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LUMPED PARAMETER MODELS OF CARDIOVASCULAR CIRCULATION IN NORMAL AND ARRHYTHMIA CASES

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ABSTRACT. A new mathematical model of pumping heart coupled to lumped compartments of blood circulation is presented. This lumped pulsatile cardiovascular model consists of eight compartments of the body that include pumping heart, the systemic circulation, and the pulmonary circulation. The governing equations for the pressure and volume in each vascular compartment are derived from the following equations: Ohm's law, conservation of volume, and the definition of compliances. The pumping heart is modeled by the time-dependent linear curves of compliances in the heart. We show that the numerical results in normal case are in agreement with corresponding data found in the literature. We extend the developed lumped model of circulation in normal case into a specific model for arrhythmia. These models provide valuable tools in examining and understanding cardiovascular diseases.

1. Introduction

A new mathematical model of pumping heart coupled to lumped compartments of blood circulation is presented. This lumped pulsatile cardiovascular model consists of eight compartments of the body that include pumping heart, the systemic circulation, and the pulmonary circulation. The heart consists of four chambers and four cardiac valves. The systemic and the pulmonary circulations consist of arteries and veins. The structure of the circulation is as follows. When the left ventricular pressure exceeds the systemic aortic pressure, the aortic valve opens and blood flows into the systemic arteries. The blood flows into the systemic veins and the veins return the blood to the right atrium.

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When the right arterial pressure exceeds the right ventricular pressure, the tricuspid valve opens and the right ventricle is filled with blood. Subsequently, blood is ejected into the pulmonary arteries through the pulmonary valve. The pulmonary arteries distribute the blood to the tissue of the lung flows into the pulmonary veins and the veins return the blood to the left atrium. This completes the circulation. In our lumped parameter model, the entire human cardiovascular circulation system is described as a network of the compliance and resistance vessels. The governing equations for the pressure and flow in each compartment are derived from the following three equations: Ohm's law, conservation of volume, and the definition of compliances. Four chambers, two arteries, and two veins are connected by the time varied activated functions for pumping heart. Our intention of this work is to develop a cardiovascular circulation model as simple as possible, in order to investigate the blood mechanisms during circulation and use this lumped model as a tool for educational purpose. For such reasons, we use linear functions to model a time-dependent compliance as a pumping source. We show that the numerical results of the pressure curves in normal case are in agreement with corresponding data found in the literature. We extend the developed lumped model of the cardiovascular circulation in normal case to a model of arrhythmia. These models provide valuable tools in examining and understanding cardiovascular diseases.

In Section 2, a new mathematical model of the cardiovascular circulation is presented. In Section 3, the numerical method and the physiological and computational parameters are described. In Section 4, the numerical results in normal and arrhythmia cases are presented. Conclusions are given in the final section.

2. Mathematical model

In this section, we introduce a new mathematical model of blood circulation in normal case. Figure 1 displays the structure of our lumped parameter model of the circulation. This lumped parameter model consists of eight compartments including pumping heart, systemic circulation, and pulmonary circulation. The heart consists of four chambers: left ventricle (LV), left atrium (LA), right ventricle (RV), and right atrium (RA). There are four cardiac valves in the heart: mitral valve (Mi), aorta valve (Ao), pulmonic valve (Pu), and tricuspid valve (Tr). The systemic and pulmonary circulations consist of arteries and veins: systemic arteries (sa), systemic veins (sv), pulmonary arteries (pa), and pulmonary veins (pv).

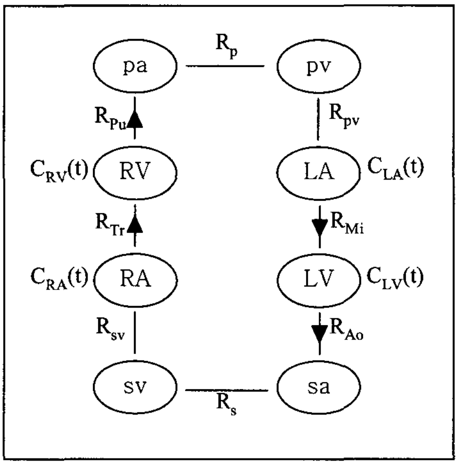

FIGURE 1. A lumped model of the circulation consists of eight compartments, including pumping heart, systemic circulation, and the pulmonary circulation.

The lumped parameter model can be derived from the analogies between electrical current and blood flow and between voltage and blood pressure. In such models resistances represent the ratios of pressure to flow, capacitors represent volume compliances. In our model, all vessels are treated as resistance and compliance vessels. In order to derive a system of differential equations for blood circulation, the following three principles from physiology and physics are used: Ohm's law, definition of compliance, and volume conservation. The cardiac valves are modeled by step functions.

From the Ohm's law, we obtain the relation between pressures and flow as follows:

$$Q(t) = \frac{\Delta P(t)}{R}, \tag{1}$$

where $Q(t)$ is the flow, $\Delta P(t)$ is the pressure difference between the successive compartments, and R is constant flow resistance.

The equation of volume conservation is

$$\frac{dV(t)}{dt} = Q_{in}(t) - Q_{out}(t), \tag{2}$$

where $V(t)$ is the volume, $Q_{in}(t)$ is flow in, and $Q_{out}(t)$ is flow out.

Next, consider the compliance relation for each compartment.

$$V(t) = V_d + C(t)P(t), \tag{3}$$

where V_d is the dead volume when the pressure is zero. The parameter $C(t)$ is the compliance. Note that the compliances are considered as the time-dependent functions at the four cardiac chambers and constants at other compartments of the circulation.

We introduce the cardiac valves in the lumped parameter model. The function of cardiac valves is to ensure to get a unidirectional flow by preventing the reversal flow. The step function, $S(t)$, is used to model a cardiac valve as follows: $S(t) = 1$ if valve is open and $S(t) = 0$ otherwise. For example, the mitral valve, $S_{Mi}(t) = 1$ if $P_{LA}(t) > P_{LV}(t)$ and $S_{Mi}(t) = 0$ otherwise.

Now, consider the eight compartments for whole circulation and combine the equations (1)–(3) and valves together. Then we can derive the following eight ordinary differential equations (ODEs) for the pressures:

$$\begin{cases}
\frac{d(C_{LA}(t)P_{LA}(t))}{dt} = \frac{P_{pv}(t)-P_{LA}(t)}{R_{pv}} - \frac{S_{Mi}(t)(P_{LA}(t)-P_{LV}(t))}{R_{Mi}}, \\
\frac{d(C_{LV}(t)P_{LV}(t))}{dt} = \frac{S_{Mi}(t)(P_{LA}(t)-P_{LV}(t))}{R_{Mi}} - \frac{S_{Ao}(t)(P_{LV}(t)-P_{sa}(t))}{R_{Ao}}, \\
\frac{C_{sa}dP_{sa}(t)}{dt} = \frac{S_{Ao}(t)(P_{LV}(t)-P_{sa}(t))}{R_{Ao}} - \frac{P_{sa}(t)-P_{sv}(t)}{R_s}, \\
\frac{C_{sv}dP_{sv}(t)}{dt} = \frac{P_{sa}(t)-P_{sv}(t)}{R_s} - \frac{P_{sv}(t)-P_{RA}(t)}{R_{sv}}, \\
\frac{d(C_{RA}(t)P_{RA}(t))}{dt} = \frac{P_{sv}(t)-P_{RA}(t)}{R_{sv}} - \frac{S_{Tr}(t)(P_{RA}(t)-P_{RV}(t))}{R_{Tr}}, \\
\frac{d(C_{RV}(t)P_{RV}(t))}{dt} = \frac{S_{Tr}(t)(P_{RA}(t)-P_{RV}(t))}{R_{Tr}} - \frac{S_{Pu}(t)(P_{RV}(t)-P_{pa}(t))}{R_{Pu}}, \\
\frac{C_{pa}dP_{pa}(t)}{dt} = \frac{S_{Pu}(t)(P_{RV}(t)-P_{pa}(t))}{R_{Pu}} - \frac{P_{pa}(t)-P_{pv}(t)}{R_p}, \\
\frac{C_{pv}dP_{pv}(t)}{dt} = \frac{P_{pa}(t)-P_{pv}(t)}{R_p} - \frac{P_{pv}(t)-P_{LA}(t)}{R_{pv}}.
\end{cases} \tag{4}$$

In order to get the motion of blood circulation, we need to generate the pumping heart. The time-dependent compliances are used as a pumping source in our model. We use the definition of compliances to activate the heart. Recall the definition of compliance: $V(t) = V_d + C(t)P(t)$.

Let's consider the compliance at the left ventricle. The figure in the top panel of Figure 2 displays the pressure and volume diagram for the left ventricle. The seven phases over one cardiac cycle are

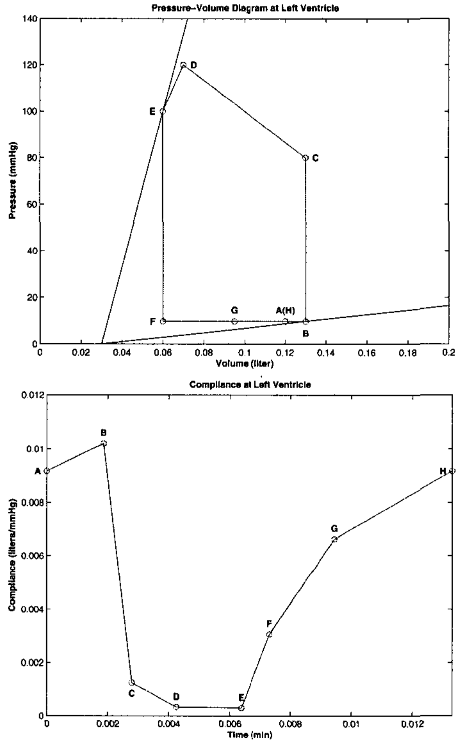

FIGURE 2. The pressure-volume diagram in the left ventricle is displayed in the top panel. The time-dependent compliance in the left ventricle is displayed in the bottom panel. This compliance, which is a sum of eight linear functions, is derived from the definition of compliance at the given eight points in the pressure-volume diagram.

named as follows: AB=Atrial Contraction, BC=Isovolumetric Contraction, CD=Rapid Ejection, DE=Reduced Ejection, EF=Isovolumetric Relaxation, FG=Rapid Filling, and GH=Reduced Filling. The values of compliances are calculated at eight points that are chosen at the end of each phase period. The periodic time-dependent compliance consists of seven linear functions by connecting the eight points. The figure in the bottom panel of Figure 2 displays the compliance at the left ventricle. The time-dependent compliances at left atrium, right ventricle, and right atrium are also chosen in the same way. The values of volumes, pressures, and compliances at four chambers are given in the Table 1.

Note that we assume the periodic motion for heart pumping, the compliance functions at heart are periodic functions: $C(t+T) = C(t)$ *for a period* T *and time* t.

TABLE 1. The values of volumes (liter), pressures (mmHg), and compliances (liter/mmHg) at eight points in the pressure-volume diagram for the left ventricle, left atrium, right ventricle, and right atrium are displayed.

Time (min)		LV			LA		
		V	P	C	V	P	C
A	0	0.12	9.8	0.0092	0.07	10	0.0040
B	0.0019	0.13	9.8	0.0102	0.06	10	0.0030
C	0.0028	0.13	80	0.0013	0.06	10	0.0030
D	0.0043	0.07	120	0.0003	0.10	10	0.0070
E	0.0064	0.06	100	0.0003	0.125	10	0.0095
F	0.0073	0.06	9.8	0.0031	0.13	10	0.0100
G	0.0094	0.095	9.8	0.0066	0.095	10	0.0065
H	0.0133	0.12	9.8	0.0092	0.07	10	0.0040

Time (min)		RV			RA		
		V	P	C	V	P	C
A	0	0.12	4.8	0.0187	0.07	5	0.0080
B	0.0019	0.13	4.8	0.0208	0.06	5	0.0060
C	0.0028	0.13	15	0.0067	0.06	5	0.0060
D	0.0043	0.07	25	0.0016	0.10	5	0.0140
E	0.0064	0.06	20	0.0015	0.125	5	0.0190
F	0.0073	0.06	4.8	0.0063	0.13	5	0.0200
G	0.0094	0.095	4.8	0.0135	0.095	5	0.0130
H	0.0133	0.12	4.8	0.0187	0.07	5	0.0080

In the following section, we introduce the numerical method to solve a system of ODEs in our lumped model and also present the physiological and computational parameters.

3. Numerical method and parameters

In this section, we present a numerical method and physiological and computational parameters to find a numerical solution of the system of ODEs (4). The backward Euler method is used to solve the system of ODEs in our simulations. Let superscripts denote the timestep index and let the time proceed in steps of duration Δt. The differential equations are discretized at time $t = n\Delta t : P^n = P(n\Delta t)$, where $n = 0, 1, \ldots$. Our goal is to compute the updated pressures P^{n+1} at each compartment from given P^n. This is done as follows:

$$
(5)\begin{cases}
\frac{C_{LA}^{n+1}P_{LA}^{n+1}-C_{LA}^{n}P_{LA}^{n}}{\Delta t} = \frac{P_{pv}^{n+1}-P_{LA}^{n+1}}{R_{pv}} - \frac{S_{Mi}^{n+1}(P_{LA}^{n+1}-P_{LV}^{n+1})}{R_{Mi}},\\
\frac{C_{LV}^{n+1}P_{LV}^{n+1}-C_{LV}^{n}P_{LV}^{n}}{\Delta t} = \frac{S_{Mi}^{n+1}(P_{LA}^{n+1}-P_{LV}^{n+1})}{R_{Mi}} - \frac{S_{Ao}^{n+1}(P_{LV}^{n+1}-P_{sa}^{n+1})}{R_{Ao}},\\
\frac{C_{sa}(P_{sa}^{n+1}-P_{sa}^{n})}{\Delta t} = \frac{S_{Ao}^{n+1}(P_{LV}^{n+1}-P_{sa}^{n+1})}{R_{Ao}} - \frac{P_{sa}^{n+1}-P_{sv}^{n+1}}{R_{s}},\\
\frac{C_{sv}(P_{sv}^{n+1}-P_{sv}^{n})}{\Delta t} = \frac{P_{sa}^{n+1}-P_{sv}^{n+1}}{R_{s}} - \frac{P_{sv}^{n+1}-P_{RA}^{n+1}}{R_{sv}},\\
\frac{C_{RA}^{n+1}P_{RA}^{n+1}-C_{RA}^{n}P_{RA}^{n}}{\Delta t} = \frac{P_{sv}^{n+1}-P_{RA}^{n+1}}{R_{sv}} - \frac{S_{Tr}^{n+1}(P_{RA}^{n+1}-P_{RV}^{n+1})}{R_{Tr}},\\
\frac{C_{RV}^{n+1}P_{RV}^{n+1}-C_{RV}^{n}P_{RV}^{n}}{\Delta t} = \frac{S_{Tr}^{n+1}(P_{RA}^{n+1}-P_{RV}^{n+1})}{R_{Tr}} - \frac{S_{Pu}^{n+1}(P_{RV}^{n+1}-P_{pa}^{n+1})}{R_{Pu}},\\
\frac{C_{pa}(P_{pa}^{n+1}-P_{pa}^{n})}{\Delta t} = \frac{S_{Pu}^{n+1}(P_{RV}(t)-P_{pa}^{n+1})}{R_{Pu}} - \frac{P_{pa}^{n+1}-P_{pv}^{n+1}}{R_{p}},\\
\frac{C_{pv}(P_{pv}^{n+1}-P_{pv}^{n})}{\Delta t} = \frac{P_{pa}^{n+1}-P_{pv}^{n+1}}{R_{p}} - \frac{P_{pv}^{n+1}-P_{LA}^{n+1}}{R_{pv}}.
\end{cases}
$$

In order to update the unknown pressures at eight compartments from the discretized equations (5), we need to solve a 8×8 system. If the states of the cardiac valves were known, this would be a 8×8 linear system in the eight unknowns for pressures. It is actually a nonlinear system, since the values of valves depend on the value of pressure at the same time. However, we solve the 8×8 linear system in our computation. The states of valves are determined as known values by trial and error as follows: First, start with any state of the valves and solve the linear system for the pressures. Then we reset the valves according to the calculated pressures. Repeat this process until the calculated values of valves are the same as the starting values of valves. Since there are four valves and each valve has the state either open or close, there are sixteen possible valve states: $(S_{Mi}(t), S_{Ao}(t), S_{Tr}(t), S_{Pu}(t)) = (0,0,0,0), (1,0,0,0), \ldots, (1,1,1,1)$, where 1 and 0 represent open and close, respectively. Note that the four time-dependent compliances at heart are known values as pumping functions (see the previous section).

The initial values for pressures at the eight compartments are chosen as $P_{LA} = 10$, $P_{LV} = 9.8$, $P_{sa} = 100$, $P_{sv} = 5$, $P_{RA} = 5$, $P_{RV} = 4.8$, $P_{pa} = 20$, and $P_{pv} = 10$. The computational parameters are as follows: the period, T, is 0.0133 min, the timestep, Δt, is $0.01 \cdot T$ min, and the total simulated duration is chosen as 1 min. The physiological resting values for resistances, constant compliances, and dead volumes at each compartment are displayed in Table 2. These parameters are based on the literatures [3, 4].

TABLE 2. The resting values for resistances (mmHg/(liter/min)), compliances (liter/mmHg), and dead volumes (liter) in normal case are displayed.

R		C	V_d	
$R_{Mi} = 0.01$	$R_s = 16.964$	$C_{sa} = 0.0018$	$V_{LAd} = 0.03$	$V_{RAd} = 0.03$
$R_{Ao} = 0.01$	$R_{sv} = 0.05$	$C_{sv} = 0.7$	$V_{LVd} = 0.03$	$V_{RVd} = 0.03$
$R_{Tr} = 0.01$	$R_p = 1.786$	$C_{pa} = 0.0046$	$V_{sad} = 0.825$	$V_{pad} = 0.038$
$R_{Pu} = 0.01$	$R_{pv} = 0.05$	$C_{pv} = 0.04$	$V_{svd} = 0$	$V_{pvd} = 0$

4. Results and discussion

In this section, we first discuss the numerical results of the computed pressure and volume curves from the lumped model in normal case. Next, this developed lumped model in normal case is modified to a model for arrhythmia case. We also present some numerical results for arrhythmia case.

4.1. Normal case

The numerical results that we obtained by solving the system of the discretized equations (5) are presented. In Figure 3, the computed systemic arterial pressure during the simulated duration, 1 min, is given in the top panel and the computed pressure-volume relation in the left ventricle during the same simulated duration is displayed in the bottom panel. The computed systemic arterial pressures are 109.5/77.6 mmHg after the periodic steady state. These numerical results elegantly approximate representative curves for normal human case as found in [2, 4]. The stoke volume is around 0.07 liter and the heart rate is 1.25 Hz, resulting in a cardiac output equal to 5.41 liter/min. These computed results are also close to data for a normal person found in [1, 6]. In Figure 4, the computed left ventricular pressure (solid curve), left atrium pressure (dashed curve), and the systemic arterial pressure (dashdot curve) are plotted in the top panel and the computed right ventricular pressure (solid curve), right atrium pressure (dashed curve), and the pulmonary arterial pressure (dashdot curve) are plotted in the bottom panel. All curves in Figure 4 approximate representative curves for normal human case as found in [2, 5]. Since we use linear functions for the cardiac compliances as pumping source, the pressure curves are not smooth enough. However, we could obtain the qualitatively good numerical results that can be verified with data found in the literature [2, 5].

4.2. Arrhythmia case

A patient who has an arrhythmia suffers from the irregular heartbeats. In our lumped model of arrhythmia, we use a random number in order to model the irregular heartbeats.

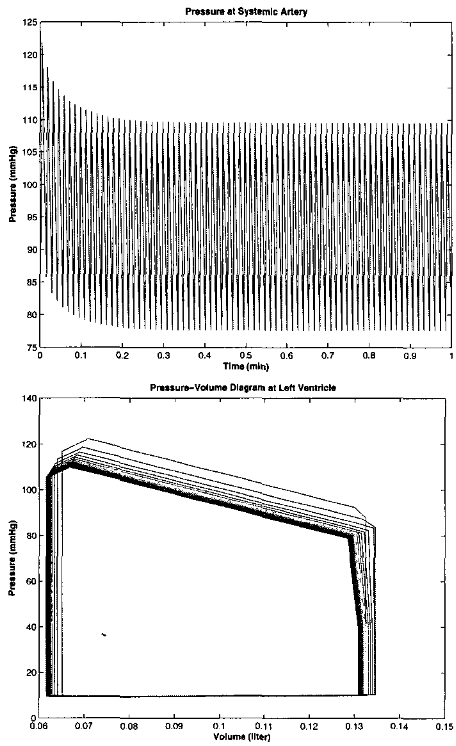

FIGURE 3. Normal Case: The computed systemic arterial pressure during the simulated duration, 1 min, is displayed in the top panel and the pressure-volume relation in the left ventricle is displayed in the bottom panel.

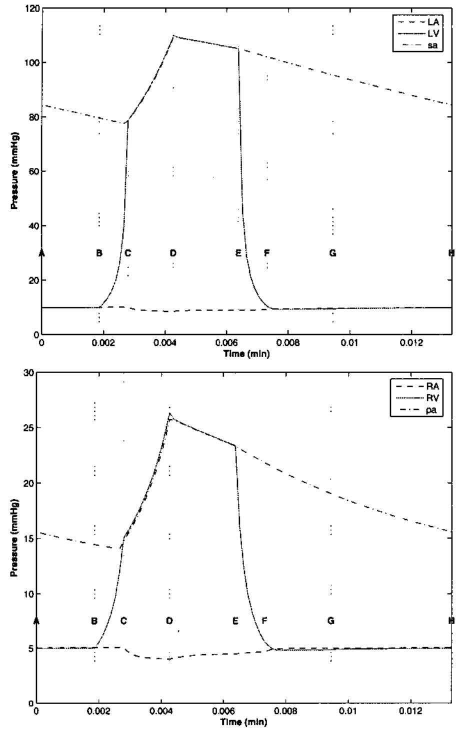

FIGURE 4. Normal Case: The computed pressures at LV (solid curve), LA (dashed curve), and sa (dashdot curve) are plotted in the top panel and the computed pressures at RV (solid curve), RA (dashed curve), and pa (dashdot curve) are plotted in the bottom panel.

If a random number is less than 0.05 (5 out of 100), then we apply the modified compliance at the left ventricle (see Table 3) instead of one in normal case. Other cardiac compliances are chosen as the same as normal case. Once random number is chosen less than 0.05, the period is reduced as $(2/3) \cdot T$ for the first two cycles and then the period, T, for normal case is applied to the rest of cycles. If random number is great than equal to 0.05, then the normal conditions are applied to the model. In Figure 5, the computed systemic arterial pressure during simulated duration, 0.9646 min, is plotted in the top panel and the pressure-volume relation in the left ventricle is displayed in the bottom panel. In our simulation the computed systemic arterial pressures have 4 peaks that are obtained when the random numbers are chosen less than 0.05. The four random numbers less than 0.05 are chosen at 8^{th}, 26^{th}, 35^{th}, and 48^{th} beats. The reason of the shorter total simulated time is that the period is reduced for the first two cycles after the random number is chosen less than 0.05. The total simulated time is obtained as follows: $0.9646 \text{ min} = 1 - 4 \cdot 2 \cdot ((2/3 \cdot T)) \text{ min}$, where $T = 0.0133$ min. Both the computed systemic arterial pressures and the pressure-volume curves in the left ventricle show the irregular heartbeats.

TABLE 3. The time-dependent compliances at the left ventricle for normal and arrhythmia cases are displayed.

Time (min)		Normal: LV			Arrhythmia: LV		
		V	P	C	V	P	C
A	0	0.12	9.8	0.0092	0.12	9.8	0.0092
B	0.0019	0.13	9.8	0.0102	0.13	9.8	0.0102
C	0.0028	0.13	80	0.0013	0.13	80	0.0013
D	0.0043	0.07	120	0.0003	0.09	106.5	0.0006
E	0.0064	0.06	100	0.0003	0.085	100	0.0006
F	0.0073	0.06	9.8	0.0031	0.085	9.8	0.0056
G	0.0094	0.095	9.8	0.0066	0.095	9.8	0.0066
H	0.0133	0.12	9.8	0.0092	0.12	9.8	0.0092

5. Conclusions

We have presented a lumped parameter model of the human circulation in normal and arrhythmia cases. This lumped model of the circulation consists of eight compartments including pumping heart, the systemic circulation, and the pulmonary circulation. The governing equations for the pressure and volume in each vascular compartment amount to the three equations characterizing motion by Ohm's law, conservation of volume, and the definition of compliances. We have introduced the time-dependent compliances at the cardiac compartments as pumping heart. One of our goals in this research is to develop a model of the circulation as simple as possible. For such reason, the linear functions are used for the compliances at the four cardiac chambers as pumping source in our model. In our knowledge, this is a new try to construct such a simple pumping source without losing important features of the circulation. So we are sure that this lumped model of the circulation will

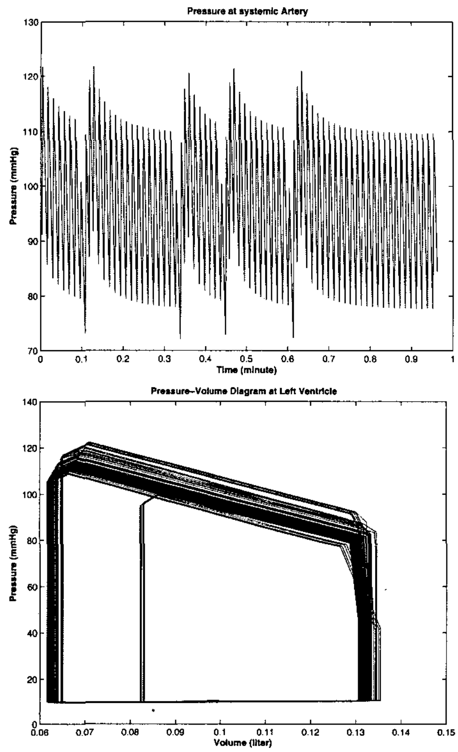

FIGURE 5. Arrhythmia Case: The computed systemic arterial pressure during the simulated duration, 0.9646 min, is displayed in the top panel and the pressure-volume relation in the left ventricle during the same simulated duration is displayed in the bottom panel.

be a great tool for understanding the mechanisms of blood circulation especially for person who does not have much background on mathematics or who is an undergraduate student.

We first showed that the computed pressures and volumes in our lumped model of the circulation in normal case approximated representative curves for normal human case as found in [2, 5]. We have modified the developed our model to get a model of arrhythmia. A random number was introduced for modeling the irregular heartbeats. The compliance at the left heart and period were changed when the certain random numbers were chosen. We could observe the irregular heartbeats from this modified lumped model. Such models are valuable tools in examining and understanding cardiovascular diseases not only arrhythmia but also hypertension, weak and enlarged hearts, and many others. Moreover, these models may also give new insights into cardiovascular functions and blood circulation.

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