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A Mixture of Multivariate Distributions with Pareto in Reliability Models

Awad El-Gohary
Department of Statistics and O.R., College of Science
King Saud University, P.O. Box 2455 Riyadh 11451, Saudi Arabia

Abstract. This paper presents a new class of multivariate distributions with Pareto where dependence among the components is characterized by a latent random variable. The new class includes several multivariate and bivariate models of Marshall and Olkin type. It is found the bivariate distribution with Pareto is positively quadrant dependent and its mixture. Some important structural properties of the bivariate distributions with Pareto are discussed. The distribution of minimum in a competing risk Pareto model is derived.

Key Words : *mixture, multivariate distributions with Pareto, positively quadrant dependent, competing risk models, distribution of minimum.*

1. INTRODUCTION

Sometimes failure rate can occur for more than one reason and the mixture of multivariate distribution is a nice tool for modeling such situation. For example, assume that T_1 and T_2 are the times at that two specific components of an electronic system fail. If these components will fail at the same time with probability p, then their common failure time may be distributed according to some univariate distributions. On the other hand these components will fail at different times with probability $1-p$, and in this case their failure times should be distributed according to some bivariate distributions.

The reliability analysis and electronics widely use the univariate distributions such as exponential, Pareto and linear failure rate, see for example Sarhan and El-Gohary (2001) and El-Gohary (2005).

Corresponding Author.
E-mail address: aigohary@ksu.edu.sa
Permanent address: Department of Mathematics, College of Science, Mansoura University, Mansoura 35516. EGYPT

Mardia (1962) introduced two families of bivariate Pareto distributions with the property that both of marginal distributions are univariate Pareto form. Also he discussed the estimation of parameters in the bivariate distributions with Pareto. Arnold (1993) studied two classes of k-dimensional distributions with generalized Pareto conditionals. Such study introduced the general functional equations characterizing distributions with generalized Pareto conditional and identify and characterize two subclasses of such distributions. Gupta (2001) studied a class of bivariate distributions with Pareto conditional from a reliability point of view. In this study, failure rate and mean residual life function of marginal distributions and their monotonic properties are investigated.

Several basic multivariate parametric families of distributions such as multivariate exponential, linear failure rate distributions, and shock models that give rise to them are considered by Barlow and Proschan (1981), El-Gohary (2004). Earlier, Marshall and Olkin (1967a) considered a shock model to derive a bivariate exponential distribution. Generalization of bivariate exponential distribution is proposed by Marshall and Olkin (1967b).

The present paper is organized as follows. Section two deals with the multivariate mixture of Pareto distributions with a Pareto latent random variable. Section three introduces the bivariate Pareto distributions. The joint probability density function of a new class of bivariate Pareto distributions is derived. The bivariate dependence of this class is investigated. Section four deals with the mixture of bivariate Pareto distributions and its bivariate dependence. Section five presents competing risk Pareto models. Finally, some properties of mixture of bivariate Pareto distributions are presented.

2. THE PARETO MODEL

This section concerns with the mixture of the second kind pareto distributions and we derive a multivariate distribution where dependence among the components is characterized by a latent Pareto random variable independently distributed of the individual component. Also we develop a bivariate pareto distribution with a latent random variable independently distributed of the individual components.

We consider an n-component system where the lifetime of $i-th$ component, namely X_i has a mixture of pareto distributions, $i = 1, 2, \dots, n$. That is

$$X_i \sim \textstyle\sum_{j=1}^{k} a_{ij} X_{ij},\ X_{ij} \sim P(a, \theta_{ij}),\ j = 1, 2, \dots, k \tag{2.1}$$

where the notation $P(a, \theta_{ij})$ means a random variable, say X_{ij}, having a pareto distribution with the parameters (a, θ_{ij}) and its density function is given as

$$f_{X_{ij}}(x) = \Big(\frac{\theta_{ij}}{a+x}\Big)\bar{F}_{X_{ij}}(x),\ x \geq 0,\ a > 0,\ \theta_{ij} > 0, \tag{2.2}$$

where $\bar{F}_{X_{ij}}(x)$ is the survival function of the random variable X_{ij} which is given by

$$\bar{F}_{X_{ij}}(x) = \Big(\frac{a}{a+x}\Big)^{\theta_{ij}},\ x \geq 0,\ a > 0,\ \theta_{ij} > 0, \tag{2.3}$$

and $\vec{a_i} = (a_{i1}, \dots, a_{ik})$ is the vector of mixing probabilities corresponding to $i-th$ component. That is

$$\sum_{j=1}^{k} a_{ij} = 1,\ \text{and}\, a_{ij} \geq 0,\ \forall\, i, j \tag{2.4}$$

Next, we introduce a pareto random variable Z, with parameters a and θ which is independent from X_{ij} for all i, j. The random variable Z will be used as a latent random variable to introduce dependence among $X_i's$. The density function of this latent variable is given by

$$f_Z(z) = \Big(\frac{\theta}{a+x}\Big)\bar{F}_Z(z),\ z \geq 0,\ a > 0,\ \theta > 0. \tag{2.5}$$

where $\bar{F}_Z(z)$ is the survival function of Z which given by

$$\bar{F}_Z(z) = \Big(\frac{a}{a+x}\Big)^{\theta},\ z \geq 0,\ a > 0,\ \theta > 0. \tag{2.6}$$

Using the assumption of our model the latent random variable Z is also independent of X_i for all $(i = 1, 2, \dots, n)$, we define the vector of multivariate distribution $\vec{S} = (S_1, S_2, \dots, S_n)$ where $S_i = \min(X_i, Z)$ for all $(i = 1, 2, \dots, n)$ and obviously they are dependent as they commonly share the influence of the latent random variable Z.

In what follows we introduce the joint of a multivariate survival function of the random variables $S_1, S_2, \dots, S_n$.

Corollary 2.1 The joint survival function of $S_1, S_2, \dots, S_n$ is given by

$$\bar{F}(s_1, s_2, \dots, s_n) = \prod_{i=1}^{n}\left\{\sum_{j=1}^{k} a_{ij}\Big(\frac{a}{a+s_i}\Big)^{\theta_{ij}}\Big(\frac{a}{a+s_0}\Big)^{\theta/n}\right\}, \tag{2.7}$$

where $s_0 = \max(s_1, s_2, \dots, s_n) > 0$.

Proof. The survival function of $S_1, S_2, \dots, S_n$ is defined by

$$\bar{F}(s_1, \dots, s_n) = P(S_1 > s_1, \dots, S_n > s_n)$$

Then using the definitions of $S_i = \min(X_i, Z)$ we get

$$\bar{F}(s_1, \dots, s_n) = P(X_1 > s_1)P(X_2 > s_2)\dots P(X_n > s_n)P(Z > s_0)$$

$$= \Big(\frac{a}{a+s_0}\Big)^{\theta}\prod_{i=1}^{n}\sum_{j=1}^{k} a_{ij}\Big(\frac{a}{a+s_i}\Big)^{\theta_{ij}} \tag{2.8}$$

But X_i is a mixture of $X_{ij} \sim P(a, \theta_{ij})$ hence one can write the above relation as given by (2.7), which completes the proof.

Obviously, the presence of the latent variable Z makes it is very difficult to calculate the multivariate density function of $S_1, S_2, \ldots, S_n$ as we have to take mixed derivatives over all possible partitions of the sample space.

3. BIVARIATE PARETO DISTRIBUTIONS

This section is concerned with the bivariate Pareto distributions. The bivariate Pareto density function (pdf) and its marginates will be derived. Also the bivariate dependence and many other important properties of the bivariate Pareto will be discussed in details.

3.1 Joint pdf of Pareto distributions

The following theorem provides an approach of obtaining the joint bivariate density when the component of the random variables can be equal with positive probability. This theorem presents the bivariate Pareto density function of the bivariate Pareto random variable (X_1, X_2).

Theorem 3.1 If the bivariate survival function $\bar{F}_{X_1,X_2}(x_1, x_2)$ of the bivariate random variable (X_1, X_2) with Pareto takes the following form:

$$\bar{F}_{X_1,X_2}(x_1, x_2) = \left(\frac{a}{a+x_1}\right)^{\theta_1} \left(\frac{a}{a+x_2}\right)^{\theta_2} \left(\frac{a}{a+z}\right)^{\theta}, \text{ where } z = \max(x_1, x_2) \quad (3.1)$$

then the joint bivariate density function of (X_1, X_2) is given by

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & x_1 > x_2 \\ f_2(x_1, x_2), & x_1 < x_2 \\ f_0(x_1, x_2), & x_1 = x_2 \end{cases} \quad (3.2)$$

where

$$f_1(x_1, x_2) = \frac{\theta_2(\theta_1 + \theta)}{a^2} \left(\frac{a}{a+x_1}\right)^{\theta_1+\theta+1} \left(\frac{a}{a+x_2}\right)^{\theta_2+1}$$

$$f_2(x_1, x_2) = \frac{\theta_1(\theta_2 + \theta)}{a^2} \left(\frac{a}{a+x_1}\right)^{\theta_1+1} \left(\frac{a}{a+x_2}\right)^{\theta_2+\theta+1}$$

$$f_0(x_1, x_2) = \frac{\theta}{a} \left(\frac{a}{a+x}\right)^{\theta+\theta_1+\theta_2+1}, \ x_1 = x_2 = x \quad (3.3)$$

Proof. The proof of this theorem is based on obtaining the forms of $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ by differentiating the joint survival $\bar{F}_{X_1,X_2}(x_1, x_2)$ that given by (3.1) with respect to both x_1 and x_2 twice times. In the other hand the function $f_0(x,x)$ will be obtained using the identity

$$\int_0^\infty \int_0^{x_1} f_1(x_1,x_2)dx_2dx_1 + \int_0^\infty \int_{x_1}^\infty f_2(x_1,x_2)dx_2dx_1 + \int_0^\infty f_0(x,x)dx = 1, \quad (3.4)$$

$$\int_0^\infty \int_0^{x_1} f_1(x_1,x_2)dx_2dx_1 = 1 - \int_0^\infty \Big(\frac{\theta+\theta_1}{a}\Big)\Big(\frac{a}{a+x_1}\Big)^{\theta+\theta_1+\theta_2+1} dx_1, \quad (3.5)$$

and

$$\int_0^\infty \int_{x_1}^\infty f_2(x_1,x_2)dx_2dx_1 = \int_0^\infty \Big(\frac{\theta_1}{a}\Big)\Big(\frac{a}{a+x_1}\Big)^{\theta+\theta_1+\theta_2+1} dx_1, \quad (3.6)$$

Substituting from (3.5) and (3.6) into (3.4) we get

$$\int_0^\infty \Big\{f_0(x,x) - \Big(\frac{\theta}{a}\Big)\Big(\frac{a}{a+x}\Big)^{\theta+\theta_1+\theta_2+1}\Big\}dx = 0, \quad (3.7)$$

Therefore, since this integral satisfies for all positive value of x, then the function $f_0(x,x)$ is given by

$$f_0(x,x) = \Big(\frac{\theta}{a}\Big)\Big(\frac{a}{a+x}\Big)^{\theta+\theta_1+\theta_2+1}, x > 0 \quad (3.8)$$

which completes the proof.

Note that joint density $f_{X_1,X_2}(x_1,x_2)$ of the bivariate random variable (X_1,X_2) is constant over intersections of lines $x_1 + a =$ const. and $x_2 + a =$ const.

Lemma 3.1 The marginal density function of the random variable X_i, $(i = 1,2)$ is given by:

$$f_{X_i}(x_i) = \Big(\frac{\theta+\theta_i}{a}\Big)\Big(\frac{a}{a+x_i}\Big)^{(\theta_i+\theta+1)}, \ i = 1,2 \quad (3.9)$$

which has a Pareto form.

The proof of this lemma can be done by integrating the joint pdf of (X_1,X_2) with respect to x_i.

From (3.9) we find that, the marginal probability density functions of X_i are also Pareto distributed with parameters $(\theta_i+\theta)$, $i = 1,2$. Also we can easily verified that the random variable X_i has decreasing failure rate which is given by

$$r_i(t) = \frac{\theta+\theta_i}{a+t}, \ t > 0, \ a, \ \theta, \ \theta_i > 0, \imath = 1,2$$

In a great many reliability situations, the random variables of interest are non-independent, but rather are associated for example structures in which components share the load, so the failure of some components results in increased load on every of the remaining components.

Theorem 3.2 The two random variables (X_1, X_2) which are defined by the survival function (3.1) are associate.

Proof. To prove the two random variables X_1 and X_2 are associate we must prove the

$$P(X_1 > x_1, X_2 > x_2) \geq \prod_{i=1}^{2} P(X_i > x_i) \tag{3.10}$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2) \geq \prod_{i=1}^{2} P(X_i \leq x_i). \tag{3.11}$$

Using the definitions of X_1 and X_2 we have

$$\Big(\frac{a}{a+z}\Big)^{\theta} \prod_{i=1}^{2} P(X_i > x_i) = \Big(\frac{a}{a+x_1}\Big)^{\theta}\Big(\frac{a}{a+x_2}\Big)^{\theta} P(X_1 > x_1, X_2 > x_2); \; z = \max(x, y) \tag{3.12}$$

Now, using the definition of z we can easily verify that the inequality (3.10) is satisfies for all $x_1, x_2 > 0$.

Using a similar manner we can easily prove the inequality (3.11) is also satisfies for all $x_1, x_2 > 0$, that is (X_1, X_2) are associate random variables which completes the proof.

Corollary 3.2 With the nonlinear transformation $X_1 = a(e^{U_1} - 1)$, $X_2 = a(e^{U_2} - 1)$ we conclude from (3.2) that the joint density of bivariate (X_1, X_2) is bivariate exponentially density function of Marshall and Olkin type with parameters (θ_1, θ_2).

Proof. Calculate the Jacobian $J(x_1, x_2)$ of the transformation

$$x_1 = a(e^{u_1} - 1),\, x_2 = a(e^{u_2} - 1) \tag{3.13}$$

and substitute into the density equation

$$f_{X,Y}(x, y) = |J|\, f_{U_1,U_2}(u_1, u_2)$$

using (3.2) and (3.3) the resulting density function function $f_{U_1,U_2}(u_1, u_2)$ of (U_1, U_2) is given by

$$f_{U_1,U_2}(u_1, u_2) = \begin{cases} \theta_2(\theta_1 + \theta) \exp\Big[-(\theta_1 + \theta)u_1 - \theta_2 u_2\Big], & u_1 > u_2 \\ \theta_1(\theta_2 + \theta) \exp\Big[-\theta_1 u_1 - (\theta_2 + \theta)u_2\Big], & u_1 < u_2 \\ \theta \exp\Big[-(\theta_1 + \theta_2 + \theta)u\Big], & u_1 = u_2 = u \end{cases} \tag{3.14}$$

which is the bivariate exponential of Marshall and Olkin type, which completes the proof.

3.2 Bivariate dependence of distributions with Pareto

In this subsection we will study the bivariate dependence of the bivariate Pareto distributions.

Lemma 3.3 The bivariate random variable (X_1, X_2) defined by the bivariate distribution

$$F_{X_1,X_2}(x_1,x_2) = \Big[1-\Big(\frac{a}{a+x_1}\Big)^{\theta_1}\Big]\Big[1-\Big(\frac{a}{a+x_2}\Big)^{\theta_2}\Big]\Big[1-\Big(\frac{a}{a+z_0}\Big)^{\theta}\Big],\ z_0 = \min(x_1,x_2) \tag{3.15}$$

is positively quadrant dependent (PQD).

Proof. A bivariate random variable (X_1, X_2) is said to be (PQD) (Tong, 1980) if

$$P\Big(X_1 \le x_1, X_2 \le x_2\Big) \ge P\Big(X_1 \le x_1\Big)\,P\Big(X_2 \le x_2\Big),\ \forall x_1, x_2. \tag{3.16}$$

Using the distribution function of (X_1, X_2) and marginal distributions of X_1 and X_2 we find that:

$$\Big[1-\Big(\frac{a}{a+x_1}\Big)^{\theta_1}\Big]\Big[1-\Big(\frac{a}{a+x_2}\Big)^{\theta_2}\Big]P\Big(X_1 \le x_1, X_2 \le x_2\Big)$$

$$= \Big[1-\Big(\frac{a}{a+z_0}\Big)^{\theta}\Big]P\Big(X_1 \le x_1\Big)P\Big(X_2 \le x_2\Big),\ \forall,\ x_1, x_2,\ z_0 = \min(x_1,x_2). \tag{3.17}$$

Note that from (3.17) we can easily verify that the inequality (3.16) holds for all x_1, x_2 , which completes the proof.

Lemma 3.4 The covariance of the bivariate Pareto distribution (X_1, X_2) is given by:

$$\mathrm{Cov}(X_1,X_2) = \frac{a^2\theta}{(\theta+\theta_1-1)(\theta+\theta_2-1)(\theta+\theta_1+\theta_2-2)} > 0, \tag{3.18}$$

where

$$\theta+\theta_i > 1,\ i=1,2,\ \theta+\theta_1+\theta_2 > 2.$$

Proof. The proof of this lemma can be reach by calculate the expectations X_i, $(i = 1, 2)$ and X_1X_2 and substituting in the covariance definition

$$\mathrm{Cov}(X_1,X_2) = E(X_1X_2) - E(X_1)E(X_2). \tag{3.19}$$

Now the expectation of X_i, $(i = 1, 2)$ can be calculate from the expectation definition in the form:

$$E(X_i) = \int_0^\infty\int_0^{x_1} x_i f_1(x_1,x_2)dx_2dx_1 + \int_0^\infty\int_{x_1}^\infty x_i f_2(x_1,x_2)dx_2dx_1 + \int_0^\infty x f_0(x,x)dx \tag{3.20}$$

Substituting (3.3) into (3.20) and after some algebraic manipulation we get

$$E(X_i) = \frac{a}{\theta + \theta_i - 1}, \; i = 1, 2. \tag{3.21}$$

Also the expectation of X_1X_2 can be obtained using

$$E(X_1X_2) = \int_0^\infty \int_0^{x_1} x_1x_2f_1(x_1,x_2)dx_2dx_1 + \int_0^\infty \int_{x_1}^\infty x_1x_2f_2(x_1,x_2)dx_2dx_1 + \int_0^\infty x^2 f_0(x,x)dx. \tag{3.22}$$

Substituting (3.3) into (3.22) and after lengthy algebraic manipulation we get the expectation of X_1X_2 in the form:

$$E(X_1X_2) = \frac{a^2(2\theta + \theta_1 + \theta_2 - 2)}{(\theta + \theta_1 - 1)(\theta + \theta_2 - 1)(\theta + \theta_1 + \theta_2 - 2).} \tag{3.23}$$

Substituting (3.21) and (3.23) into (3.19), one gets the formula (3.18) of the covariance of (X_1, X_2). Hence the covariance of (X_1, X_2) is positive which completes the proof.

Lemma 3.5 The random variable X_1 is a left-tail decreasing function of the random variable X_2 if $X_1 < X_2$. Further also the random variable X_2 is a left-tail decreasing function of X_1 if $X_1 > X_2$.

Proof. The proof of this lemma can be achieved using definition of a left-tail of X_1 and X_2. The random variable X_1 is a left-tail decreasing function of X_2 if

$$P\Big(X_1 \le x | X_2 \le x_2\Big) \tag{3.24}$$

is non-increasing function of the random variable X_2.

Using the above definition and the bivariate distribution (3.15) one can easily gets:

$$P\Big(X_1 \le x_1 | X_2 \le x_2\Big) = \Big[1 - \Big(\frac{a}{a + x_1}\Big)^{\theta_1}\Big]\Big[1 - \Big(\frac{a}{a + z_0}\Big)^{\theta}\Big]\Phi_1(x_2) \tag{3.25}$$

where

$$\Phi_1(x_2) = \Big[1 - \Big(\frac{a}{a + x_2}\Big)^{\theta}\Big]^{-1}. \tag{3.26}$$

We note that the function $\Phi_1(x_2)$ is non-increasing function of x_2. Since, if we assume that $x_2 < x_2^*$, then we can easily verify that $\Phi_1(x_2) \ge \Phi_1(x_2^*)$ which leads to $\Phi_1(x_2)$ non-increasing function of x_2 if $X_1 < X_2$ for all x_1, therefore X_1 is a left-tail decreasing function of x_2. Similarly we can also verify that X_2 is also a left-tail decreasing function of x_1 if $X_1 > X_2$.

Next, the rest of this paper we will concerned with the mixture of bivariate Pareto distributions and Pareto competing risk models.

4. MIXTURE OF BIVARIATE PARETO

This section deals with the mixture of bivariate Pareto distributions with a latent variable which is also Pareto distributed. Also, the dependence of mixture of bivariate Pareto is proved to be positive as expected.

4.1 Mixture of bivariate Pareto distributions

In this subsection we consider the case $n = k = 2$ for simplicity, that is under the plan of the bivariate two component mixture of Pareto distributions. Then, from (2.8) it follows that the joint survival function of the mixture S_1 and S_2 will take the following form

$$\begin{aligned}\bar{F}(s_1, s_2) &= P(S_1 > s_1, S_2 > s_2) = P(X_1 > s_1)P(X_2 > s_2)P(Z > s_0)\\ &= \Big(\frac{a}{a+s_0}\Big)^{\theta} \Big\{p_{11}\Big(\frac{a}{a+s_1}\Big)^{\theta_{11}}\Big(\frac{a}{a+s_2}\Big)^{\theta_{21}} + p_{12}\Big(\frac{a}{a+s_1}\Big)^{\theta_{11}}\Big(\frac{a}{a+s_2}\Big)^{\theta_{22}}\\ &+ p_{21}\Big(\frac{a}{a+s_1}\Big)^{\theta_{12}}\Big(\frac{a}{a+s_2}\Big)^{\theta_{21}} + p_{22}\Big(\frac{a}{a+s_1}\Big)^{\theta_{12}}\Big(\frac{a}{a+s_2}\Big)^{\theta_{22}}\Big\}\end{aligned} \tag{4.1}$$

where X_1 and X_2 have mixture of Pareto distributions that denoted by

$$\begin{aligned}X_1 &\sim [a_1 P(a, \theta_{11}) + (1-a_1)P(a, \theta_{12})],\\ X_1 &\sim [a_2 P(a, \theta_{21}) + (1-a_2)P(a, \theta_{22})],\end{aligned} \tag{4.2}$$

and

$$p_{ij} = a_1^{2-i} a_2^{2-j}(1-a_1)^{i-1}(1-a_2)^{j-1},\ \forall\, i, j \in \{1, 2\}. \tag{4.3}$$

Form the relation (4.1) we can conclude that

1. For $i, j \in \{1, 2\}$, $p_{ij} \geq 0$ and $p_{11} + p_{12} + p_{21} + p_{22} = 1$.
2. Every term of the right hand side of $\bar{F}(s_1, s_2)$ which given by equation (4.1) has a survival function of a bivariate Pareto distributions.

Therefore we can easily conclude that, the survival function (4.1) can be considered as the joint survival function of a mixture of four bivariate Pareto distributions.

Now the following Theorem gives the joint probability density function of the mixture S_1 and S_2.

Theorem 4.1 Using the joint survival (4.1) of the mixture S_1, S_2, then the joint pdf of S_1, S_2 say $f(s_1, s_2)$ is given by

$$f(s_1, s_2) = \begin{cases} f_1(s_1, s_2) & s_1 > s_2\\ f_2(s_1, s_2) & s_1 < s_2\\ f_0(s_0, s_0) & s_1 = s_2 = s_0\end{cases} \tag{4.4}$$

where

$$f_1(s_1,s_2)=a^{-2}\Big\{p_{11}\theta_{21}(\theta_{11}+\theta)\Big(\frac{a}{a+s_1}\Big)^{\theta_{11}+1}\Big(\frac{a}{a+s_2}\Big)^{\theta_{21}+1}$$
$$+p_{12}\theta_{22}(\theta_{11}+\theta)\Big(\frac{a}{a+s_1}\Big)^{\theta_{11}+1}\Big(\frac{a}{a+s_2}\Big)^{\theta_{22}+1}$$
$$+p_{21}\theta_{21}(\theta_{12}+\theta)\Big(\frac{a}{a+s_1}\Big)^{\theta_{12}+1}\Big(\frac{a}{a+s_2}\Big)^{\theta_{21}+1}$$
$$+p_{22}\theta_{21}(\theta_{22}+\theta)\Big(\frac{a}{a+s_1}\Big)^{\theta_{12}+1}\Big(\frac{a}{a+s_2}\Big)^{\theta_{22}+1}\Big\}\Big(\frac{a}{a+s_1}\Big)^{\theta}.$$

$$f_2(s_1,s_2)=a^{-2}\Big\{p_{11}\theta_{11}(\theta_{21}+\theta)\Big(\frac{a}{a+s_1}\Big)^{\theta_{11}+1}\Big(\frac{a}{a+s_2}\Big)^{\theta_{21}+1}$$
$$+p_{12}\theta_{11}(\theta_{22}+\theta)\Big(\frac{a}{a+s_1}\Big)^{\theta_{11}+1}\Big(\frac{a}{a+s_2}\Big)^{\theta_{22}+1}$$
$$+p_{21}\theta_{12}(\theta_{21}+\theta)\Big(\frac{a}{a+s_1}\Big)^{\theta_{12}+1}\Big(\frac{a}{a+s_2}\Big)^{\theta_{21}+1}$$
$$+p_{22}\theta_{12}(\theta_{22}+\theta)\Big(\frac{a}{a+s_1}\Big)^{\theta_{12}+1}\Big(\frac{a}{a+s_2}\Big)^{\theta_{22}+1}\Big\}\Big(\frac{a}{a+s_2}\Big)^{\theta}$$

$$f_0(s_0,s_0)=\theta\,a^{-2}\Big\{p_{11}\Big(\frac{a}{a+s_0}\Big)^{\theta_{11}+\theta_{21}+\theta+1}+p_{12}\Big(\frac{a}{a+s_0}\Big)^{\theta_{11}+\theta_{22}+\theta+1}$$
$$+p_{21}\Big(\frac{a}{a+s_0}\Big)^{\theta_{12}+\theta_{21}+\theta+1}+p_{22}\Big(\frac{a}{a+s_0}\Big)^{\theta_{12}+\theta_{22}+\theta+1}\Big\}. \tag{4.5}$$

Proof. The forms of $f_1(s_1,s_2)$ and $f_2(s_1,s_2)$ can be obtained by differentiating the joint survival function $\bar{F}(s_1,s_2)$ with respect to s_1 and s_2. But the function $f_0(s_0,s_0)$ can not be derived in a similar method. In fact to derive the function $f_0(s_0,s_0)$ we will use an identity similar to (3.4).

Corollary 4.1 The expectations of $S_i\,(i=1,2)$ and S_1S_2 are given by :

$$E(S_i)=a\Big[\frac{a_i}{\theta+\theta_{i1}-1}+\frac{1-a_i}{\theta+\theta_{i2}-1}\Big],\ (i=1,2) \tag{4.6}$$

and

$$E(S_1S_2)=a^2\Big\{\frac{p_{11}(2\theta+\theta_{11}+\theta_{21}-2)}{(\theta+\theta_{11}-1)(\theta+\theta_{21}-1)(\theta+\theta_{11}+\theta_{21}-2)}$$
$$+\frac{p_{12}(2\theta+\theta_{11}+\theta_{22}-2)}{(\theta+\theta_{11}-1)(\theta+\theta_{22}-1)(\theta+\theta_{11}+\theta_{22}-2)}$$
$$+\frac{p_{21}(2\theta+\theta_{12}+\theta_{21}-2)}{(\theta+\theta_{12}-1)(\theta+\theta_{21}-1)(\theta+\theta_{12}+\theta_{21}-2)}$$
$$+\frac{p_{22}(2\theta+\theta_{12}+\theta_{22}-2)}{(\theta+\theta_{12}-1)(\theta+\theta_{22}-1)(\theta+\theta_{12}+\theta_{22}-2)}\Big\}. \tag{4.7}$$

Proof. The proof of this Corollary can be reached using directly the definition of the mixture expectations taking into considerations the bivariate expectations that given by (3.21) and (3.23).

Corollary 4.2 The mixture of bivariate Pareto that defined by the density function (4.4) is positively dependent.

The proof of this Corollary can be achieved by substituting from (4.6) and (4.7) into

$$\mathrm{Cov}(S_1, S_2) = E(S_1 S_2) - E(S_1)E(S_2). \tag{4.8}$$

After lengthy algebraic manipulation one gets:

$$\begin{aligned}\mathrm{Cov}(S_1, S_2) = a^2\theta \Bigg[& \frac{p_{11}}{(\theta+\theta_{11}-1)(\theta+\theta_{21}-1)(\theta+\theta_{11}+\theta_{21}-2)} \\ & + \frac{p_{12}}{(\theta+\theta_{11}-1)(\theta+\theta_{22}-1)(\theta+\theta_{11}+\theta_{22}-2)} \\ & + \frac{p_{21}}{(\theta+\theta_{12}-1)(\theta+\theta_{21}-1)(\theta+\theta_{12}+\theta_{21}-2)} \\ & + \frac{p_{22}}{(\theta+\theta_{12}-1)(\theta+\theta_{22}-1)(\theta+\theta_{12}+\theta_{22}-2)}\Bigg] > 0, \;\; 3\theta + \sum_{i=1}^{2}\sum_{j=1}^{2}\theta_{ij} > 4 \end{aligned} \tag{4.9}$$

Therefore the mixture of bivariate Pareto distributions is positively dependent.

5. COMPETING RISK PARETO MODELS

In this section we propose a Pareto competing risk models. These models arise in situation in which fail of the components is due to several different causes. In such situations every system failure is caused by only of the competing risks. In the present work we consider each competing risk has a mixture of Pareto and the latent variable is also Pareto distributed.

Assume that an item may fail due to any one of the mutually exclusive causes $\{C_1, \ldots, C_k\}$, that is the item fail due to the cause C_i then the item did not fail due to any other cause $\{C_j\}$, $j \neq i$. Risks $\{C_j\}$, $j \neq i$ as well as the risk of failure due to the cause C_i are called competing risk. Such a situation can arise when an item under test has k different components and the item fails as soon as any one of the components fails. Therefore k different components can designated as k causes $\{C_1, \ldots, C_k\}$ of fail and the i-component to fail would be due to the cause C_i.

Now we develop the distribution of the minimum. Assume that the random variable X_i be a mixture of $X_{i1}, X_{i2}, \ldots, X_{ik}$ every of them say X_{ij} has a Pareto distribution with parameters (a, θ_{ij}) and the mixing probability are $a_{i1}, a_{i2}, \ldots, a_{ik}$, that is $\sum_{j=}^{k} a_{ij} = 1$ and $a_{ij} > 0 \forall i = 1, 2, \ldots, n$. Now, consider one latent random variable Z which is Pareto distributed with parameters (a, θ) and it is independent of $X_1, X_2, \ldots, X_n$. Also we define the lifetime of the system T as

$$T = \min(X_1, X_2, \ldots, X_n, Z) \tag{5.1}$$

Therefore, the survival function of system is defined as:

$$P\Big(T > t\Big) = P\Big[\min(X_1, X_2, \ldots, X_n, Z) > t\Big] = P(Z > t) \prod_{i=1}^{n} P\Big(X_i > t\Big) \tag{5.2}$$

Since every X_i is a mixture of Pareto and Z also, one gets

$$P\Big(T > t\Big) = \Big(\frac{a}{a+t}\Big)^{\theta} \prod_{i=1}^{n} \sum_{j=1}^{k} a_{ij} \Big(\frac{a}{a+t}\Big)^{\theta_{ij}} = \prod_{i=1}^{n} P(T_i > t) \tag{5.3}$$

where T_i is a mixture of Pareto distributions with parameters $(a, \frac{\theta}{n} + \theta_{ij})$ and mixing probability $a_{i1}, a_{2k}, \ldots, a_{ik}$.

Thus, we find that the survival function of the system can be expressed as:

$$\bar{F}_T(t) = \prod_{i=1}^{n} \bar{F}_{T_i}(t) \tag{5.4}$$

Therefore, the probability density function of the lifetime T of the system can be obtained as follows:

$$f_T(t) = -\frac{d}{dt} P(T > t) = -P(T > t) \sum_{i=1}^{n} \frac{1}{P(T_i > t)} \frac{d}{dt} P(T_i > t) =$$

$$f_T(t) = \sum_{i=1}^{n} \frac{P(T > t)}{P(T_i > t)} f_{T_i}(t) = \bar{F}_T(t) \sum_{i=1}^{n} \frac{f_{T_i}(t)}{\bar{F}_{T_i}(t)} = \bar{F}_T(t) \sum_{i=1}^{n} h_{T_i}(t) \tag{5.5}$$

and so the hazard function of T is given by

$$h_T(t) = \frac{f_T(t)}{\bar{F}_T(t)} = \sum_{i=1}^{n} h_{T_i}(t). \tag{5.6}$$

The average of the mean time to failure of the system is defined

$$E(T) = \int_0^{\infty} t\, f_T(t) dt = -\int_0^{\infty} t \frac{d}{dt} \bar{F}_T(t) dt = -\int_0^{\infty} t\, d\bar{F}_T(t) \tag{5.7}$$

Using the integration by parts and properties of $\bar{F}_T(t)$, one gets

$$E(T) = \int_0^{\infty} \bar{F}_T(t) dt = \int_0^{\infty} \prod_{i=1}^{n} \bar{F}_{T_i}(t) dt = \int_0^{\infty} \prod_{i=1}^{n} e^{-\int_0^t h_{T_i}(\tau) d\tau} dt$$

$$= \int_0^\infty \prod_{i=1}^n e^{-H_{T_i}(t)}dt = \int_0^\infty e^{-\sum_{i=1}^n H_{T_i}(t)}dt, \tag{5.8}$$

where $H_{T_i}(t) = \int_0^t h_{T_i}(\tau)d\tau$ is the integrated hazard rate function of T_i, $(i = 1, \dots, n)$

5.1 Mixture Pareto model

In this subsection we will develop the distribution of the minimum and the corresponding hazard function for the mixture of Pareto distributions.

Now we define $S = \min(S_1, \dots, S_n)$, as the minimum of mixture of multivariate distributions with Pareto, then the survival function of S is given by

$$\bar{F}(s) = P(S > s) = P\Big[\min(S_1, \dots, S_n) > s\Big] = P\Big[S_1 > s, S_2 > s, \dots, S_n > s_n\Big]$$

$$= P(Z > s)\prod_{i=1}^n P(X_i > s) = \prod_{i=1}^n \sum_{j=1}^k a_{ij}\Big(\frac{a}{a+s}\Big)^{(\theta_{ij}+\frac{\theta}{n})}$$

$$= \prod_{i=1}^n P(T_i > s) = \prod_{i=1}^n \bar{F}_{T_i}(s) \tag{5.9}$$

where the density function of T_i, $(i = 1, \dots, n)$ is given by

$$f_{T_i}(t) = \sum_{j=1}^k a_{ij}\Big(\frac{n\theta_{ij} + \theta}{na}\Big)\left(\frac{a}{a+t}\right)^{(\theta_{ij}+\frac{\theta}{n}+1)}, \; t > 0, \; (i = 1, \dots, n) \tag{5.10}$$

Hence, the density function of the minimum S takes the form

$$f_T(s) = \bar{F}_S(s)\sum_{i=1}^n \frac{f_{T_i}(s)}{\bar{F}_{T_i}(s)} \tag{5.11}$$

Therefore the hazard function corresponding to the minimum S reduces to

$$h_S(s) = \sum_{i=1}^n h_{T_i}(s)$$

where

$$h_{T_i}(s) = \frac{\sum_{j=1}^k a_{ij}\left(\frac{n\theta_{ij}+\theta}{na}\right)\left(\frac{a}{a+s}\right)^{(\theta_{ij}+\frac{\theta}{n}+1)}}{\sum_{j=1}^k a_{ij}\left(\frac{a}{a+s}\right)^{(\theta_{ij}+\frac{\theta}{n})}}, \; (i = 1, \dots, n) \tag{5.12}$$

is the hazard function of T_i, $i = 1, \dots, n$.

In what follows we will obtained a special case that occurs when we put $k = 2$ and $n = 2$ in (5.9). This case represents a mixture of bivariate Pareto distribution with Pareto latent random variable. Then the density function of T_1 and T_2 are given by

$$f_{T_1}(t) = a_1\Big(\frac{2\theta_{11}+\theta}{2a}\Big)\left(\frac{a}{a+t}\right)^{(\theta_{11}+\frac{\theta}{2}+1)} + (1-a_1)\Big(\frac{2\theta_{12}+\theta}{2a}\Big)\left(\frac{a}{a+t}\right)^{(\theta_{12}+\frac{\theta}{2}+1)}$$

$$f_{T_2}(t) = a_2\Big(\frac{2\theta_{21}+\theta}{2a}\Big)\left(\frac{a}{a+t}\right)^{(\theta_{21}+\frac{\theta}{2}+1)} + (1-a_2)\Big(\frac{2\theta_{22}+\theta}{2a}\Big)\left(\frac{a}{a+t}\right)^{(\theta_{22}+\frac{\theta}{2}+1)} \tag{5.13}$$

and the survival function of T is given by

$$\bar{F}_T(t) = p_{11}\left(\frac{a}{a+t}\right)^{(\theta_{11}+\theta_{21}+\theta)} + p_{12}\left(\frac{a}{a+t}\right)^{(\theta_{11}+\theta_{22}+\theta)}$$

$$+p_{21}\left(\frac{a}{a+t}\right)^{(\theta_{12}+\theta_{21}+\theta)} + p_{22}\left(\frac{a}{a+t}\right)^{(\theta_{12}+\theta_{22}+\theta)}, \tag{5.14}$$

which represent a mixture of four pareto distributions with parameters $(a, \theta_{ij} + \theta/2), i, j = 1, 2$ and a mixture of bivariate Pareto distributions with parameters (a, θ_{ij}) with mixing probabilities p_{ij}, $i, j = 1, 2$.

6. CONCLUSION

The new class of models developed in this paper has many different applications in different fields. In this paper we present a new class of multivariate Pareto distributions. The obtained class includes bivariate models including Marslall and Olkin type. The bivariate Pareto is associate and positively quadrant dependent. The approach in this paper is based on the introducing a Pareto distributed latent random variable. The distribution of minimum in a competing risk reliability model is discussed.

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