J. Korean Math. Soc. **43** (2006), No. 4, pp. 899–909

# ON WEYL'S THEOREM FOR QUASI-CLASS $A$ OPERATORS

Bhagwati P. Duggal, In Ho Jeon, and In Hyoun Kim

Abstract. Let $T$ be a bounded linear operator on a complex infinite dimensional Hilbert space $\mathscr{H}$. We say that $T$ is a quasi-class $A$ operator if $T^*|T^2|T \geq T^*|T|^2T$. In this paper we prove that if $T$ is a quasi-class $A$ operator and $f$ is a function analytic on a neighborhood of the spectrum of $T$, then $f(T)$ satisfies Weyl's theorem and $f(T^*)$ satisfies a-Weyl's theorem.

## 1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded linear operators on a complex infinite dimensional Hilbert space $\mathscr{H}$. Recall ([3], [9], [15], [29]) that $T \in \mathscr{L}(\mathscr{H})$ is called *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for $p \in (0,1]$, $T$ is called *paranormal* if $||T^2x|| \geq ||Tx||^2$ for all unit vector $x \in \mathscr{H}$, and $T$ is called *normaloid* if $||T^n|| = ||T||^n$ for $n \in \mathbb{N}$ (equivalently, $||T|| = r(T)$, the spectral radius of $T$). Following [10] and [9] we say that $T \in \mathscr{L}(\mathscr{H})$ belongs to *class A* if $|T^2| \geq |T|^2$. Recall ([17], [26], [28]) that $T$ is called *p-quasihyponormal* if $T^*(T^*T)^pT \geq T^*(TT^*)^pT$ for $p \in (0,1]$. For brevity, we shall denote classes of $p$-hyponormal operators, $p$-quasihyponormal operators, paranormal operators, normaloid operators, and class $A$ operators by $\mathcal{H}(p)$, $\mathcal{QH}(p)$, $\mathcal{PN}$, $\mathcal{N}$ and $\mathcal{A}$, respectively. It is well known that

(1) $\mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{PN} \subset \mathcal{N}$ and $\mathcal{H}(p) \subset \mathcal{QH}(p) \subset \mathcal{PN} \subset \mathcal{N}$.

In [16] Jeon and Kim considered an extension of the notion of class $A$ operators, similar in spirit to the extension of the notion of $p$-hyponormality to $p$-quasihyponormality.

Received September 16, 2005.

2000 Mathematics Subject Classification: Primary 47A53, 47B20.

Key words and phrases: quasi-class $A$ operators, Weyl's theorem, a-Weyl's theorem, a-Browder theorem.

This work of the second and third authors was supported by a grant (R14-2003-006-01001-0) from the Korea Science and Engineering Foundation.

DEFINITION 0.1. We say that $T \in \mathscr{L}(\mathscr{H})$ is *quasi-class* $A$ operator if

$$T^*|T^2|T \geq T^*|T|^2T.$$

For brevity, we shall denote the set of quasi-class $A$ operators by $\mathcal{QA}$. As shown in [16], the class of quasi-class $A$ operators properly contains classes of class $A$ operators and $p$-quasihyponormal operators, i.e., the following inclusions holds;

$$\mathcal{H}(p) \subset \mathcal{QH}(p) \subset \mathcal{QA} \text{ and } \mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{QA}. \tag{2}$$

In view of inclusions (1), it seems reasonable to expect that the operators in class $\mathcal{QA}$ are paranormal or at least normaloid: the following example shows that one would be wrong in such an expectation.

EXAMPLE 0.2. ([16]) We consider unilateral weighted shift operators on $\ell^2$. Recall that given a bounded sequence of positive numbers $\alpha$ : $\alpha_0, \alpha_1, \ldots$ (called weights), the unilateral weighted shift $W_\alpha$ associated with $\alpha$ is the operator on $\ell^2$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2$. Straightforward calculations show that $W_\alpha$ belongs to $\mathcal{QA}$ if and only if

$$W_\alpha = \begin{pmatrix} 0 & & & & \\ \alpha_0 & 0 & & & \\ & \alpha_1 & 0 & & \\ & & \alpha_2 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}, \tag{3}$$

where $\alpha_0$ is arbitrary and $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots$ . So if we consider $W_\alpha$ having weights $\alpha_0 = 2$ and $\alpha_i = \frac{1}{2}$ $(i \geq 1)$, then we easily see that $W_\alpha$ is quasi-class $A$ but not normaloid because $||W_\alpha|| = 2 \neq 1 = r(W_\alpha)$.

We shall denote the set of all complex numbers and the complex conjugate of a complex number $\lambda$ by $\mathbb{C}$ and $\overline{\lambda}$, respectively. The closure of a set $\mathcal{M}$ will be denoted by $\overline{\mathcal{M}}$ and we shall henceforth shorten $T-\lambda I$ to $T - \lambda$. If $T \in \mathscr{L}(\mathscr{H})$ we shall write ker$T$ and ran$T$ for the null space and range of $T$, respectively. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *Fredholm* if it has closed range, finite dimensional null space (i.e., $\alpha(T) :=$ dim ker$T < \infty$), and its range has finite co-dimensional (i.e., $\beta(T) :=$ dim ker$T^* < \infty$). We consider the sets

$$\Phi_+(\mathscr{H}) := \{T \in \mathscr{L}(\mathscr{H}) : \text{ran}T \text{ is closed and } \alpha(T) < \infty\},$$

$$\Phi_-(\mathscr{H}) := \{T \in \mathscr{L}(\mathscr{H}) : \text{ran}T \text{ is closed and } \beta(T) < \infty\},$$

$$\Phi(\mathscr{H}) := \Phi_+(\mathscr{H}) \cap \Phi_-(\mathscr{H}),$$

and

$$\Phi_{\pm}(\mathscr{H}) := \Phi_{+}(\mathscr{H}) \cup \Phi_{-}(\mathscr{H}).$$

We say that $T \in \mathscr{L}(\mathscr{H})$ is *semi-Fredholm* if $T \in \Phi_{\pm}(\mathscr{H})$. Evidently,

$$T \text{ is Fredholm if and only if } T \in \Phi(\mathscr{H}).$$

If $T \in \Phi_{\pm}(\mathscr{H})$, then the *index* of $T$, denoted ind$(T)$, is given by

$$\text{ind}\,(T) = \alpha(T) - \beta(T).$$

The index is an integer or $\{\pm\infty\}$. The ascent of $T \in \mathscr{L}(\mathscr{H})$, denote asc$(T)$, is the least non-negative integer $n$ such that $\ker T^n = \ker T^{n+1}$ and the descent of $T$, denote dsc$(T)$, is the least non-negative integer $n$ such that $\text{ran} T^n = \text{ran} T^{n+1}$. We say that $T \in \mathscr{L}(\mathscr{H})$ is of finite ascent (resp. finite descent) if asc$(T-\lambda) < \infty$ (resp. dsc$(T-\lambda) < \infty$) for all $\lambda \in \mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is called *Weyl* if it is Fredholm of index zero, and $T \in \mathscr{L}(\mathscr{H})$ is called *Browder* if it is Fredholm of "finite ascent and descent": equivalently [11, Theorem 7.9.3] if $T$ is Fredholm and $T-\lambda$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. We denote the spectrum of $T \in \mathscr{L}(\mathscr{H})$ by $\sigma(T)$, and the sets of isolated points and accumulation points of $\sigma(T)$ are denoted by iso$\sigma(T)$ and acc$\sigma(T)$, respectively. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, and the Browder spectrum $\sigma_b(T)$ of $T \in \mathscr{L}(\mathscr{H})$ are defined by

$$\begin{aligned}
\sigma_e(T) &= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Fredholm}\},\\
\sigma_w(T) &= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Weyl}\},\\
\sigma_b(T) &= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Browder}\}.
\end{aligned}$$

It is well known [11] that

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}\sigma(T).$$

Let $\mathscr{K}(\mathscr{H})$ denote the ideal of compact operators in $\mathscr{B}(\mathscr{H})$, and consider the following spectral subsets:

$$\begin{aligned}
&\sigma_a(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not bounded below}\},\\
&\sigma_{aw}(T) := \bigcap\{\sigma_a(T+K) : K \in \mathscr{K}(\mathscr{H})\},\\
&\sigma_{ab}(T) := \bigcap\{\sigma_a(T+K) : K \in \mathscr{K}(\mathscr{H}) \text{ and } TK = KT\},\\
&\sigma_s(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not surjective}\},\\
&\sigma_{sw}(T) := \bigcap\{\sigma_s(T+K) : K \in \mathscr{K}(\mathscr{H})\},\\
&\sigma_p(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not injective}\},\\
&\pi_0(T) := \sigma(T) \setminus \sigma_b(T),\\
&\pi_{00}(T) := \{\lambda \in \mathbb{C} : \lambda \in \text{iso}\sigma(T) \text{ and } 0 < \alpha(T-\lambda) < \infty\},
\end{aligned}$$

and

$$\pi_{a0}(T) := \{\lambda \in \mathbb{C} : \lambda \in \text{iso}\sigma_a(T) \text{ and } 0 < \alpha(T-\lambda) < \infty\}.$$

Evidently

$$\pi_0(T) \subseteq \pi_{00}(T) \subseteq \pi_{a0}(T),$$

and [23] that

$$\sigma_{aw}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(\mathscr{H})\},$$

where $\Phi_+^-(\mathscr{H}) := \{T \in \Phi_+(\mathscr{H}) : \mathrm{ind}(T) \leq 0\}$.

Following [12], [5], [24], and [6] we say that $T \in \mathscr{L}(\mathscr{H})$ *satisfies Browder's theorem* if

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T),$$

$T$ *satisfies Weyl's theorem* if there is equality

$$\sigma(T) \setminus \sigma_b(T) = \pi_{00}(T),$$

$T$ *satisfies* a-*Weyl's theorem* if there is equality

$$\sigma_a(T) \setminus \sigma_{aw}(T) = \pi_{a0}(T),$$

and $T$ *satisfies* a-*Browder's theorem* if there is equality

$$\sigma_{aw}(T) = \sigma_{ab}(T).$$

It is well known ([6], [12]) that

$$\begin{array}{ccccc} & \nearrow & \text{Weyl's theorem} & \searrow & \\ \text{a-Weyl's theorem} & & & & \text{Browder's theorem} \\ & \searrow & \text{a-Browder's theorem} & \nearrow & \end{array}$$

Let $H(\sigma(T))$ be the set of all analytic functions on an open neighborhood of $\sigma(T)$. In [21], Lee and Lee showed that if $T$ is hyponormal and $f \in H(\sigma(T))$, then Weyl's theorem holds for $f(T)$. Recently, this result was extended to $p$-quasihyponormal and class $A$ operators in [28] and [27], respectively.

In this paper we prove that if $T \in \mathcal{QA}$ and $f \in H(\sigma(T))$, then $f(T)$ satisfies Weyl's theorem and $f(T^*)$ satisfies a-Weyl's theorem, respectively. This completely extends earlier results proved in [28] and [27], respectively. During the course of proving these results, we prove also that if $T \in \mathcal{QA}$, then $T$ and $T^*$ satisfy a-Browder's theorem.

## 1. Results

### 1.1. Weyl's theorem for $f(T)$

We begin by recalling some basic properties of $\mathcal{QA}$ operators from [16].

PROPOSITION 1.1. *Let $T \in \mathcal{QA}$ and $T$ not have dense range. Then*

$$T = \begin{pmatrix} A & * \\ 0 & 0 \end{pmatrix} \quad on \quad \mathscr{H} = \overline{\text{ran}T} \oplus \ker T^*,$$

*where $A = T|_{\overline{\text{ran}T}}$, the restriction of $T$ to $\overline{\text{ran}T}$, belongs to $\mathcal{A}$. Moreover, $\sigma(T) = \sigma(A) \cup \{0\}$.*

PROPOSITION 1.2. *Let $T \in \mathcal{QA}$ and let $\lambda \neq 0$. Then*

$$(T - \lambda)x = 0 \;\Rightarrow\; (T - \lambda)^* x = 0; \; x \in \mathscr{H}.$$

PROPOSITION 1.3. *Let $T \in \mathcal{QA}$. If $\lambda_\circ (\neq 0) \in \text{iso}\sigma(T)$ and $E$ is the Riesz idempotent for $\lambda_\circ$, then $E$ is self-adjoint and*

$$\text{ran}E = \ker(T - \lambda_\circ) = \ker(T - \lambda_\circ)^*. \tag{4}$$

A proof of Proposition 1.3 appears in [16]. However, in view of Proposition 1.2, the following argument provides a quick proof of Proposition 1.3. Observe that the non-zero isolated points of $\sigma(T)$ are isolated points of $\sigma(A)$, and hence eigenvalues of $T$ (see Lemma 1.8 below). Proposition 1.2 implies that the non-zero eigenvalues of $T$ are normal (i.e., if $0 \neq \lambda_\circ$ is an eigenvalue of $T$, then $\ker(T - \lambda_\circ)$ reduces $T$). Hence $\mathscr{H} = \text{ran}E \oplus \ker E$, $\lambda_\circ$ is a pole of the resolvent of $T$, the (Riesz) projection $E$ is self-adjoint and $\text{ran}E = \ker(T - \lambda_\circ) = \ker(T - \lambda_\circ)^*$.

In this subsection we prove the following.

THEOREM 1.4. *If $T \in \mathcal{QA}$, then $f(T)$ satisfies Weyl's theorem for every $f \in H(\sigma(T))$.*

To prove Theorem 1.4 we need following series of lemmas.

The class of operators having finite ascent is considerably large, and very important. For example, generalized scalar operators, subscalar operators, and operators satisfying Bishop's property $(\beta)$ are of finite ascent. Therefore, in particular, hyponormal operators, $p$-hyponormal operators [29], $p$-quasihyponormal operators, and class $A$ operators [4] are of finite ascent. The following result say that every $T \in \mathcal{QA}$ also is of finite ascent.

LEMMA 1.5. *Let $T \in \mathcal{QA}$. Then $T$ is of finite ascent.*

*Proof.* To prove this lemma we shall show that $\ker(T-\lambda)^2 = \ker(T-\lambda)^3$. Since we can easily conclude from Proposition 1.2 that

$$\ker(T - \lambda) = \ker(T - \lambda)^2 \text{ for } \lambda \neq 0,$$

it suffices to prove that $\ker T^2 = \ker T^3$. Now assume that $T^3x = 0$ but $Tx \neq 0$ because if $Tx = 0$ then we obviously get the conclusion. Using Hölder-McCarthy inequality [22], we have

$$\begin{aligned} 0 = \|T^3x\| = \langle T^3x, T^3x\rangle^{\frac{1}{2}} &= \langle |T^2|^2Tx, Tx\rangle^{\frac{1}{2}} \\ &\geq \langle |T^2|Tx, Tx\rangle \|Tx\|^{-1} \\ &\geq \langle |T|^2Tx, Tx\rangle \|Tx\|^{-1} = \|T^2x\|^2\|Tx\|^{-1}, \end{aligned}$$

which implies $\ker T^2 \supseteq \ker T^3$. Consequently, $\ker T^2 = \ker T^3$. Hence the proof is complete. □

It was shown in [27] that Weyl's theorem holds for class $A$ operators. We can prove more:

LEMMA 1.6. *If $T \in \mathcal{QA}$, then $T$ satisfies Weyl's theorem.*

*Proof.* To prove this lemma we use the fact [14, Theorem 2] that if $T$ has finite ascent, then Weyl's theorem holds for $T$ if and only if $\operatorname{ran}(T - \lambda)$ has closed range for $\lambda \in \pi_{00}(T)$. Assume that $\lambda \in \pi_{00}(T)$ and , see Proposition 1.1, let

$$T = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \mathscr{H} = \overline{\operatorname{ran}T} \oplus \ker T^*.$$

Case 1. $\lambda \neq 0$: if $0 \neq \lambda \in \pi_{00}(T)$, then $0 < \dim\ker(T - \lambda) < \infty$. So $0 < \dim\ker(T - \lambda)^* < \infty$ by Proposition 1.3. Hence $T - \lambda$ has closed range.

Case 2. $\lambda = 0$: if $0 \in \pi_{00}(T)$, we see that $0 \in \pi_{00}(A)$ or $0 \notin \sigma(A)$. If $0 \in \pi_{00}(A)$, then $A$ is Weyl because Weyl's theorem holds for $A$ [27]. So $A$ can be perturbed by a compact operator $K$ to an invertible operator $U$, i.e., A=U+K. Thus we have that

$$T = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U + K & S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U & S \\ 0 & -I \end{pmatrix} + \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix},$$

where $\begin{pmatrix} U & S \\ 0 & -I \end{pmatrix}$ is invertible with the inverse matrix $\begin{pmatrix} U^{-1} & U^{-1}S \\ 0 & -I \end{pmatrix}$ and $\begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix}$ is compact. Hence $T$ is also Weyl, and so $T$ has closed range. On the other hand, if $0 \notin \sigma(A)$ (i.e., $A$ is invertible), then $T$ has a generalized inverse. Indeed, we have that

$$T = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & A^{-1}S \\ 0 & -I \end{pmatrix} \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix}.$$

In Hilbert space context, it is well known [11] that $T$ has a generalized inverse if and only if $T$ has closed range, and hence this completes the proof. □

REMARK 1.7. A proof of Lemma 1.6 may also be obtained by combining results from the papers [19] and [20] on Weyl's theorem for upper triangular operator matrices by W. Y. Lee. Thus, since Weyl's theorem holds for $A$, $A$ is isoloid and $\sigma_w(A) \cap \sigma_w(0)$ has no interior, Weyl's theorem holds for $A \oplus 0$ [20, Corollary 9]; again, since $\sigma(0)$ has no pseudoholes (and $A$ is isoloid, and Weyl's theorem holds for both $A$ and $A \oplus 0$), Weyl's theorem holds for $T$ [19, Theorem 2.4]. However, we have in the above given a direct proof of Lemma 1.6 using an alternative argument based upon the results of [14].

LEMMA 1.8. *Operators $T \in \mathcal{QA}$ are isoloid.*

*Proof.* Assume that $\lambda \in \mathrm{iso}\sigma(T)$. Then $\lambda \in \mathrm{iso}\sigma(A)$ or $\lambda = 0$ if $\lambda \notin \sigma(A)$. If $\lambda \in \mathrm{iso}\sigma(A)$, $\lambda \in \sigma_p(A)$ because $A$ is isoloid. This easily implies that $\lambda \in \sigma_p(T)$.

On the other hand, if $0 \in \mathrm{iso}\sigma(T)$ and $0 \notin \sigma(A)$, then $\ker T^* \neq \{0\}$. So we can take a non-zero vector $y \in \ker T^*$, and then $A^{-1}Sy \oplus y$ is an eigenvector of $T$. Therefore, $0 \in \sigma_p(T)$. □

LEMMA 1.9. *Let $T \in \mathcal{QA}$. Then*

$$(5) \qquad \tau(f(T)) = f(\tau(T)) \quad \text{for every } f \in H(\sigma(T)),$$

*where $\tau(T)$ denotes either of $\sigma_w(T)$ or $\sigma_{aw}(T)$ or $\sigma_{sw}(T)$.*

*Proof.* Recall that a semi-Fredholm operator $T$ is said to have *stable index* if either $\mathrm{ind}(T - \lambda) \geq 0$ or $\mathrm{ind}(T - \lambda) \leq 0$ for all complex $\lambda$ such that $T - \lambda$ is semi-Fredholm. Recall also from Schmoeger [25, Theorems 2, 4 and 5] (see also [12]) that if $T$ is of stable index for all $\lambda$ such that: (i) $T - \lambda \in \Phi_+(\mathscr{H})$, then $\sigma_{aw}(f(T)) = f(\sigma_{aw}(T))$; (ii) $T - \lambda \in \Phi_-(\mathscr{H})$, then $\sigma_{sw}(f(T)) = f(\sigma_{sw}(T))$, and (iii) $T - \lambda \in \Phi(\mathscr{H})$, then $\sigma_w(f(T)) = f(\sigma_w(T))$ for every $f \in H(\sigma(T))$. Hence, since the finite ascent property of $T \in \mathcal{QA}$ implies $\mathrm{ind}(T - \lambda) \leq 0$ for all complex $\lambda$ [13, Proposition 38.5], a proof of the lemma follows from Lemma 1.5. □

We are now ready to prove Theorem 1.4.

*Proof.* Recall [21, Lemma] that if $A \in \mathscr{L}(\mathscr{H})$ is isoloid, then

$$f\big(\sigma(A) \setminus \pi_{00}(A)\big) = \sigma(f(A)) \setminus \pi_{00}(f(A)) \quad \text{for every } f \in H(\sigma(A)).$$

Thus it follows from Lemmas 1.6, 1.8, 1.9 that

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f\big(\sigma(T) \setminus \pi_{00}(T)\big) = f(\sigma_w(T)) = \sigma_w(f(T)),$$

which implies that $f(T)$ satisfies Weyl's theorem. □

### 1.2. $a$-Weyl's theorem for $f(T^*)$

In this subsection, it will be deduced that $f(T^*)$ satisfies Weyl's theorem from the more general result that $f(T^*)$ satisfies $a$-Weyl's theorem. We say that $T \in \mathscr{L}(\mathscr{H})$ has the *single valued extension property* (say, SVEP) at $\lambda_\circ \in \mathbb{C}$ if, for a neighborhood $U$ of $\lambda_\circ$, $f \equiv 0$ is the only analytic function $f : U \to \mathscr{H}$ satisfying $(T - \lambda)f(\lambda) = 0$. Also, we say that $T$ has SVEP if $T$ has this property at every $\lambda \in \mathbb{C}$. It is well known [18] that the finite ascent property implies SVEP. Thus we see that operators $T \in \mathcal{QA}$ have SVEP from Lemma 1.5.

LEMMA 1.10. *If $T \in \mathcal{QA}$, then $T$ and $T^*$ satisfy $a$-Browder's theorem.*

*Proof.* Recall that an operator $T$ satisfies $a$-Browder's theorem if and only if $\sigma_{ab}(T) = \sigma_{aw}(T)$. Since $\mathrm{asc}(T - \lambda) < \infty$ for every $\lambda \in \sigma(T)$, $\lambda \notin \sigma_{aw}(T) \Longrightarrow T - \lambda \in \Phi_+^-(\mathscr{H})$ and $\mathrm{asc}(T - \lambda) < \infty \Longrightarrow \lambda \notin \sigma_{ab}(T)$. Since $\sigma_{aw}(T) \subseteq \sigma_{ab}(T)$ for every operator $T$, $\sigma_{aw}(T) = \sigma_{ab}(T) \Longrightarrow T$ satisfies $a$-Browder's theorem. Again, if $\overline{\lambda} \notin \sigma_{aw}(T^*)$, then $T^* - \overline{\lambda} \in \Phi_+^-(\mathscr{H}) \Longrightarrow T - \lambda \in \Phi_-(\mathscr{H})$ and $\mathrm{ind}(T - \lambda) \geq 0$. Since $\mathrm{asc}(T - \lambda) < \infty \Longrightarrow \mathrm{ind}(T-\lambda) \leq 0$, $T - \lambda \in \Phi_-(\mathscr{H})$ and $\mathrm{ind}(T - \lambda) = 0 \Longrightarrow T - \lambda \in \Phi(\mathscr{H})$, $\mathrm{ind}(T - \lambda) = 0$ and $\mathrm{asc}(T - \lambda) < \infty \Longrightarrow T - \lambda \in \Phi(\mathscr{H})$ and $\mathrm{asc}(T - \lambda) = \mathrm{dsc}(T - \lambda) < \infty$ [13, Proposition 38.6] $\Longleftrightarrow T^* - \overline{\lambda} \in \Phi(\mathscr{H})$ and $\mathrm{asc}(T^* - \overline{\lambda}) = \mathrm{dsc}(T^* - \overline{\lambda}) < \infty$. Hence $\overline{\lambda} \notin \sigma_{ab}(T^*) \Longrightarrow \sigma_{aw}(T^*) = \sigma_{ab}(T^*)$. This completes the proof. □

REMARK 1.11. Lemma 1.10 is a particular case of a more general result: If either $T$ or $T^*$ has SVEP for a Banach space operator $T \in \mathscr{L}(\mathscr{X})$, then both $T$ and $T^*$ satisfy $a$-Browder's theorem [2, Corollary 2.4]. A necessary and sufficient condition for $T \in \mathscr{L}(\mathscr{X})$ to satisfy $a$-Browder's theorem is that $T$ has SVEP at points $\lambda \in \sigma_a(T) \setminus \sigma_{aw}(T)$ [7, Lemma 2.18].

LEMMA 1.12. *Points $\overline{\lambda} \in \pi_{00}(T^*)$ for a $T \in \mathcal{QA}$ are poles of the resolvent.*

*Proof.* If $0 \neq \overline{\lambda} \in \pi_{00}(T^*)$, then $\lambda \in \mathrm{iso}\sigma(T) \Longrightarrow \lambda$ is a normal eigenvalue of $T$ (by Proposition 1.2), and hence a simple pole of the resolvent of $T$ (Proposition 1.3). If, instead, $\lambda = 0$, then $\dim \ker T^* < \infty \Longrightarrow \mathrm{ran} T^*$ is closed (see the proof of Lemma 1.6) and hence $T^* \in$

$\Phi_+(\mathscr{H}) \Longrightarrow T \in \Phi_-(\mathscr{H})$. Since both $T$ and $T^*$ have SVEP at 0, it follows that $\mathrm{asc}(T) = \mathrm{dsc}(T) < \infty$ (see [1, Theorem 2.3] or [2, Theorem 1.2]) $\Longrightarrow$ 0 is a pole of the resolvent of $T \Longrightarrow$ 0 is a pole of the resolvent of $T^*$. □

THEOREM 1.13. *If $T \in \mathcal{QA}$, then $f(T^*)$ satisfies a-Weyl's theorem for every $f \in H(\sigma(T))$.*

*Proof.* Recall from [1, Theorem 3.6] (see also [8]) that for a Banach space operator $T$ with SVEP, $T^*$ satisfies Weyl's theorem if and only if $T^*$ satisfies $a$-Weyl's theorem. Since $T$ has SVEP implies $f(T)$ has SVEP for every $f \in H(\sigma(T))$ [18, Theorem 3.3.6], it will suffice to prove that $f(T^*)$ satisfies Weyl's theorem. Observe that if $T \in \mathcal{QA}$, then $\mathrm{asc}(T - \lambda) < \infty \Longrightarrow \mathrm{ind}(T - \lambda) \leq 0 \Longrightarrow \mathrm{ind}(T^* - \overline{\lambda}) \geq 0$ for every $\lambda$; hence (since $\sigma_w(T) = \overline{\sigma_w(T^*)}$ and $\sigma(f(T^*)) = \sigma(f(T)^*)$) it follows from Lemma 1.8 and the proof of Theorem 1.4 that it will suffice to prove that $T^*$ satisfies Weyl's theorem. Since $T^*$ satisfies Browder's theorem (by Lemma 1.10), $\sigma(T^*) \setminus \sigma_w(T^*) = \pi_0(T^*) \subseteq \pi_{00}(T^*)$. Let $\overline{\lambda} \in \pi_{00}(T^*)$; then $\overline{\lambda} \in \pi_0(T^*)$ (by Lemma 1.12). Hence $\pi_0(T^*) = \pi_{00}(T^*) \Longrightarrow T^*$ satisfies Weyl's theorem. □

Observe that $T \in \mathcal{QA}$ may not satisfy $a$-Weyl's theorem: consider for example the forward unilateral shift. The following theorem gives a sufficient condition for $T \in \mathcal{QA}$ to satisfy $a$-Weyl's theorem.

THEOREM 1.14. *Let $T \in \mathcal{QA}$. Then a sufficient condition for $f(T)$ to satisfy a-Weyl's theorem for every $f \in H(\sigma(T))$ is that $T^*$ has SVEP.*

*Proof.* If $T^*$ has SVEP, then $\sigma(T) = \sigma_a(T)$; hence $\sigma_{aw}(T) = \sigma_w(T)$ and $\pi_{a0}(T) = \pi_{00}(T)$. Since $T$ satisfies Weyl's theorem, $T$ satisfies $a$-Weyl's theorem. The conclusion that $f(T)$ satisfies $a$-Weyl's theorem now follows since $T$ is isoloid (Lemma 1.8) and $\sigma_{aw}(T)$ satisfies the spectral mapping theorem (Lemma 1.9). □

## References


[1] P. Aiena, *Classes of operators satisfying a-Weyl's theorem*, Studia Math. **169** (2005), no. 2, 105–122.

[2] P. Aiena, C. Carpintero and E. Rosas, *Some characterizations of operators satisfying a-Browder's theorem*, J. Math. Anal. Appl. **311** (2005), no. 2, 530–544.

[3] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory **13** (1990), no. 3, 307–315.

[4] M. Cho and T. Yamazaki, *An operator transform from class A to the class of hyponormal operators and its application*, Integral Equations Operator Theory **53** (2005), 497–508.

[5] L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.
[6] S. V. Djordjević and Y. M. Han, *Browder's theorem and spectral continuity*, Glasgow Math. J. **42** (2000), no. 3, 479–486.
[7] B. P. Duggal, *Hereditarily normaloid operators*, Extracta Math. **20** (2005), 203–217.
[8] ———, *Weyl's theorem for totally hereditarily normaloid operators*, Rend. Circ. Mat. Palermo (2) **53** (2004), no. 3, 417–428.
[9] T. Furuta, *Invitation to Linear Operators*, Taylor and Francis, London, 2001.
[10] T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Scientiae Mathematicae **1** (1998), 389–403.
[11] R. E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Marcel Dekker, New York, 1988.
[12] R. E. Harte and W. Y. Lee, *Another note on Weyl's theorem*, Trans. Amer. Math. Soc. **349** (1997), no. 5, 2115–2124.
[13] H. Heuser, *Functional Analysis*, Marcel Dekker, New York, 1982.
[14] I. H. Jeon, *Weyl's theorem and quasi-similarity*, Integral Equations Operator Theory **39** (2001), no. 2, 214–221.
[15] I. H. Jeon and B. P. Duggal, *On operators with an absolute value condition*, J. Korean Math. Soc. **41** (2004), no. 4, 617–627.
[16] I. H. Jeon and I. H. Kim, *On operators satisfying* $T^*|T^2|T \geq T^*|T|^2T$, Lin. Alg. Appl. (to appear)
[17] I. H. Jeon, J. I. Lee and A. Uchiyama, *On p-quasihyponormal operators and quasisimilarity*, Math. Ineq. Appl. **6** (2003), no. 2, 309–315.
[18] K. B. Laursen and M. N. Neumann, *Introduction to local spectral theory*, Clarendon Press, Oxford, 2000.
[19] W. Y. Lee, *Weyl's theorem for operator matrices*, Integral Equations Operator Theory **32** (1998), 319–331.
[20] ———, *Weyl spectra of operator matrices*, Proc. Amer. Math. Soc. **129** (2001), no. 1, 131–138.
[21] W. Y. Lee and S. H. Lee, *A spectral mapping theorem for the Weyl spectrum*, Glasgow Math. J. **38** (1996), no. 1, 61–64.
[22] C. A. McCarthy, $c_\rho$, Israel J. Math. **5** (1967), 249–271.
[23] V. Rakočević, *On one subset of M. Scechter's essential spectrum*, Mat. Vesnik **5** (1981), no. 4, 389–391.
[24] ———, *Operators obeying a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl. **34** (1989), no. 10, 915–919.
[25] C. Schmoeger, *The spectral mapping theorem for the essentail approximate point spectrum*, Collquium Math. **74** (1997), 167–176.
[26] A. Uchiyama, *Inequalities of Putnam and Berger-Shaw for p-quasihyponormal operators*, Integral Equations Operator Theory **34** (1999), no. 1, 91–106.
[27] ———, *Weyl's theorem for class A operators*, Math. Inequal. Appl. **4** (2001), no. 1, 143–150.
[28] A. Uchiyama and S. V. Djordjević, *Weyl's theorem for p-quasihyponormal operators* (preprint).

[29] R. Yingbin and Y. Zikun, *Spectral structure and subdecomposability of p-hyponormal operators*, Proc. Amer. Math. Soc. **128** (2000), no. 7, 2069–2074.

Bhagwati P. Duggal
8 Redwood Grove
Northfields Avenue
Ealing, London W54SZ, U.K.
*E-mail*: bpduggal@yahoo.co.uk

In Ho Jeon
Department of Mathematics Education
Seoul National University of Education
Seoul 137-742, Korea
*E-mail*: jihmath@snue.ac.kr

In Hyoun Kim
Department of Mathematics
Seoul National University
Seoul 151-747, Korea
*E-mail*: ihkim@math.snu.ac.kr