Kangweon-Kyungki Math. Jour. **14** (2006), No. 1, pp. 13–33

HARMONIC MORPHISMS AND STABLE MINIMAL SUBMANIFOLDS

Gundon Choi and Gabjin Yun*†

Abstract. In this article, we study the relations of horizontally conformal maps and harmonic morphisms with the stability of minimal fibers. Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion. There is a tensor T measuring minimality or totally geodesics of fibers of φ. We prove that if T is parallel and the horizontal distribution is integrable, then any minimal fiber of φ is volume-stable. As a corollary, we obtain that any fiber of a submersive harmonic morphism whose fibers are totally geodesics and the horizontal distribution is integrable is volume-stable. As a consequence, we obtain if $\varphi : (M^n, g) \to (N^2, h)$ is a submersive harmonic morphism of minimal fibers from a compact Riemannian manifold M into a surface N, T is parallel and the horizontal distribution is integrable, then φ is energy-stable.

1. Introduction

The theory of harmonic morphisms is one of particularly interesting subclasses of harmonic maps. A harmonic map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is a critical point of the energy functional defined on each compact domain of M. A harmonic morphism between Riemannian manifolds is a map preserving harmonic structure. In other words, a map $\varphi : (M^n, g) \to (N^m, h)$ is called a harmonic morphism if for any harmonic function f defined on an open subset $V \subset N$ such

---

Received November 28, 2005.
2000 Mathematics Subject Classification: 53C43, 58E20.
Key words and phrases: harmonic morphism, stable minimal submanifold.
* Corresponding author.
†This work was supported by Korea Research Foundation Grant (KRF-2002-015-CP0029).

that $\varphi^{-1}(V) \neq \emptyset$, the composition $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{R}$ is also harmonic. Harmonic morphisms are characterized as harmonic maps which are horizontally (weakly) conformal ([3], [6]).

Let $\varphi : (M^n, g) \to (N^m, h)$ be a harmonic morphism between Riemannian manifolds. Then it is well-known ([1]) that if $\dim(N) = m = 2$, the regular fibers of φ are minimal submanifolds of M and if $\dim(N) = m \geq 3$, φ has minimal fibers if and only if it is horizontally homothetic.

A minimal submanifold of a Riemannian manifold is a submanifold whose mean curvature defined as the trace of the second fundamental form determined by a normal vector field is vanishing. Or, equivalently, a minimal submanifold is a critical point of the volume functional defined on the variation of each compact domain. On the other hand, a minimal submanifold of a Riemannian manifold is called stable (or volume-stable) if the second derivative of the volume functional is non-negative for any normal variation with compact support. Not much results for stable minimal submanifolds are known compared with minimal submanifolds.

In this paper, we studied the stability of minimal fibers of harmonic morphisms and horizontally (weakly) conformal maps between Riemannian manifolds. Given a horizontally (weakly) conformal map $\varphi : (M^n, g) \to (N^m, h)$ with $n \geq m$, there is a $(2, 1)$-tensor T defined originally by O'Neill ([12]) measuring whether the fibers of φ are minimal or totally geodesic (see section 2 for definition). In fact, it is easy observation that the fibers of φ are totally geodesic if and only if T vanishes. We say the tensor T is parallel if the covariant derivative of T with respect to any vector field is vanishing. Thus, if a horizontally (weakly) conformal map $\varphi : (M^n, g) \to (N^m, h)$ has totally geodesic fibers, then T is automatically parallel.

Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion. If the dimension of N is one and T is parallel, then we could show any minimal fiber is volume-stable. In case the dimension of N is two, if T is parallel and the horizontal distribution is integrable, then any regular fiber is volume-stable. And in case $\dim(N) = m > 2$, if the horizontal distribution is integrable, any minimal fiber is volume-stable. Consequently, if $\varphi : (M^n, g) \to (N^m, h)$ is a submersive harmonic morphism with totally geodesic fibers and the horizontal distribution is integrable, then all the fibers are volume-stable.

Finally, we would like to mention a result due to Montaldo ([11]) that if a submersive harmonic morphism $\varphi : (M^n, g) \to (N^2, h)$ from a compact Riemannian manifold to a surface has volume-stable minimal fibers, then φ is energy-stable, i.e., the second derivative of the energy functional is non-negative for any variation. Therefore, we could conclude that if $\varphi : (M^n, g) \to (N^2, h)$ is a submersive harmonic morphism to a surface, and if T is parallel and the horizontal distribution is integrable, then φ is energy-stable.

Finally, we would like to remark that our main result can be deduced from the theory of foliation (cf. [4], [13]).

2. Preliminaries

In this section, we shall describe basic notions for horizontally (weakly) conformal maps and stability of minimal submanifolds.

Let $\varphi : (M^n, g) \to (N^m, h)$ be a smooth map between Riemannian manifolds (M, g) and (N, h). For a point $x \in M$, we set $\mathcal{V}_x = \ker(d\varphi_x)$. The space $\mathcal{V}_x$ is called the vertical space at x. Let $\mathcal{H}_x$ denote the orthogonal complement of $\mathcal{V}_x$ in the tangent space T_xM. For a tangent vector $X \in T_xM$, we denote $X^{\mathcal{V}}$ and $X^{\mathcal{H}}$, respectively, the vertical component and the horizontal component of X. Let $\mathcal{V}$ and $\mathcal{H}$ denote the corresponding vertical and horizontal distributions in the tangent bundle TM. We say that φ is horizontally (weakly) conformal if, for each point $x \in M$ at which $d\varphi_x \neq 0$, the restriction $d\varphi_x|_{\mathcal{H}_x} : \mathcal{H}_x \to T_{\varphi(x)}N$ is conformal and surjective. Thus there exists a non-negative function λ on M such that

$$h(d\varphi(X), d\varphi(Y)) = \lambda^2 g(X, Y)$$

for horizontal vectors X, Y. The function λ is called the dilation of φ. Note that λ^2 is smooth and is equal to $|d\varphi|^2/m$, where $m = \dim(N)$.

Let $\varphi : M^n \to N^m$ be a horizontally (weakly) conformal map between Riemannian manifolds (M, g) and (N, h). Denote the set of critical points of φ by $C_\varphi = \{x \in M : d\varphi_x = 0\}$ and let $M^* = M - C_\varphi$. We define two tensors T and A over M^* by

$$T_E F = (\overline{\nabla}_{E^{\mathcal{V}}} F^{\mathcal{V}})^{\mathcal{H}} + (\overline{\nabla}_{E^{\mathcal{V}}} F^{\mathcal{H}})^{\mathcal{V}}$$

and

$$A_E F = (\overline{\nabla}_{E^{\mathcal{H}}} F^{\mathcal{H}})^{\mathcal{V}} + (\overline{\nabla}_{E^{\mathcal{H}}} F^{\mathcal{V}})^{\mathcal{H}}$$

for vector fields E and F on M. Here $\overline{\nabla}$ denotes the Levi-Civita connection on M.

A smooth map $\varphi : (M^n, g) \to (N^m, h)$ between Riemannian manifolds M and N of dimensions n and m, respectively, is called a harmonic morphism if φ preserves the harmonic structures of (M, g) and (N, h). In other words, $\varphi : (M^n, g) \to (N^m, h)$ is a harmonic morphism if it pull backs local harmonic functions to harmonic functions. It is well-known ([3], [6]) that a smooth map $\varphi : M \to N$ is a harmonic morphism if and only if φ is both harmonic and horizontally (weakly) conformal.

Let M^n be an n-dimensional complete Riemannian manifold and let P be a k-dimensional immersed submanifold of M. Then the tangent space of M can be decomposed into

$$TM = TP \oplus TP^{\perp}.$$

Define, for two tangent vectors X, Y on P, i.e., sections of TP, the symmetric 2-tensor $B(X, Y)$ by

$$B(X, Y) = (\overline{\nabla}_X Y)^{\perp} = (\overline{\nabla}_X Y)^{\mathcal{H}} = T_X Y,$$

where $\overline{\nabla}$ is the Levi-Civita connection on M and $\perp$ denotes the normal component. We say P is minimal if the mean curvature

$$\sum_{i=1}^{k} B(e_i, e_i) = \sum_{i=1}^{k} T_{e_i} e_i = 0,$$

where $\{e_i, \cdots, e_k\}$ is a local orthonormal frame on P.

Let E be a normal vector field on P with compact support. Then the second derivative of the volume functional $\mathcal{A}$ in the direction E ([9]) is given by

$$\mathcal{A}''_E(0) = \int_P \langle -\Delta E + \overline{\mathcal{R}}(E) - \mathcal{B}(E), E\rangle. \tag{2.1}$$

Introducing a local orthonormal basis $\{e_1, \cdots, e_k, \xi_{k+1}, \cdots, \xi_n\}$ on TM such that $\{\xi_{k+1}, \cdots, \xi_n\}$ is a local orthonormal frame on $TP^{\perp}$, the equation (2.1) becomes

$$\mathcal{A}''_E(0) = \int_P |\nabla^{\perp} E|^2 - \sum_{i=1}^{k} \langle \overline{R}(e_i, E)E, e_i\rangle - \sum_{i,j=1}^{k} \langle B(e_i, e_j), E\rangle^2 \tag{2.2}$$

Here $\nabla^\perp$ denotes the normal connection on $TP^\perp$. In other words,

$$\overline{\mathcal{R}}(E) = \sum_{i=1}^{n-2} \overline{R}(e_i, E)e_i$$

and

$$\langle \mathcal{B}(E), E \rangle = \sum_{i,j=1}^{k} \langle B(e_i, e_j), E \rangle^2. \tag{2.3}$$

Generally, the operator $\mathcal{B}$ is given by

$$\langle \mathcal{B}(E), F \rangle = \sum_{i,j=1}^{k} \langle B(e_i, e_j), E \rangle \langle B(e_i, e_j), F \rangle$$

and it is well-known that P is totally geodesic if and only if $B = 0$ and hence $\mathcal{B} = 0$.

We say a minimally immersed submanifold P of M is *stable* (or volume-stable) if, for any normal variation E with compact support, the second derivative of the volume functional in the direction E is non-negative, i. e.,

$$\mathcal{A}''_E(0) \geq 0.$$

3. Basic Formulae

Using curvature formula for horizontally (weakly) conformal maps, we shall derive an integral formula for the second derivative of the volume functional for the fibers of horizontally conformal maps.

Let $\varphi : M^n \to N^m$ be a horizontally (weakly) conformal map between Riemannian manifolds (M, g) and (N, h). Suppose for a point $z \in N$, the fiber $P := \varphi^{-1}(z)$ is a k-dimensional minimal submanifold of M. Then the tangent vectors to P correspond vertical vectors of φ and normal vectors to P correspond to horizontal vectors of φ. From now we shall carry out some computations for the integrands in the second derivative of the volume functional for the minimal submanifold P. First of all, note that if E is a normal vector field on P,

$$\left|\nabla^\perp E\right|^2 = \sum_{i=1}^{k} \left|\nabla^\perp_{e_i} E\right|^2 = \sum_{i=1}^{k} \left|\left(\overline{\nabla}_{e_i} E\right)^\perp\right|^2 = \sum_{i=1}^{k} \left|\left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}}\right|^2. \tag{3.1}$$

Next, recall that

$$\begin{aligned}\langle\overline{\mathcal{R}}(E), E\rangle &= \sum_{i=1}^{k}\langle\overline{R}(e_i, E)e_i, E\rangle = -\sum_{i=1}^{k}\langle\overline{R}(e_i, E)E, e_i\rangle \\ &= -|E|^2\sum_{i=1}^{k} K_M\left(\frac{E}{|E|}\wedge e_i\right),\end{aligned}$$

where $K_M\left(\frac{E}{|E|}\wedge e_i\right)$ is the sectional curvature of the plane spanned by $\frac{E}{|E|}$ and e_i on M.

By [5] or [8], the sectional curvature is given by

$$\begin{aligned}K_M\left(\frac{E}{|E|}\wedge e_i\right) &= \left|A_{\frac{E}{|E|}}e_i\right|^2 - \left|T_{e_i}\frac{E}{|E|}\right|^2 + \left\langle\left(\overline{\nabla}_{\frac{E}{|E|}}T\right)_{e_i} e_i, \frac{E}{|E|}\right\rangle \\ &\quad -\frac{1}{2}\left\langle\nabla\log\lambda^2, e_i\right\rangle^2 + \frac{1}{2}\left\langle\overline{\nabla}_{e_i}\left(\nabla\log\lambda^2\right)^{\mathcal{V}}, e_i\right\rangle,\end{aligned} \tag{3.2}$$

where ∇ denotes the gradient on M. So the gradient on P is the vertical component of the gradient on M. In other words,

$$\nabla_P f = (\nabla f)^{\mathcal{V}}.$$

Thus from now on, we shall use the confused notation for the gradient on M and P since there are no ambiguities.

By definitions of tensors A and T, one obtains

$$A_{\frac{E}{|E|}}e_i = \left(\overline{\nabla}_{\frac{E}{|E|}}e_i\right)^{\mathcal{H}} = \frac{1}{|E|}\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}} = \frac{1}{|E|}A_E e_i. \tag{3.3}$$

So,

$$\left|A_{\frac{E}{|E|}}e_i\right|^2 = \frac{1}{|E|^2}\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}}\right|^2 = \frac{1}{|E|^2}\left|\left(\overline{\nabla}_E e_i\right)^{\perp}\right|^2. \tag{3.4}$$

$$T_{e_i}\frac{E}{|E|} = \left(\overline{\nabla}_{e_i}\frac{E}{|E|}\right)^{\mathcal{V}} = \left(\frac{1}{|E|}\overline{\nabla}_{e_i}E + e_i\left(\frac{1}{|E|}\right)E\right)^{\mathcal{V}} \tag{3.5}$$

$$= \frac{1}{|E|}\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{V}} \tag{3.6}$$

So,

$$\left|T_{e_i}\frac{E}{|E|}\right|^2 = \frac{1}{|E|^2}\left|\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{V}}\right|^2 = \frac{1}{|E|^2}\left|T_{e_i}E\right|^2. \tag{3.7}$$

Lemma 3.1.

$$\left(\overline{\nabla}_{\frac{E}{|E|}} T\right)_{e_i} e_i = \frac{1}{|E|}\left(\overline{\nabla}_E T\right)_{e_i} e_i. \tag{3.8}$$

Proof. It follows from the fact that T is a tensor. In fact,

$$\begin{aligned}\left(\overline{\nabla}_{\frac{E}{|E|}} T\right)_{e_i} e_i &= \overline{\nabla}_{\frac{E}{|E|}} T_{e_i} e_i - T_{\overline{\nabla}_{\frac{E}{|E|}} e_i} e_i - T_{e_i} \overline{\nabla}_{\frac{E}{|E|}} e_i \\ &= \frac{1}{|E|}\overline{\nabla}_E T_{e_i} e_i - \frac{1}{|E|} T_{\overline{\nabla}_E e_i} e_i - T_{e_i}\left(\frac{1}{|E|}\overline{\nabla}_E e_i\right)\end{aligned}$$

Also since T is a tensor, we have

$$T_{e_i}\left(\frac{1}{|E|}\overline{\nabla}_E e_i\right) = \frac{1}{|E|} T_{e_i}\overline{\nabla}_E e_i.$$

Therefore,

$$\left(\overline{\nabla}_{\frac{E}{|E|}} T\right)_{e_i} e_i = \frac{1}{|E|}\left(\overline{\nabla}_E T\right)_{e_i} e_i.$$

□

Hence

(3.9)

$$\begin{aligned}\langle \overline{\mathcal{R}}(E), E\rangle = &-\sum_{i=1}^{k}\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}}\right|^2 + \sum_{i=1}^{k}\left|\left(\overline{\nabla}_{e_i} E\right)^{\mathcal{V}}\right|^2 - \left\langle \left(\overline{\nabla}_E T\right)_{e_i} e_i, E\right\rangle \\ &+ \frac{|E|^2}{2}\sum_{i=1}^{k}\left\langle \nabla \log\lambda^2, e_i\right\rangle^2 - \frac{|E|^2}{2}\sum_{i=1}^{k}\left\langle \overline{\nabla}_{e_i}\left(\nabla\log\lambda^2\right)^{\mathcal{V}}, e_i\right\rangle.\end{aligned}$$

Finally, by (2.3) and definition of the second fundamental form B,

$$\begin{aligned}\langle \mathcal{B}(E), E\rangle &= \sum_{i,j=1}^{k}\langle B(e_i,e_j), E\rangle^2 = \sum_{i,j=1}^{k}\langle \left(\overline{\nabla}_{e_i} e_j\right)^{\mathcal{H}}, E\rangle^2 \\ &= \sum_{i,j=1}^{k}\langle \overline{\nabla}_{e_i} e_j, E\rangle^2 = \sum_{i,j=1}^{k}\langle e_j, \overline{\nabla}_{e_i} E\rangle^2 = \sum_{i=1}^{k}\left|\left(\overline{\nabla}_{e_i} E\right)^{\mathcal{V}}\right|^2. \end{aligned} \tag{3.10}$$

Therefore, from (2.2), (3.1), (3.9) and (3.10),

(3.11)

$$\mathcal{A}''_E(0) = \int_P \sum_{i=1}^{k} \left(\left| \left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}} \right|^2 - \left| \left(\overline{\nabla}_{E} e_i\right)^{\mathcal{H}} \right|^2 \right) + \int_P \sum_{i=1}^{k} \left\langle \left(\overline{\nabla}_E T\right)_{e_i} e_i, E \right\rangle$$
$$+ \int_P \left\{ \frac{|E|^2}{2} \sum_{i=1}^{k} \langle \nabla \log \lambda^2, e_i \rangle^2 - \frac{|E|^2}{2} \sum_{i=1}^{k} \left\langle \overline{\nabla}_{e_i} \left(\nabla \log \lambda^2\right)^{\mathcal{V}}, e_i \right\rangle \right\}$$

Before closing this section, we shall prove a basic property for covariant derivatives of a horizontal vector field and a vertical vector field of a horizontally weakly conformal map. This fact will be used in later sections.

LEMMA 3.2. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion. If X is a horizontal vector field and V is a vertical vector field on M, then*

$$\left| \left(\overline{\nabla}_V X\right)^{\mathcal{H}} \right|^2 = \left| \left(\overline{\nabla}_X V\right)^{\mathcal{H}} \right|^2 .$$

Proof. Since X is a horizontal vector field and V is a vertical vector field, it is easy to see that

$$[X, V] = \overline{\nabla}_X V - \overline{\nabla}_V X$$

is a vertical vector field and so

$$\left(\overline{\nabla}_X V - \overline{\nabla}_V X\right)^{\mathcal{H}} = 0.$$

□

4. Codimension 1 and 2

In this section, we consider horizontally conformal submersions or submersive harmonic morphisms having vertical fibers of codimension one or two. In low codimensional case, one can obtain some conditions that a minimal fiber of a harmonic morphism or horizontally conformal submersion is to be volume-stable.

First, let $\varphi : (M^n, g) \to (N^1, h)$ be a horizontally conformal submersion with dilation λ, where N is an one-dimensional manifold. Suppose for a point $t \in N$, $\varphi^{-1}(t) := P$ is a minimal hypersurface in M. Introducing a local orthonormal frame $\{e_1, \cdots, e_{n-1}, \xi\}$ on M such that ξ is an unit normal vector field to P, the equation (3.11) becomes

(4.1)

$$\begin{aligned}\mathcal{A}''_E(0) &= \int_P \sum_{i=1}^{n-1} \left(\left| \left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}} \right|^2 - \left| \left(\overline{\nabla}_E e_i\right)^{\mathcal{H}} \right|^2 \right) \\ &\quad + \int_P \sum_{i=1}^{n-1} \left\langle \left(\overline{\nabla}_E T\right)_{e_i} e_i, E \right\rangle \\ &\quad + \int_P \left\{ \frac{|E|^2}{2} \sum_{i=1}^{n-1} \langle \nabla \log \lambda^2, e_i \rangle^2 - \frac{|E|^2}{2} \sum_{i=1}^{n-1} \left\langle \overline{\nabla}_{e_i} \left(\nabla \log \lambda^2\right)^{\mathcal{V}}, e_i \right\rangle \right\}\end{aligned}$$

for any normal vector field E with compact support. We may assume ξ is a horizontal vector field on P with unit length, i. e.,

$$d\varphi(\xi) = \lambda \frac{d}{dt}, \tag{4.2}$$

where t is the standard coordinate so that $\frac{d}{dt}$ is an unit vector field on N.

Lemma 4.1. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion with dilation λ. Assume $P := \varphi^{-1}(z), z \in N$ is a submanifold of M. Then*

$$\lambda^2 \Delta_P \left(\frac{1}{\lambda^2} \right) = \left| \left(\nabla \log \lambda^2\right)^{\mathcal{V}} \right|^2 - \operatorname{div}_P (\nabla \log \lambda^2)^{\mathcal{V}},$$

where Δ_P and div_P denote the Laplacian and divergence on P, respectively, and ∇ denotes the gradient on M.

Proof. With the notation as in (4.2) and a local orthonormal frame $\{e_1, \cdots, e_{n-1}, \xi\}$ on M, let $X = \frac{1}{\lambda}\xi$ so that $\lambda^2 |X|^2 = 1$. Then the derivative in the direction e_i becomes

$$0 = e_i(\lambda^2 |X|^2)$$

and so

$$e_i(|X|^2) = -\frac{e_i(\lambda^2)}{\lambda^2} |X|^2 = -|X|^2 e_i(\log \lambda^2).$$

Thus,

$$\left(\nabla |X|^2\right)^{\mathcal{V}} = -|X|^2 \left(\nabla \log \lambda^2\right)^{\mathcal{V}},$$

In fact, the following identity holds between M and P:

$$\left(\nabla f\right)^{\mathcal{V}} = \nabla_P f = \sum_{i=1}^{n-1} e_i(f)e_i \qquad \text{(locally)}$$

for any function f defined on M.

Now by definition of the Laplacian,

$$\Delta_P|X|^2 = \mathrm{div}_P\left(\nabla|X|^2\right)^{\mathcal{V}} = -\langle\left(\nabla|X|^2\right)^{\mathcal{V}}, \left(\nabla\log\lambda^2\right)^{\mathcal{V}}\rangle - |X|^2\mathrm{div}_P\left(\nabla\log\lambda^2\right)^{\mathcal{V}}.$$

Since $\lambda^2\left(\nabla\left(\frac{1}{\lambda^2}\right)\right)^{\mathcal{V}} = -\left(\nabla\log\lambda^2\right)^{\mathcal{V}}$, one obtains

$$\left(\nabla|X|^2\right)^{\mathcal{V}} = \left(\nabla\left(\frac{1}{\lambda^2}\right)\right)^{\mathcal{V}} = -\frac{1}{\lambda^2}\left(\nabla\log\lambda^2\right)^{\mathcal{V}}. \tag{4.3}$$

Hence

$$\Delta_P\left(\frac{1}{\lambda^2}\right) = \Delta_P(|X|^2) = \frac{1}{\lambda^2}\left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2 - \frac{1}{\lambda^2}\mathrm{div}_P\left(\nabla\log\lambda^2\right)^{\mathcal{V}},$$

That is,

$$\lambda^2\Delta_P\left(\frac{1}{\lambda^2}\right) = \left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2 - \mathrm{div}_P\left(\nabla\log\lambda^2\right)^{\mathcal{V}}.$$

□

THEOREM 4.2. *Let $\varphi : (M^n, g) \to (N^1, h)$ be a horizontally conformal submersion and suppose $P = \varphi^{-1}(t), t \in N$ is a minimal hypersurface of M. If T is parallel, then P is volume-stable.*

Proof. Writing $E = f\xi$, one has

$$\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{H}} = e_i(f)\xi + f\left(\overline{\nabla}_{e_i}\xi\right)^{\mathcal{H}}.$$

Since the codimension of P is one and ξ is an unit vector field,

$$\langle\xi, \overline{\nabla}_{e_i}\xi\rangle = 0$$

and so

$$\left(\overline{\nabla}_{e_i}\xi\right)^{\mathcal{H}} = 0.$$

Thus,

$$\sum_{i=1}^{n-1}\left|\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{H}}\right|^2 = \left|(\nabla f)^{\mathcal{V}}\right|^2.$$

Now since ξ is a horizontal vector field, it follows from Lemma 3.2 that
$$\left(\overline{\nabla}_\xi e_i\right)^{\mathcal{H}} = \left(\overline{\nabla}_{e_i}\xi\right)^{\mathcal{H}} = 0$$
and so
$$\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}}\right|^2 = f^2\left|\left(\overline{\nabla}_\xi e_i\right)^{\mathcal{H}}\right|^2 = 0.$$
Therefore by Lemma 4.1 and the equation (4.1),
$$\begin{aligned}\mathcal{A}''_E(0) &= \int_P \left|(\nabla f)^{\mathcal{V}}\right|^2 + \frac{1}{2}\int_P f^2\left[\left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2 - \operatorname{div}_P(\nabla\log\lambda^2)^{\mathcal{V}}\right]\\ &= \int_P \left|(\nabla f)^{\mathcal{V}}\right|^2 + \frac{1}{2}\int_P f^2\lambda^2\Delta\left(\frac{1}{\lambda^2}\right).\end{aligned}\tag{4.4}$$
Now assume P is compact without boundary or P is non-compact and f has compact support. Applying the integration by parts, one has
$$\begin{aligned}\frac{1}{2}\int_P f^2\lambda^2\Delta\left(\frac{1}{\lambda^2}\right) &= -\frac{1}{2}\int_P\left\langle\nabla(f^2\lambda^2),\nabla\left(\frac{1}{\lambda^2}\right)\right\rangle\\ &= -\frac{1}{2}\int_P\left\langle\lambda^2\nabla f^2 + f^2\nabla\lambda^2,\nabla\left(\frac{1}{\lambda^2}\right)\right\rangle\\ &= \frac{1}{2}\int_P\left\langle\nabla f^2,\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right\rangle - \frac{1}{2}\int_P f^2\left\langle\nabla\lambda^2,\nabla\left(\frac{1}{\lambda^2}\right)\right\rangle\\ &= \int_P f\left\langle\nabla f,\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right\rangle + 2\int_P f^2\frac{|\nabla\lambda|^2}{\lambda^2}\\ &= \int_P f\left\langle\nabla f,\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right\rangle + \frac{1}{2}\int_P f^2\left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2.\end{aligned}$$
Here ∇ denotes the gradient on P (Actually, we used a confused notation for gradients on P and M, but there are no ambiguities).

By Schwarz inequality,
$$\begin{aligned}&\int_P f\left\langle\nabla f,\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right\rangle + \frac{1}{2}\int_P f^2\left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2\\ &\geq -\frac{1}{2}\int_P\left(|\nabla f|^2 + f^2\left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2\right) + \frac{1}{2}\int_P f^2\left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2\\ &= -\frac{1}{2}\int_P|\nabla f|^2\end{aligned}$$
Hence by (4.4),
$$\mathcal{A}''_E(0) = \frac{1}{2}\int_P\left|(\nabla f)^{\mathcal{V}}\right|^2 \geq 0.$$

□

COROLLARY 4.3. Let $\varphi : (M^n, g) \to (N^1, h)$ be a horizontally conformal submersion whose fibers are totally geodesics. Then every fiber P is volume-stable.

Now, let $\varphi : (M^n, g) \to (N^2, h)$ be a horizontally conformal submersion with dilation λ, and N is a two-dimensional Riemannian manifold. Suppose for a point $t \in N$, $\varphi^{-1}(t) := P$ is a minimal submanifold of M with codimension 2. Introducing a local orthonormal frame $\{e_1, \cdots, e_{n-2}, \xi_1, \xi_2\}$ on M such that ξ_1 and ξ_2 are unit normal vector fields to P, the equation (3.11) becomes

(4.5)
$$\mathcal{A}''_E(0) = \int_P \sum_{i=1}^{n-2} \left(\left| \left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}} \right|^2 - \left| \left(\overline{\nabla}_E e_i\right)^{\mathcal{H}} \right|^2 \right) + \int_P \sum_{i=1}^{n-2} \left\langle \left(\overline{\nabla}_E T\right)_{e_i} e_i, E \right\rangle$$
$$+ \int_P \left\{ \frac{|E|^2}{2} \sum_{i=1}^{n-2} \langle \nabla \log \lambda^2, e_i \rangle^2 - \frac{|E|^2}{2} \sum_{i=1}^{n-2} \left\langle \overline{\nabla}_{e_i} \left(\nabla \log \lambda^2\right)^{\mathcal{V}}, e_i \right\rangle \right\}$$

for any normal vector field E on P with compact support.

Let $\check{X}_1$ and $\check{X}_2$ be a local orthonormal frame on N and $\overline{X}_1$ and $\overline{X}_2$ be their horizontal lifts, respectively. Define

$$\xi_i = \frac{\overline{X}_i}{|\overline{X}_i|}.$$

Since $\overline{X}_j$ (and so ξ_j) is horizontal vector field and e_i is vertical, it follows from Lemma 3.2 that

$$\left(\overline{\nabla}_{e_i} \xi_j\right)^{\mathcal{H}} = \left(\overline{\nabla}_{\xi_j} e_i\right)^{\mathcal{H}}. \tag{4.6}$$

Also the following lemma is well-known ([5], [8]).

LEMMA 4.4. *One has*

$$\left(\overline{\nabla}_{\xi_1} \xi_2\right)^{\mathcal{V}} = \frac{1}{2}[\xi_1, \xi_2]^{\mathcal{V}}.$$

Note that, in fact, $\left(\overline{\nabla}_{\xi_1} \xi_2\right)^{\mathcal{V}} = A_{\xi_1} \xi_2$.

LEMMA 4.5.

$$\left\langle \overline{\nabla}_{e_i} \xi_1, \xi_2 \right\rangle = -\frac{1}{2} \left\langle [\xi_1, \xi_2]^{\mathcal{V}}, e_i \right\rangle.$$

Proof. Since e_i is vertical (tangent to P) and ξ_j is horizontal (normal to P), it follows from Lemma 4.4 and the equation (4.6) that

$$\begin{aligned}\langle \overline{\nabla}_{e_i}\xi_1, \xi_2\rangle &= \left\langle \left(\overline{\nabla}_{e_i}\xi_1\right)^{\mathcal{H}}, \xi_2\right\rangle = \left\langle \left(\overline{\nabla}_{\xi_1}e_i\right)^{\mathcal{H}}, \xi_2\right\rangle = \langle \overline{\nabla}_{\xi_1}e_i, \xi_2\rangle \\ &= -\left\langle \left(\overline{\nabla}_{\xi_1}\xi_2\right)^{\mathcal{V}}, e_i\right\rangle = -\frac{1}{2}\left\langle [\xi_1,\xi_2]^{\mathcal{V}}, e_i\right\rangle.\end{aligned}$$

□

Let E be a normal vector field on P and write

$$E = f_1\xi_1 + f_2\xi_2$$

on P. Then

$$\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{H}} = \sum_{j=1}^{2} e_i(f_j)\xi_j + \sum_{j=1}^{2} f_j\left(\overline{\nabla}_{e_i}\xi_j\right)^{\mathcal{H}}.$$

Since ξ_1 and ξ_2 are unit normal vector fields,

$$\langle \xi_j, \overline{\nabla}_{e_i}\xi_j\rangle = 0 \qquad (j=1,2). \tag{4.7}$$

So,

$$\begin{aligned}\left|\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{H}}\right|^2 &= \sum_{j=1}^{2}\left\{ e_i(f_j)^2 + f_j^2\left|\left(\overline{\nabla}_{e_i}\xi_j\right)^{\mathcal{H}}\right|^2\right\} \\ &\quad + \sum_{j=1}^{2}\left\{2f_2e_i(f_1)\langle \xi_1, \overline{\nabla}_{e_i}\xi_2\rangle + 2f_1e_i(f_2)\langle \xi_2, \overline{\nabla}_{e_i}\xi_1\rangle\right\} \\ &\quad + \sum_{j=1}^{2}\left\{2f_1f_2\left\langle \left(\overline{\nabla}_{e_i}\xi_1\right)^{\mathcal{H}}, \left(\overline{\nabla}_{e_i}\xi_2\right)^{\mathcal{H}}\right\rangle\right\}.\end{aligned}$$

On the other hand, since $\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}} = \sum_{j=1}^{2} f_j\left(\overline{\nabla}_{\xi_j}e_i\right)^{\mathcal{H}}$, one has

$$\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}}\right|^2 = \sum_{j=1}^{2} f_j^2\left|\left(\overline{\nabla}_{\xi_j}e_i\right)^{\mathcal{H}}\right|^2 + 2f_1f_2\left\langle \left(\overline{\nabla}_{\xi_1}e_i\right)^{\mathcal{H}}, \left(\overline{\nabla}_{\xi_2}e_i\right)^{\mathcal{H}}\right\rangle.$$

Thus

$$\begin{aligned}&\left|\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{H}}\right|^2-\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}}\right|^2\\&\quad=\sum_{j=1}^{2}e_i(f_j)^2+2(f_1e_i(f_2)-f_2e_i(f_1))\left\langle\overline{\nabla}_{e_i}\xi_1,\xi_2\right\rangle\end{aligned}$$

and so

$$\begin{aligned}&\sum_{i=1}^{n-2}\left|\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{H}}\right|^2-\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}}\right|^2\\&\quad=\sum_{j=1}^{2}|\nabla f_j|^2+2\sum_{i=1}^{n-2}(f_1e_i(f_2)-f_2e_i(f_1))\left\langle\overline{\nabla}_{e_i}\xi_1,\xi_2\right\rangle.\end{aligned}$$

Moreover, it follows from Lemma 4.5 that

$$\begin{aligned}&\sum_{i=1}^{n-2}(f_1e_i(f_2)-f_2e_i(f_1))\left\langle\overline{\nabla}_{e_i}\xi_1,\xi_2\right\rangle\\&\quad=-\frac{1}{2}\sum_{i=1}^{n-2}(f_1e_i(f_2)-f_2e_i(f_1))\left\langle[\xi_1,\xi_2]^{\mathcal{V}},e_i\right\rangle\\&\quad=-\frac{1}{2}\left\langle[\xi_1,\xi_2]^{\mathcal{V}},f_1\nabla f_2-f_2\nabla f_1\right\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}&\textstyle\sum_{i=1}^{n-2}\left|\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{H}}\right|^2-\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}}\right|^2\\&\quad=\textstyle\sum_{j=1}^{2}|\nabla f_j|^2-\left\langle[\xi_1,\xi_2]^{\mathcal{V}},f_1\nabla f_2-f_2\nabla f_1\right\rangle.\end{aligned}\tag{4.8}$$

Consequently, by (4.1), (4.8) and Lemma 4.1,

(4.9)

$$\begin{aligned}\mathcal{A}_E''(0)=&\int_P\left\{\sum_{j=1}^{2}|\nabla f_j|^2-\left\langle[\xi_1,\xi_2]^{\mathcal{V}},f_1\nabla f_2-f_2\nabla f_1\right\rangle+\frac{|E|^2}{2}\lambda^2\Delta\left(\frac{1}{\lambda^2}\right)\right\}\\&+\int_P\sum_{i=1}^{n-2}\left\langle\left(\overline{\nabla}_E T\right)_{e_i}e_i,E\right\rangle\end{aligned}$$

and $|E|^2=f_1^2+f_2^2$.

THEOREM 4.6. *Let $\varphi : (M^n, g) \to (N^2, h)$ be a horizontally conformal submersion with dilation λ, where N is a 2-dimensional Riemannian manifold. Suppose $P = \varphi^{-1}(z), z \in N$ is a minimal submanifold of M. If T is parallel and the horizontal distribution $\mathcal{H}$ is integrable, then P is volume-stable.*

Proof. Let E be a local normal vector field on P and assume E has compact support if P is non-compact. By hypotheses, one has

$$\overline{\nabla}_E T = 0 \qquad \text{and} \qquad [\xi_1, \xi_2]^{\mathcal{V}} = 0, \tag{4.10}$$

where ξ_1, ξ_2 are horizontal vector fields on M which are projectable. Thus, writing $E = f_1\xi_1 + f_2\xi_2$, one obtains, by (4.9),

$$\mathcal{A}''_E(0) = \int_P \left\{ \sum_{j=1}^{2} |\nabla f_j|^2 + \frac{f_1^2 + f_2^2}{2} \lambda^2 \Delta \left(\frac{1}{\lambda^2} \right) \right\}. \tag{4.11}$$

Now as in the proof of Theorem 4.2, one can obtain, for each $j = 1, 2$,

$$\frac{1}{2} \int_P f_j^2 \lambda^2 \Delta \left(\frac{1}{\lambda^2} \right) \geq -\frac{1}{2} \int_P |\nabla f_j|^2.$$

Therefore,

$$\mathcal{A}''_E(0) \geq \frac{1}{2} \int_P \sum_{j=1}^{2} |\nabla f_j|^2 \geq 0.$$

□

REMARK 4.7. Since the singular point of a smooth map $\varphi : (M^n, g) \to (N^2, h)$ is discrete, Theorem 4.6 could hold without the condition that φ is a submersion. In fact, if $\varphi : (M^n, g) \to (N^2, h)$ is a horizontally weakly conformal map with dilation λ, and a fiber $P = \varphi^{-1}(z), z \in N$ is a minimal submanifold of M, then any fiber near z is a smooth submanifold of M. Thus, every notion like a local frame and local vector fields is well-defined. Therefore, in this case, if T is parallel and the horizontal distribution $\mathcal{H}$ is integrable, then P is still volume-stable.

COROLLARY 4.8. Let $\varphi : (M^n, g) \to (N^2, h)$ be a submersive harmonic morphism from an n-dimensional Riemannian manifold M^n to a 2-dimensional Riemannian manifold N^2. If T is parallel and the horizontal distribution $\mathcal{H}$ is integrable, then any fiber is volume-stable.

Proof. Since $\dim(N) = 2$ and φ is a harmonic morphism, it is well-known ([1]) that any fiber is an $(n-2)$-dimensional minimal submanifold. Thus the corollary follows from Theorem 4.6. □

Corollary 4.9. Let $\varphi : (M^n, g) \to (N^2, h)$ be a submersive harmonic morphism with totally geodesic fibers. If the horizontal distribution $\mathcal{H}$ is integrable, then any fiber is volume-stable.

Remark 4.10. In Theorem 4.6, Corollary 4.8 or Corollary 4.9, the condition that the horizontal distribution $\mathcal{H}$ is integrable is indispensable. For instance, the Hopf map $\varphi : S^3 \to S^2$ is a submersive harmonic morphism with totally geodesic fibers, but the fibers are not volume-stable.

In [11], Montaldo proved if a submersive harmonic morphism $\varphi : (M^n, g) \to (N^2, h)$ from a compact Riemannian manifold to a surface has volume-stable minimal fibers, then φ is energy-stable, that is, the second derivative of the energy functional is non-negative. Thus Corollary 4.8 and Corollary 4.9 imply the following corollaries.

Corollary 4.11. Let $\varphi : (M^n, g) \to (N^2, h)$ be a submersive harmonic morphism from a compact n-dimensional Riemannian manifold M^n to a 2-dimensional Riemannian manifold N^2. If T is parallel and the horizontal distribution $\mathcal{H}$ is integrable, then φ is energy-stable harmonic map.

Corollary 4.12. Let $\varphi : (M^n, g) \to (N^2, h)$ be a submersive harmonic morphism with totally geodesic fibers and M is compact. If the horizontal distribution $\mathcal{H}$ is integrable, then φ is energy-stable.

The converse for Corollary 4.11 or Corollary 4.12 is not true anymore. In fact, it is known ([11]) that the quotient map $\overline{\varphi} : \mathbb{R}P^3 \to S^2$ of the Hopf map $\varphi : S^3 \to S^2$ is energy-stable, but the horizontal distribution of $\overline{\varphi}$ is not integrable.

5. Higher Codimensions and Harmonic p-forms

In the previous section, we considered a horizontally conformal submersion or a harmonic morphism of codimension one or two. In this section, we shall consider general higher codimensional case.

Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion with dilation λ from an n-dimensional Riemannian manifold M^n to an m-dimensional Riemannian manifold N^m. Suppose $n \geq m \geq 3$. In case that φ is a harmonic morphism, it is well-known ([1]) that φ has minimal fibers if and only if φ is horizontally homothetic, i.e., the horizontal component of the gradient of the dilation is vanishing.

Let $\check{X}_1, \cdots, \check{X}_m$ be a local orthonormal frame on N and let $\overline{X}_1, \cdots, \overline{X}_m$ be their horizontal lifts. Set

$$\xi_j = \frac{X_j}{|X_j|} = \frac{X_j}{\lambda}$$

so that $\xi_1, \cdots, \xi_m$ form a local orthonormal frame on horizontal distribution $\mathcal{H}$. Let $k = n - m$ and assume $P := \varphi^{-1}(z), z \in N$ is a smooth submanifold of M. Then near z the fibers of φ are also smooth submanifolds of M. Choose a local orthonormal frame $\{e_1, \cdots, e_k\}$ on vertical distribution $\mathcal{V}$ near z so that $\{e_1, \cdots, e_k, \xi_1, \cdots, \xi_m\}$ forms a local orthonormal frame on TM.

Let E be a normal vector field on P with compact support and write

$$E = \sum_{j=1}^{m} f_j \xi_j.$$

As in the previous section, using Lemma 4.5, one can compute

$$\begin{aligned} &\sum_{i=1}^{k} \left(\left| \left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}} \right|^2 - \left| \left(\overline{\nabla}_{E} e_i\right)^{\mathcal{H}} \right|^2 \right) \\ &= \sum_{j=1}^{m} |\nabla f_j|^2 + \sum_{\substack{j,l=1 \\ j \neq l}}^{n} \left\langle [\xi_j, \xi_l]^{\mathcal{V}}, f_j \nabla f_j - f_j \nabla f_l \right\rangle, \end{aligned}$$

where ∇ denotes the gradient on P. So, by Lemma 4.1 and the equation (3.11),

$$\begin{aligned} \mathcal{A}''_E(0) = &\sum_{j=1}^{m} \int_P \left\{ |\nabla f_j|^2 + \frac{1}{2} f_j^2 \lambda^2 \Delta \left(\frac{1}{\lambda^2} \right) \right\} \\ &+ \sum_{\substack{j,l=1 \\ j \neq l}}^{n} \int_P \left\langle [\xi_j, \xi_l]^{\mathcal{V}}, f_j \nabla f_j - f_j \nabla f_l \right\rangle + \int_P \sum_{i=1}^{k} \left\langle \left(\overline{\nabla}_E T\right)_{e_i} e_i, E \right\rangle. \end{aligned}$$

As in the proof of Theorem 4.2, applying the integration by parts and Schwarz inequality, one has, for each $j = 1, \cdots, m$,

$$\int_P f_j^2 \lambda^2 \Delta \left(\frac{1}{\lambda^2} \right) \geq - \int_P |\nabla f_j|^2.$$

Hence

$$(5.1)\quad \mathcal{A}''_E(0) \geq \frac{1}{2}\sum_{j=1}^{m}\int_P |\nabla f_j|^2 + \sum_{\substack{j,l=1\\ j\neq l}}^{n}\int_P \left\langle [\xi_j,\xi_l]^{\mathcal{V}}, f_j\nabla f_j - f_j\nabla f_l\right\rangle + \int_P \sum_{i=1}^{k}\left\langle \left(\overline{\nabla}_E T\right)_{e_i} e_i, E\right\rangle.$$

Therefore one obtains the following theorem.

THEOREM 5.1. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion from an n-dimensional Riemannian manifold M^n to an m-dimensional Riemannian manifold N^m $(n \geq m \geq 3)$. Suppose a fiber of φ, $P = \varphi^{-1}(z), z \in N$ is a minimal submanifold of M. If the horizontal distribution $\mathcal{H}$ is integrable and the tensor T is parallel, then P is volume-stable.*

COROLLARY 5.2. Let $\varphi : (M^n, g) \to (N^m, h)$ be a submersive harmonic morphism with totally geodesic fibers. If the horizontal distribution $\mathcal{H}$ is integrable, then every fiber is volume-stable.

REMARK 5.3. Let $\varphi : (M^n, g) \to (N^2, h)$ be a horizontally conformal submersion with dilation λ to a 2-dimensional manifold N and let $P = \varphi^{-1}(z), z \in N$ is a smooth submanifold of M. Let E be a normal vector field on P with compact support. Recall, by the equation (2.2), the second derivative of the volume functional in the direction is given by

$$\mathcal{A}''_E(0) = \int_P |\nabla^{\perp} E|^2 - \sum_{i=1}^{n-2}\langle \overline{R}(e_i, E)E, e_i\rangle - \sum_{i,j=1}^{n-2}\langle B(e_i, e_j), E\rangle^2,$$

where $\{e_1, \cdots, e_{n-2}\}$ is a local orthonormal frame on vertical distribution of φ.

Let $\{\xi_1, \xi_2\}$ be a local orthonormal frame on the horizontal distribution. Then it follows from the definition of Ricci curvature and $\dim(N) = 2$ that

$$\sum_{i=1}^{n-2}\langle \overline{R}(e_i, E)E, e_i\rangle = \overline{Ric}(E, E) + |E|^2\langle \overline{R}(\xi_1, \xi_2)\xi_2, \xi_1\rangle,$$

where $\overline{Ric}$ and $\overline{R}$ denote the Ricci curvature and Riemannian curvature tensor of M, respectively. Thus,

$$\mathcal{A}''_E(0) = \int_P |\nabla^\perp E|^2 - \overline{Ric}(E,E) + |E|^2 \langle \overline{R}(\xi_1,\xi_2)\xi_2,\xi_1\rangle - \sum_{i,j=1}^{n-2} \langle B(e_i,e_j),E\rangle^2,$$

On the other hand, if φ is horizontally homothetic, one can compute ([5], [8]) that

$$\langle \overline{R}(\xi_1,\xi_2)\xi_2,\xi_1\rangle = \lambda^2 K_N \circ \varphi - \frac{1}{4}\left|\left(\overline{\nabla}\log\lambda^2\right)^{\mathcal{V}}\right| - \frac{3}{4}\left|[\xi_1,\xi_2]^{\mathcal{V}}\right|^2,$$

where K_N denotes the sectional curvature of N. Hence if P is totally geodesic, one obtains

$$\begin{aligned} (5.2)\quad \mathcal{A}''_E(0) &= \int_P |\nabla^\perp E|^2 - \overline{Ric}(E,E) \\ &\quad + |E|^2 \left(\lambda^2 K_N \circ \varphi - \frac{1}{4}\left|\left(\overline{\nabla}\log\lambda^2\right)^{\mathcal{V}}\right| - \frac{3}{4}\left|[\xi_1,\xi_2]^{\mathcal{V}}\right|^2\right). \end{aligned}$$

Assume the Ricci curvature of M is non-positive and the sectional curvature of N is non-negative. An easy observation is that if the horizontal distribution is integrable and the dilation is constant, then the fiber is volume-stable. In higher codimensional case, we can also a similar formula as the equation (5.2).

Finally, we shall consider p-harmonic morphisms ($p \geq 2$). A smooth map $\varphi : (M^n, g) \to (N^m, h)$ between Riemannian manifolds is called a p-harmonic map ($p \geq 2$) if φ is a critical point of the p-energy functional

$$E_p(\varphi;\Omega) = \frac{1}{p}\int_\Omega |d\varphi|^p \, dv_g$$

for all compact subsets $\Omega \subset M$.

A map $\varphi : (M^n, g) \to (N^m, h)$ is called a p-harmonic morphism if it pulls back (local) p-harmonic functions on N to (local) p-harmonic functions on M. In [10], Loubeau proved that a map $\varphi : (M^n, g) \to (N^m, h)$ is a p-harmonic morphism if and only if it is a horizontally weakly conformal p-harmonic map.

On the other hand, Jin and Mo ([7]) proved that if $\varphi : (M^n, g) \to (N^m, h)$ is a horizontally conformal submersion whose fibers are all volume-stable minimal submanifolds of a compact manifold M, then φ is an energy-stable m-harmonic morphism, where $m = \dim(N)$.

Also, Baird and Gudmundsson ([2]) proved a horizontally conformal submersion $\varphi : (M^n, g) \to (N^m, h)$ is an m-harmonic map if and only if all the fibers of φ are minimal submanifolds of M. Thus, we have the following result.

THEOREM 5.4. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a submersive m-harmonic morphism from a compact Riemannian manifold M. If T is parallel and the horizontal distribution $\mathcal{H}$ is integrable, then φ is energy-stable.*

Proof. It follows from [2] that all the fibers of φ are minimal submanifolds of M. By Theorem 5.1, all fibers are volume-stable and so the conclusion follows from [7]. □

COROLLARY 5.5. Let $\varphi : (M^n, g) \to (N^m, h)$ be a submersive m-harmonic morphism with totally geodesic fibers from a compact Riemannian manifold M. If the horizontal distribution $\mathcal{H}$ is integrable, then φ is energy-stable.

References

[1] P. Baird and J. Eells, A conservation law for harmonic maps, Lecture Notes in Mathematics, 894, Springer(1981), pp. 1-25.
[2] P. Baird and S. Gudmundsson, p-harmonic maps and minimal submanifolds, Math. Ann., 294 (1992), 611-624.
[3] B. Fuglede, Harmonic morphisms between Riemannian manifolds, Ann. Inst. Fourier, Grenoble 28 (1978), 107-144.
[4] R. Harvey and H. B. Lawson, Calibrated Geometries, Acta Math., 148 (1982), 47-157.
[5] S. Gudmundsson, The Geometry of Harmonic Morphisms, Ph. D Thesis, 1992.
[6] T. Ishihara, A mappings of Riemannian manifolds which preserves harmonic functions, J. Math. Kyoto Univ. 19 (1979), 215-229.
[7] H. Jin and X. Mo, On submersive p-harmonic morphisms and their stability, Contemporary Mathematics, 308 (2002), 205-209.
[8] A. Kasue and T. Washio, Growth of equivariant harmonic maps and harmonic morphisms, Osaka Jour. of Math., 27 (1990), 899-928.
[9] H. B. Lawson, Lectures on minimal submanifolds, Mathematics Lecture Series, 9, Publish or Perish, 1980.
[10] E. Loubeau, On p-harmonic morphisms, Differential Geom. Appl., 12 (2000), 219-229.
[11] S. Montaldo, Stability of harmonic morphisms to a surface, Intern. J. Math., 9 (1998), 865-875.
[12] B. O'Neill, The fundamental equations of a submersion, Mich. Math. J., 13 (1966), 459-469.

[13] H. Rummler, Quelques notions simples en géométrie riemannienne et leurs applications aus fluilletages compacts, Comment. Math. Helv., 54 (1979), 224-239.

Gabjin Yun
Department of Mathematics
Myong Ji University
Yongin, 449–728, Korea
E-mail: gabjin@mju.ac.kr

Gundon Choi
GARC and Department of Mathematics
Seoul National University
Seoul, 151–747, Korea
E-mail: cgd@math.snu.ac.kr