contradiction.

This completes the proof of the theorem. □

References

[1] F. Bagemihl, *Curvilinear cluster sets of arbitrary function*, Proc. Nat. Acad. Sci. U.S.A. 41(1955), 379-382

[2] F. Bagemihl, *A Curvilinear Extension of the Maximum Modulus Principle*, Proc-Nat. Acad. Sci. U.S.A. (1969), 36-37

[3] H. Blumberh, *A Theorem on semi-continuous function*, Bull. Amer. Math. Soc. 24(1918), 381-383

[4] H. Blumberg, *Exceptional sets*, Fund. Math. 32(1939), 3-32

[5] H. Blumberg, *A theorem on arbitrary function of two variables with application*, Fund. Math. 16(1930), 17-24

[6] O. H. Choi, *On the boundary behavior of analytic functions*, Bull. of the Korean Math. Soc. 26(1989), 165-168

[7] Bo-Hyun Chung, *Some results for the extremal lengths of curve families (II)*, J. Appl. Math. and Computing., 15(No. 1-2)(2004), 495-502.

[8] Bo-Hyun Chung, *A note on geometric applications of extremal lengths (I)*, J. Appl. Math. and Computing., 18(No. 1-2)(2005), 603-611.

[9] E. F. Collingwood, *Cluster sets of arbitrary functions*, Proc. Nat. Acad. Sci., U. S. A., 46(1960), 1236-1242

[10] E. F. Collingwood and A. J. Lohwater, **The theory of cluster sets**, Cambridege Univ. Press, 1966

[11] V. Jarnik, *Sur les functions de deux variables reelles*, Fund. Math. 27(1936), 147-150

[12] K. Noshiro, *Cluster sets*, Spring-verlag, Berlin-Goetlingen-Heidelberg, 1960

[13] C. L. Schnitzer, *Cluster sets of arbitrary real functions : a partial survey*, Real Analysis Exchange I(1976), 7-20

[14] W. H. Young, *On the distinction of right and lift at points of discontinuity*, Quart. J. Pure Apple. Math. Soc. 8(1908), 67-83

[15] W. H. Young, *On some applications of semi-continuous functions*, Atti. del. IV Congresso-Ibternationale dei Matematici (Rome 1908), Roma(1909), 49-60

[16] W. H. Young, *On the discontinuities of a function of one or more real variables*, proc. London. Math. Soc. 8(1909), 117-124

[17] W. H. Young, *La symetrie de structure des functions de variable reelles*, Bull. Sci. math. 52(1928), 265-280

Mathematics Section
College of Science and Technology
Hongik University
Jochiwon, 339–701, Korea
E-mail: bohyun@hongik.ac.kr