J. Korean Math. Soc. **43** (2006), No. 4, pp. 871–883

ON COLUMN INVARIANT AND INDEX OF COHEN-MACAULAY LOCAL RINGS

JEE KOH AND KISUK LEE

ABSTRACT. We show that the Auslander index is the same as the column invariant over Gorenstein local rings. We also show that Ding's conjecture ([3]) holds for an isolated non-Gorenstein ring A satisfying a certain condition which seems to be weaker than the condition that the associated graded ring of A is Cohen-Macaulay.

0. Introduction

A Cohen-Macaulay approximation was defined by Auslander and Buchweitz for a Gorenstein local ring A as follows: Let M be an A-module. An exact sequence of A-modules, $0 \to Y_M \xrightarrow{\varphi} X_M \xrightarrow{\phi} M \to 0$, is called a Cohen-Macaulay approximation of M if $\text{projdim}_R Y_M < \infty$ and X_M is a maximal Cohen-Macaulay module. In [2], Auslander introduced the numerical invariant $\delta(M)$ for any finite module M over a Gorenstein local ring A: $\delta(M)$ is defined to be the maximal rank of free summands of X_M in a minimal Cohen-Macaulay approximation of M. In [3, 4], Ding has studied the δ-invariant of cyclic modules $A/\mathfrak{m}^i (i \geq 1)$ and defined a new invariant index(A). He also conjectured that index(A) is the same as $\ell\ell(A)$, the generalized Loewy length of A.

It was proved in [8] that there are certain restrictions on the entries of the maps in the minimal free resolutions of finitely generated modules of infinite projective dimension over Noetherian local rings A. Using these restrictions, some new invariants were introduced in [9]: They are (see Definition 1.1): col(A) [resp. row(A)] for a number associated with the columns [resp. rows] of the maps and crs(A) and drs(A), which are

Received August 1, 2005.
2000 Mathematics Subject Classification: 13H10, 13C14, 13D02.
Key words and phrases: column invariant, index, Loewy length, Cohen-Macaulay ring, Gorenstein ring.
The second author was supported by 2004 Sookmyung Women's University Research Grant.

associated with the cyclic modules determined by regular sequences and their Matlis duals. It was shown in [9, Proposition 1.4] that drs(A) is equal to $\ell\ell(A)$ if A is Cohen-Macaulay.

The purpose of this paper is to relate the (Auslander) index of a Cohen-Macaulay local ring to some of the invariants considered above.

In Section 1, we show that for a Cohen-Macaulay local ring A, index (A) can be described in terms of the columns of presenting matrices of maximal Cohen-Macaulay modules without free summands (Proposition 1.5). We also show that index(A) = col(A) if A is Gorenstein (Corollary 1.7). As a consequence we obtain a result in [13] on the behavior of the index of Gorenstein local rings under an extension of finite flat dimension (Corollary 1.8). We also discuss some of properties concerning the δ-invariant of Matlis dual of a module of finite length.

In Section 2, we obtain that for Gorenstein local rings A with infinite residue fields the conjecture in [9] is equivalent to Ding's conjecture in [3], which asserts that index$(A) = \ell\ell(A)$. (A surprising counterexample to Ding's conjecture when the residue field is F_2 was given in [6].) The main result (Theorem 2.1) of this paper is:

THEOREM. *Let* $(A, \mathfrak{m})$ *be a non regular Cohen-Macaulay local ring of dimension* d. *Suppose there is a system of parameters* $\mathbf{x} = x_1, \ldots, x_d$ *such that the following two conditions* $(*)$ *are satisfied: for some positive integer* r,

i) $\mathfrak{m}^{r+1} \subseteq (\mathbf{x})$, *but* $\mathfrak{m}^r \not\subseteq (\mathbf{x})$, *and*

ii) $\mathfrak{m}^{r+1} \cap I_k = \mathfrak{m}^r I_k$ *for* $k = 1, \ldots, d$, *where* $I_k = (x_1, \ldots, x_k)$.

Then $\mathrm{col}_{CM}(A) \geq r + 1$. *In particular,* $\mathrm{col}_{CM}(A) \geq \ell\ell(A)$.

This theorem immediately implies that Ding's conjecture (or the conjecture in [9]) holds for Gorenstein local rings satisfying the condition $(*)$ because $\mathrm{col}_{CM}(A) = \mathrm{index}(A)$ and the inequality in the other direction is shown to hold in [3] for Gorenstein local rings. We remark that the condition $(*)$ is at least weaker than the condition the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(A)$ of A is Cohen-Macaulay which was assumed in showing the equality in [5] because every minimal reduction satisfies $(*)$ in this case (the residue field has to be infinite so that there is a minimal reduction).

Although all rings we consider in this paper are commutative, Noetherian with identity, and all modules are unital, we emphasize the Noetherian property in our statements. We use the usual notation $E(A/\mathfrak{m})$ for the injective hull of $A/\mathfrak{m}$ and $M^\vee$ for Matlis dual, $\mathrm{Hom}_A(-, E(A/\mathfrak{m}))$.

1. col(A) =index(A) over a Gorenstein local ring A

In this section we recall the invariants defined in [9], and the Auslander index, index(A) and the generalized Loewy length $\ell\ell(A)$. We also state the basic properties of these invariants. In particular, we show that col(A) is the same as index(A) over a Gorenstein local ring A.

DEFINITION 1.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring. We denote by $\varphi_i(M)$ the ith map in a minimal resolution of a finitely generated A-module M. We also use the usual notation $\mathrm{Soc}(M) = \mathrm{Hom}_A(A/\mathfrak{m}, M)$ to denote the socle of M.

If projdim M $< \infty$, we define $\mathrm{col}(M) = 1$. If projdim M $= \infty$, we define

i) $\mathrm{col}(M) =: \inf\{t \geq 1$: each column of $\varphi_i(M)$ has an entry outside $\mathfrak{m}^t$, for all $i > 1 + \operatorname{depth} A\ \}$.
$\mathrm{col}(A) =: \sup\{\mathrm{col}(M)$: M is a finitely generated A-module $\}$.

We define the 'socle number' $s(M)$ by $s(M) := \inf\{t \geq 1 : Soc(M) \not\subseteq \mathfrak{m}^t M\}$.

ii) $\mathrm{crs}(A) =: \inf\{s(A/(x))$: $\mathbf{x}$ is a maximal regular sequence $\}$.
$\mathrm{drs}(A) =: \inf\{s((A/(x))^\vee)$: $\mathbf{x}$ is a system of parameters of A $\}$.

For a Cohen-Macaulay local ring $(A, \mathfrak{m})$, we also define $\mathrm{col}_{CM}(M)$ to be the smallest $t \geq 1$ such that each column of the presenting matrix of M contains an element outside $\mathfrak{m}^t$ for a maximal Cohen-Macaulay module M without free summands. We now define:

iii) $\mathrm{col}_{CM}(A) =: \sup\{\mathrm{col}_{CM}(M) : M$ is a maximal Cohen-Macaulay module without free summands.$\}$

We recall the definition of the generalized Loewy length of A:

$$\ell\ell(A) = \inf\{t \geq 1 : \mathfrak{m}^t \subseteq (\mathbf{x}) \text{ for some system of parameters } \mathbf{x}\}.$$

REMARKS 1.2. i) We may describe col(A) as follows ([11]):

$\mathrm{col}(A) = \inf\{t \geq 1$: each column of the presenting matrix φ of a (d+1)-st syzygy module M contains an element outside $\mathfrak{m}^t\}$.

ii) We recall that for a Cohen-Macaulay local ring A, the equality $\mathrm{drs}(A) = \ell\ell(A)$ was shown in [9, Proposition 1.4.i)].

We now recall the basic properties of Auslander δ-invariant: Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical module ω. For a finitely generated A-module X, define $f\text{-rank}(X) := r$ if $X = A^r \oplus U$, where U has no free summands. We note that this r is well defined because for any surjective map $\phi : X \to A^s$, $\phi(A^r) = A^s$ ($\phi(U) \subseteq \mathfrak{m}A^s$ since U has no free summands). We recall the definition of $\delta(M)$:

$\delta(M) :=$ inf $\{f\text{-rank}(X) : X$ is a maximal Cohen-Macaulay module and M is a homomorphic image of $X\}$.

We now recall Cohen-Macaulay approximation established in [1]: For each finitely generated A-module M, there is an exact sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow M \longrightarrow 0$, where X is a maximal Cohen-Macaulay module and Y is of finite injective dimension.

It is also known (see [2], [6] or [15]) that there is a unique (up to isomorphism) minimal one which is denoted by

$$0 \longrightarrow Y_M \longrightarrow X_M \longrightarrow M \longrightarrow 0,$$

with the property that if X is any maximal Cohen-Macaulay A-module that maps onto M, then X maps onto X_M.

The definition of $\delta(M)$ and the properties of minimal Cohen-Macaulay approximation immediately imply:

1) if $N \cong M/M_1$, then $\delta(M) \geq \delta(N)$,
2) $\delta(M) = f\text{-rank}(X_M)$, and
3) if A is Gorenstein and M is of finite projective dimension, then $\delta(M) > 0$.

We now recall:

DEFINITION 1.3. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical module. The index is define by

$$\text{index}(A) = \inf \{t \geq 1 : \delta(A/\mathfrak{m}^t) > 0\}.$$

REMARK 1.4. It was shown in [4] that index$(A) < \infty$ if and only if A is isolated non-Gorenstein. For such rings, we note that if $\mathfrak{m}^t \subseteq (\mathbf{x})$ for some system of parameters $\mathbf{x} = x_1, \ldots, x_d$, then $\delta(A/\mathfrak{m}^t) \geq \delta(A/(\mathbf{x})) > 0$ by 1) and 3) above. Hence index$(A) \leq \ell\ell(A)$.

We describe the index in terms of the columns of the presenting matrix of maximal Cohen-Macaulay modules to relate to col(A) for Gorenstein local rings A.

PROPOSITION 1.5. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical module. Then*

$$\text{index}(A) = \text{col}_{CM}(A).$$

Proof. We claim that i) $\operatorname{index}(A) = \infty$ if and only if $\operatorname{col}_{CM}(A) = \infty$, and ii) if one of them is finite, then they are the same. To prove these claims, it is enough, by the definitions of $\operatorname{col}_{CM}(\text{-})$ and index(-), to show that for any positive integer c, $\delta(A/\mathfrak{m}^c) = 0$ if and only if there is a maximal Cohen-Macaulay module X without free summands such that every entry of some column of the presenting matrix of X is contained in $\mathfrak{m}^c$.

Suppose that $\delta(A/\mathfrak{m}^c) = 0$. Then there exists a maximal Cohen-Macaulay module X without free summands such that $\epsilon : X \to A/\mathfrak{m}^c \to 0$ is an epimorphism. Let $x_1, \dots, x_{m_0}$ be minimal generators of X. Since ϵ is onto, we may assume that

(1) $\epsilon(x_1) = \bar{1}$
(2) $\epsilon(x_i) = \bar{0}$ for $i = 2, \dots, m_0$ by replacing x_i by $x_i' = x_i - a_i \cdot x_1$ if necessary, where $\epsilon(x_i) = \overline{a_i}$.

Let

$$G_\bullet : \cdots \longrightarrow A^{m_1} \xrightarrow{\psi} A^{m_0} \xrightarrow{\phi} X \longrightarrow 0$$

be a minimal resolution of X, where $\psi = [\psi_{ij}]_{m_1 \times m_0}$ and $\phi(e_i) = x_i$, where $\{e_i\}$ is a basis of A^{m_0}. Then for $(1, 0, \dots, 0) \in A^{m_1}$,

$$\begin{aligned} 0 &= \phi \circ \psi((1, 0, \dots, 0)) \\ &= \phi((\psi_{11}, \psi_{12}, \dots, \psi_{1m_0})) \\ &= \psi_{11} \cdot x_1 + \psi_{12} \cdot x_2 + \cdots + \psi_{1m_0} \cdot x_{m_0}, \end{aligned}$$

and so

$$\begin{aligned} \bar{0} &= \epsilon(\psi_{11} \cdot x_1 + \psi_{12} \cdot x_2 + \cdots + \psi_{1m_0} \cdot x_{m_0}) \\ &= \psi_{11} \cdot \bar{1} \quad \text{in } A/\mathfrak{m}^c, \text{ i.e., } \psi_{11} \in \mathfrak{m}^c. \end{aligned}$$

Similarly, we can also show that $\psi_{i1} \in \mathfrak{m}^c$ for $i = 2, \dots, m_1$. In other words, every entry of the first column of ψ belongs to $\mathfrak{m}^c$.

To prove the converse, suppose that there is a maximal Cohen-Macaulay module X without free summands such that every entry of some column, say the first column, of the presenting matrix of M is contained in $\mathfrak{m}^c$. Now, consider the following diagram with η and h described below:

$$\begin{array}{ccccc} \cdots \to A^{n_1} & \xrightarrow{\varphi_1} & A^{n_0} & \xrightarrow{\varphi_0} & X \to 0. \\ & & \eta \searrow & & \swarrow h \\ & & & A/\mathfrak{m}^c & \end{array}$$

Let $\eta : A^{n_0} \xrightarrow{p} A \xrightarrow{\pi} A/\mathfrak{m}^c$, where p is the projection onto the first summand of A^{n_0} and π is the canonical quotient map. Then clearly η is an epimorphism. Note that $\eta(\operatorname{Im} \varphi_1) = 0$ because every entry of the first column of φ_1 belongs to $\mathfrak{m}^c$. Thus $\ker\varphi_0 = \operatorname{Im}\varphi_1 \subseteq \ker\eta$. By the universal mapping property, there exists an epimorphism $h : X \longrightarrow A/\mathfrak{m}^c$. Thus $\delta(A/\mathfrak{m}^c) = 0$ by the definition of δ-invariant. This completes the proof. □

In [11], it is shown that $\operatorname{col}(A) = \operatorname{col}_{CM}(A)$ if A is a Gorenstein local ring, using the following fact:

FACT 1.6. Let $(A, \mathfrak{m})$ be a Gorenstein local ring and X a maximal Cohen-Macaulay A-module without free summands. Then for any integer $\ell \geq 0$, X is an ℓ-th syzygy.

Thus we have the following corollary:

COROLLARY 1.7. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring. Then*

$$\operatorname{index}(A) = \operatorname{col}(A).$$

Proof. It is clear since $\operatorname{index}(A) = \operatorname{col}_{CM}(A) = \operatorname{col}(A)$ by Proposition 1.5 and Theorem 3.6 in [11]. □

Using Fact 1.6 and Corollary 1.7, we may extend [9, Proposition 2.6] as follows:

COROLLARY 1.8. *Let $\varphi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism of Gorenstein local rings. If φ is of finite flat dimension, then $\operatorname{col}(A) \leq \operatorname{col}(B)$. In particular, $\operatorname{index}(A) \leq \operatorname{index}(B)$ ([13, Theorem 3.7]).*

Proof. Let q be the flat dimension of φ, i.e., $\operatorname{Tor}_i^A(M, B) = 0$ for any A-module M and $i > q$. By the theorem above, there is a maximal Cohen-Macaulay A-module M without free summands such that its minimal presenting matrix has a column consisting entirely of elements in $\mathfrak{m}^{\operatorname{col}(A)-1}$. Let $A^{n_1} \xrightarrow{\phi} A^{n_0} \to M \to 0$ be a minimal presentation of M. Since A is Gorenstein and M is a maximal Cohen-Macaulay module without free summands, we may assume that M is a $(1 + q + \dim B)$-th syzygy of some finitely generated A-module N by Fact 1.6. Write $t = q + \dim B$. From any minimal resolution $(F_\bullet, \lambda_\bullet)$ of N, we can get the following minimal resolution of N:

$$\begin{array}{ccccccccccc} G_\bullet : \cdots \to A^{n_1} \xrightarrow{\phi} & A^{n_0} & \longrightarrow & A^{m_t} \xrightarrow{\lambda_t} A^{m_{t-1}} \to \cdots \xrightarrow{\lambda_1} A^{m_0} \to 0. \\ & & \searrow \quad \nearrow & \\ & & M & \end{array}$$

Applying $(-) \otimes_A B$ to $G_\bullet$, we get the following minimal resolution of $\operatorname{Coker} \lambda_{q+1} \otimes_A 1$:

$$\cdots \to B^{n_1} \xrightarrow{\phi \otimes 1} B^{n_0} \longrightarrow B^{m_t} \xrightarrow{\lambda_t \otimes 1} B^{m_{t-1}} \to \cdots \xrightarrow{\lambda_{q+1} \otimes 1} B^{m_q} \to 0.$$

Since $\phi \otimes 1$ is the $(2 + \dim B)$-th map which has a column consisting of elements in $\mathfrak{n}^{\operatorname{col}(A)-1}$, we know $\operatorname{col}(A) \leq \operatorname{col}(B)$. In particular, $\operatorname{index}(A) \leq \operatorname{index}(B)$ since $\operatorname{index}(-) = \operatorname{col}(-)$. □

REMARK 1.9. When A is Gorenstein (with an infinite residue field), the conjecture in [9] asserts that $\operatorname{col}(A) = \operatorname{drs}(A)$. By Remark 1.2 and Corollary 1.7, the conjecture in [9] is equivalent to Ding's Conjecture which asserts that $\operatorname{index}(A) = \ell\ell(A)$ for Gorenstein local rings A (with an infinite residue field).

We close this section with a discussion concerning the δ-invariant of Matlis dual of a module of finite length. We denote the Matlis dual of M by $M^\vee$, i.e., $M^\vee = \operatorname{Hom}_A(M, E(k))$, and denote a d-th syzygy module of N by $\Omega_d(N)$.

PROPOSITION 1.10. *Let $(A, \mathfrak{m}, k)$ be a complete Gorenstein local ring of dimension d, and M a finitely generated A-module of finite length. If $\delta(M) \neq 0$, then $\Omega_d(M^\vee)$ has a free summand. In particular, $\delta(\Omega_d(M^\vee)) \neq 0$.*

Proof. Let $(F_\bullet, \varphi_\bullet)$ be a minimal resolution of $M^\vee$:

$$F_\bullet : \cdots \to A^{n_{d+1}} \xrightarrow{\varphi_{d+1}} A^{n_d} \xrightarrow{\varphi_d} A^{n_{d-1}} \to \cdots \to A^{n_0} \to M^\vee \to 0.$$

We notice that $\operatorname{Ext}^i_A(M^\vee, A) = 0$ if $i \neq d$, and $\operatorname{Ext}^d_A(M^\vee, A) \cong \operatorname{H}^0_\mathfrak{m}(M^\vee)^\vee \cong M^{\vee\vee} \cong M$ by the local duality since A is a complete Gorenstein local ring. Thus $M \cong \ker(\varphi^*_{d+1})/\operatorname{Im}(\varphi^*_d)$, where $\varphi^*_\bullet = \operatorname{Hom}_A(\varphi_\bullet, A)$. Since a truncated complex of $\operatorname{Hom}_A(F_\bullet, A)$, $0 \to \ker(\varphi^*_{d+1}) \to A^{n_d} \to A^{n_{d+1}} \to A^{n_{d+2}} \to \cdots$ is exact, $\ker(\varphi^*_{d+1})$ is a maximal Cohen-Macaulay module. By the assumption $\delta(M) \neq 0$, $\ker(\varphi^*_{d+1})$ should have a free summand. This fact implies that the number of minimal generators of $\operatorname{Im}(\varphi^*_d)$ is less than n_d, so we assume that φ^*_{d+1} has a row of zeros, i.e., φ_{d+1}, which is a transpose matrix of φ^*_{d+1}, has a column of zeros. Hence $\Omega_d(M^\vee) = \operatorname{Im}(\varphi_d)$ has a free summand. □

In [15], Yoshino has studied the Auslander's higher delta invariants, i.e., $\delta(\Omega_n(M))$, and he has shown that there is an integer t_0 such that $\delta(\Omega_n(A/\mathfrak{m}^t)) = 0$ for any $t \geq t_0$ and for any $n > 0$. The following corollary shows that the above fact does not work on the Matlis dual of $A/\mathfrak{m}^t$.

COROLLARY 1.11. *Let $(A, \mathfrak{m}, k)$ be a complete Gorenstein local ring of dimension d. If $t \geq \operatorname{index}(A)$, then $\Omega_d((A/\mathfrak{m}^t)^\vee)$ has a free summand. In particular, $\delta(\Omega_d((A/\mathfrak{m}^t)^\vee)) \neq 0$.*

Proof. It is clear by Proposition 1.10 since $\delta(A/\mathfrak{m}^t) \neq 0$. □

It is shown in [15] that if the depth of the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(A)$ is $d-1$, then $\delta(\Omega^n(A/\mathfrak{m}^t)) = 0$ for any positive integers t and n, in particular, $\Omega_d(A/\mathfrak{m}^t)$ has no free summand. Therefore, if we replace M in Proposition 1.10 by $(A/\mathfrak{m}^t)^\vee$, then we have the following corollary:

COROLLARY 1.12. *Let $(A, \mathfrak{m}, k)$ be a complete Gorenstein local ring of dimension d. Suppose that the depth of the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(A)$ is $d-1$. Then $\delta((A/\mathfrak{m}^t)^\vee) = 0$ for all $t \geq 1$.*

Proof. If $\delta((A/\mathfrak{m}^t)^\vee) \neq 0$ for some t, then $\Omega_d(A/\mathfrak{m}^t)$ would have a free summand, which contradicts the fact in the above note. Hence $\delta((A/\mathfrak{m}^t)^\vee) = 0$ for all $t \geq 1$. □

2. Index of a Cohen-Macaulay local ring

In this section we prove the following theorem which is the main result of this paper.

THEOREM 2.1. *Let $(A, \mathfrak{m})$ be a non regular Cohen-Macaulay local ring of dimension d. Suppose there is a system of parameters $\mathbf{x} = x_1, \ldots, x_d$ such that the following two conditions $(*)$ are satisfied: for some positive integer r*

i) $\mathfrak{m}^{r+1} \subseteq (\mathbf{x})$, *but* $\mathfrak{m}^r \not\subseteq (\mathbf{x})$, *and*

ii) $\mathfrak{m}^{r+1} \cap I_k = \mathfrak{m}^r I_k$ *for* $k = 1, \ldots, d$, *where* $I_k = (x_1, \ldots, x_k)$.

Then $\operatorname{col}_{CM}(A) \geq r+1$. In particular, $\operatorname{col}_M(A) \geq \ell\ell(A)$.

COROLLARY 2.2. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring satisfying the condition $(*)$ in Theorem 2.1. Then Ding's Conjecture holds, i.e. $\operatorname{index}(A) = \ell\ell(A)$.*

Proof. The conclusion follows from Remark 1.4, Theorem 1.7, and Theorem 2.1. □

REMARK 2.3. i) We remark that the condition $(*)$ is satisfied if the associated graded ring of A, $\operatorname{gr}_{\mathfrak{m}}(A)$, is Cohen-Macaulay and $A/\mathfrak{m}$ is infinite. Indeed, since $A/\mathfrak{m}$ is an infinite field and $\operatorname{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay, we can choose a maximal A-sequence $\mathbf{x} = x_1, \ldots, x_d \in \mathfrak{m} - \mathfrak{m}^2$ such that the initial form, $\widetilde{\mathbf{x}} = \widetilde{x_1}, \ldots, \widetilde{x_d}$, of $\mathbf{x}$ is a $\operatorname{gr}_{\mathfrak{m}}(A)$-sequence. If

A is not regular, then $\mathfrak{m}^{r+1} \subseteq (\mathbf{x})$, but $\mathfrak{m}^r \not\subseteq (\mathbf{x})$ for some positive integer r. It is known ([12, Corollary 1.4], or [14, Corollary 2.7]) that $\widetilde{x_1}, \ldots, \widetilde{x_t} \in \mathfrak{m}/\mathfrak{m}^2$ is a $\mathrm{gr}_{\mathfrak{m}}(A)$-sequence if and only if $(x_1, \ldots, x_t) \cap \mathfrak{m}^{i+1} = (x_1, \ldots, x_t)\mathfrak{m}^i$ for all i (from this fact, we expect that the condition $(*)$ in Theorem 2.1 is weaker than requiring the associated graded ring be Cohen-Macaulay assumed for Gorenstein local rings). Thus since $\widetilde{\mathbf{x}} = \widetilde{x_1}, \ldots, \widetilde{x_d} \in \mathfrak{m}/\mathfrak{m}^2$ is a $\mathrm{gr}_{\mathfrak{m}}(A)$-sequence, the second condition $\mathfrak{m}^{r+1} \cap I_k = \mathfrak{m}^r I_k$, where $k = 1, \ldots, d$, and $I_k = (x_1, \ldots, x_k)$, is also satisfied. We note that $\ell\ell(A)$ is attained for such system of parameters $x_1, \ldots, x_d$, and $\ell\ell(A) = r + 1$ ([7]).

ii) We also point out that the Corollary 2.2 holds for isolated non-Gorenstein local rings as long as they satisfy the condition $(*)$ (see Remark 1.4).

COROLLARY 2.4. ([5, Theorem 2.1]) *Let $(A, \mathfrak{m})$ be a Gorenstein local ring. Suppose the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay. Then*

$$\mathrm{index}(A) = \ell\ell(A).$$

Proof. Since $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay, there is some positive integer r, which satisfies the condition $(*)$ (in Theorem 2.1) by Remark 2.3. Using Proposition 1.5 and Theorem 2.1, we have the inequalities $\mathrm{index}(A) = \mathrm{col}_{CM}(A) \geq r \geq \ell\ell(A)$; hence $\mathrm{index}(A) = \ell\ell(A)$. □

Before proving Theorem 2.1, we note that the system of parameters $\mathbf{x} = x_1, \ldots, x_d$ satisfying the condition $(*)$ is of degree 1, i.e., $x_i \in \mathfrak{m} - \mathfrak{m}^2$ for each i. Indeed, the condition $(*)$ assures $\mathfrak{m}^{r+1} = (\mathbf{x})\mathfrak{m}^r$, i.e., $(\mathbf{x})$ is a reduction of $\mathfrak{m}$. If all x_i are in $\mathfrak{m}^2$, then $\mathfrak{m}^{r+1} = (\mathbf{x})\mathfrak{m}^r \subseteq \mathfrak{m}^{r+2}$, which implies $\mathfrak{m} = 0$ by Nakayama lemma. Suppose $x_1, \ldots, x_j \in \mathfrak{m} - \mathfrak{m}^2$ and $x_{j+1}, \ldots, x_d \in \mathfrak{m}^2$. Now, let $N := (x_1, \ldots, x_j)\mathfrak{m}^{r-1}$ be a submodule of $M := (x_1, \ldots, x_d)\mathfrak{m}^{r-1}$. Then it is easy to show that $\mathfrak{m}M + N = M$ since $x_{j+1}, \ldots, x_d \in \mathfrak{m}^2$ and $\mathfrak{m}M = \mathfrak{m}^{r+1}$ by assumption. Therefore, by Nakayama lemma, we have $M = N$, i.e., $(x_1, \ldots, x_d)\mathfrak{m}^{r-1} = (x_1, \ldots, x_j)\mathfrak{m}^{r-1}$, which implies $j = d$ since $x_1, \ldots, x_d$ is a system of parameters of A. Thus all x_i are in $\mathfrak{m} - \mathfrak{m}^2$.

Proof of Theorem 2.1. Let's choose $y \in \mathfrak{m}^r - (\mathbf{x})$. Since $\mathfrak{m}^{r+1} \subseteq (\mathbf{x})$ by assumption, $\bar{y} \in \mathrm{Soc}(A/\mathbf{x})$, where $\bar{y}$ is the image of y in $A/\mathbf{x}$. Thus we have a monomorphism $y : A/\mathfrak{m} \to A/\mathbf{x}$, which sends $\bar{1}$ to $\bar{y}$, and have the following commutative complexes:

$$\begin{array}{ccccccccccccc} F_\bullet : \cdots & \to & A^{n_{d+1}} & \xrightarrow{\varphi_{d+1}} & A^{n_d} & \to \cdots \to & A^{n_1} & \xrightarrow{\varphi_1} & A & \xrightarrow{\varphi_0} & A/\mathfrak{m} & \to 0 \\ & & \downarrow & & \phi_d \downarrow & & \phi_1 \downarrow & & \phi_0 \downarrow & & y \downarrow & \\ K_\bullet : \cdots & \to & 0 & \longrightarrow & A^{\binom{d}{d}} & \to \cdots \to & A^{\binom{d}{1}} & \xrightarrow{d_1} & A & \xrightarrow{d_0} & A/\mathbf{x} & \to 0, \end{array}$$

where $(F_\bullet, \varphi_\bullet)$ is a minimal resolution of $A/\mathfrak{m}$, $(K_\bullet, d_\bullet)$ is a Koszul complex of $\mathbf{x}$, and ϕ_i's are liftings of the map y. We note that the entries of ϕ_i are in $\mathfrak{m}$ since $\mathbf{x} = x_1, \ldots, x_d$ is a part of minimal generators of $\mathfrak{m}$.

Let $M(\phi_\bullet)_\bullet$ be the mapping cone of $\phi_\bullet$ such that $M(\phi_\bullet)_i = A^{\binom{d}{i}} \oplus A^{n_{i-1}}$ and the differential $\Psi_i = \begin{bmatrix} d_i & 0 \\ \phi_{i-1} & -\varphi_{i-1} \end{bmatrix}$. For examples, $\Psi_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_d \\ y \end{bmatrix}$ and $\Psi_{d+1} = \begin{bmatrix} \phi_d & -\varphi_d \end{bmatrix}$. Since all entries of d_i, φ_i, ϕ_i are in $\mathfrak{m}$, $(M(\phi_\bullet)_\bullet, \Psi_\bullet)$ is a minimal resolution of $A/(\mathbf{x}, y)$.

Now, let $\phi_\ell = [\phi^\ell_{ij}]_{n_\ell \times \binom{d}{\ell}}$, $d_\ell = [d^\ell_{ij}]_{\binom{d}{\ell} \times \binom{d}{\ell-1}}$ and $\varphi_\ell = [\varphi^\ell_{ij}]_{n_\ell \times n_{\ell-1}}$.

CLAIM 1. We may assume that $\phi^\ell_{i1} \in \mathfrak{m}^r$ for $\ell = 0, \ldots, d$ and $i = 1, \ldots, n_\ell$, i.e., every entry of the first column of each ϕ_ℓ is contained in $\mathfrak{m}^r$.

Proof of Claim 1. We first note that since $\mathfrak{m}^{r+1} \cap I_k = \mathfrak{m}^r I_k$ for $k = 1, \ldots, d$, where $I_k = (x_1, \ldots, x_k)$ we have the property that if $\sum_{i=1}^k r_i x_i \in \mathfrak{m}^{r+1}$, $1 \le k \le d$, then we can choose $r_i^* \in \mathfrak{m}^r$ such that $\sum_{i=1}^k r_i x_i = \sum_{i=1}^k r_i^* x_i$.

To prove claim 1, we use induction on ℓ.

The case $\ell = 0$ is clear since $\phi_0 = y \in \mathfrak{m}^r$. Suppose claim 1 is true for all $\ell < \ell_0$. We need to show that $\phi^{\ell_0}_{i1} \in \mathfrak{m}^r$ for $i = 1, \ldots, n_{\ell_0}$. Since $\Psi_{\ell_0+1}\Psi_{\ell_0} = 0$, we have

$$\begin{array}{lllllll} & \phi^{\ell_0}_{11}(\pm x_{\ell_0}) & + & \phi^{\ell_0}_{12}(\pm x_{\ell_0+1}) & + \cdots + & \phi^{\ell_0}_{1,d-\ell_0+1}(\pm x_d) & \\ + & (-\varphi^{\ell_0}_{11})\phi^{\ell_0-1}_{11} & + & (-\varphi^{\ell_0}_{12})\phi^{\ell_0-1}_{21} & + \cdots + & (-\varphi^{\ell_0}_{1,n_{\ell_0-1}})\phi^{\ell_0-1}_{n_{\ell_0-1},1} & = 0. \end{array}$$

Since $\phi^{\ell_0-1}_{i1} \in \mathfrak{m}^r$ for $i = 1, \ldots, n_{\ell_0-1}$ by the induction hypothesis, we know that

$$\phi^{\ell_0}_{11} \cdot x_{\ell_0} + \phi^{\ell_0}_{12} \cdot x_{\ell_0+1} + \cdots + \phi^{\ell_0}_{1,d-\ell_0+1} \cdot x_d \in \mathfrak{m}^{r+1}.$$

Thus there exist $\phi^{\ell_0 *}_{11}, \ldots, \phi^{\ell_0 *}_{1,d-\ell_0+1} \in \mathfrak{m}^r$ such that

$$(\phi^{\ell_0}_{11} - \phi^{\ell_0 *}_{11}) \cdot x_{\ell_0} + \cdots + (\phi^{\ell_0}_{1,d-\ell_0+1} - \phi^{\ell_0 *}_{1,d-\ell_0+1}) \cdot x_d = 0.$$

This implies that

$$(\phi_{11}^{\ell_0} - \phi_{11}^{\ell_0 *}, \ldots, \phi_{1,d-\ell_0+1}^{\ell_0} - \phi_{1,d-\ell_0+1}^{\ell_0 *}) \in \ker d'_1 = \operatorname{Im} d'_2,$$

where d'_i are the differentials of Koszul complex of $(x_{\ell_0}, \ldots, x_d)$. Therefore, we may write

$$\phi_{1j}^{\ell_0} = \phi_{1j}^{\ell_0 *} + \sum_{i=1}^{d-\ell_0} r_i d_{ij}^{\ell_0+1} \quad \text{for some } r_i \in \mathfrak{m} \text{ and } j = 1, \ldots, d - \ell_0 + 1.$$

For $j > d - \ell_0 + 1$, put $\phi_{1j}^{\ell_0 *} = \phi_{1j}^{\ell_0} - \sum_{i=1}^{d-\ell_0} r_i d_{ij}^{\ell_0+1}$. Using a change of basis, we can replace $\phi_{1j}^{\ell_0}$ by $\phi_{1j}^{\ell_0 *}$ for $j = 1, \ldots, \binom{d}{\ell_0}$, and thus we may assume $\phi_{11}^{\ell_0} \in \mathfrak{m}^r$. Similarly, we may assume that $\phi_{i1}^{\ell_0} \in \mathfrak{m}^r$ for $i = 2, \ldots, n_{\ell_0}$. This completes the proof of Claim 1.

We are now interested in image Ψ_d, say $\mathbf{\Omega_d}$, i.e., d-th syzygy of $A/(\mathbf{x}, y)$. Since $\operatorname{depth} A/(\mathbf{x}, y) = 0$, we know that $\mathbf{\Omega_d}$ is a maximal Cohen-Macaulay module by the Ext characterization of depth and the long exact sequence of Ext. Thus if $\mathbf{\Omega_d}$ has no free summand, $\operatorname{col}_{CM}(A) \geq r + 1$ by the definition of col_{CM}(-) since the presenting matrix of $\mathbf{\Omega_d}$ is Ψ_{d+1} and every entries of the first column of Ψ_{d+1} is in $\mathfrak{m}^r$ by claim 1.

Suppose that $\mathbf{\Omega_d}$ has some free summand and let $\mathbf{\Omega_d} = \mathbf{\Omega'_d} \oplus A$. Let $\mathbf{\Omega_d} = (\eta_1, \ldots, \eta_s)$, where η_j is a j-th row of Ψ_d and $s = 1 + n_{d-1}$. Then it is easy to show that there exists $\epsilon_0 \in \operatorname{Hom}_A(\mathbf{\Omega_d}, A)$ such that $\epsilon_0(\eta_{i_0}) = 1$ for some η_{i_0}, and $\mathbf{\Omega'_d} = (\eta'_1, \ldots, \eta'_{i_0-1}, \eta'_{i_0+1}, \ldots, \eta'_s)$, where $\eta'_k = \eta_k - \epsilon_0(\eta_k)\eta_{i_0}$ for $k = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, s$. Thus the presenting matrix of $\mathbf{\Omega'_d}$ is Ψ_{d+1}' such that Ψ_{d+1}' is obtained after deleting i_0-th column of Ψ_{d+1}, whose entries are all 0 by row-column operations.

CLAIM 2. (1) $\epsilon(\eta_1)$ is not a unit for any $\epsilon \in \operatorname{Hom}_A(\mathbf{\Omega_d}, A)$ and (2) $\pi(\eta'_1)$ is not a unit for any $\pi \in \operatorname{Hom}_A(\mathbf{\Omega'_d}, A)$.

Proof of Claim 2. If (1) is true, then (2) is clear because

$$\pi(\eta'_1) = \pi(\eta_1 - \epsilon_0(\eta_1)\eta_{i_0}) = \pi(\eta_1) - \epsilon_0(\eta_1)\pi(\eta_{i_0}) \in \mathfrak{m}$$

for $\pi \in \operatorname{Hom}_A(\mathbf{\Omega'_d}, A) \subseteq \operatorname{Hom}_A(\mathbf{\Omega_d}, A)$.

To show (1), suppose $\epsilon'(\eta_1) = 1$ for some $\epsilon' \in \operatorname{Hom}_A(\mathbf{\Omega_d}, A)$ if possible. Then we may assume that the entries of the first column of Ψ_{d+1} are all 0 by a change of basis. Thus from the minimal resolution of $A/(\mathbf{x}, y)$, we know that $\operatorname{Tor}_d^A(A/\mathbf{x}, A/(\mathbf{x}, y))$ has $A/\mathbf{x}$ as a submodule since every entry of the first row of Ψ_d is in $(\mathbf{x}) = (x_1, \ldots, x_d)$. On the other hand, from the minimal resolution of $A/\mathbf{x}$, i.e., Koszul complex of $\mathbf{x}$, we also have that $\mathrm{Tor}_d^A(A/\mathbf{x}, A/(\mathbf{x}, y)) \cong A/(\mathbf{x}, y)$. This implies that $(\mathbf{x}, y) \subseteq (\mathbf{x})$ from the monomorphism $A/\mathbf{x} \hookrightarrow \mathrm{Tor}_d^A(A/\mathbf{x}, A/(\mathbf{x}, y)) \cong A/(\mathbf{x}, y)$. This contradicts that $y \notin (\mathbf{x})$.

We can continue the process of a change of basis deleting free summands of $\mathbf{\Omega_d}$ until we have a (maximal Cohen-Macaulay) submodule X of $\mathbf{\Omega_d}$ without free summands. Then by claim 2, we know that the presenting matrix of X has a first column whose entries are still in $\mathfrak{m}^r$. Hence $\mathrm{col}_{CM}(A) \geq r+1$, and so $\mathrm{col}_{CM}(A) \geq \ell\ell(A)$. This completes the proof. □

We close this section with J. Sally's example which shows that the conjecture in [9] holds even if the condition $(*)$ fails. We recall the definitions of $\mathrm{crs}(A)$ and $\mathrm{row}(A)$: if $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring, we define $\mathrm{row}(M) := \inf\ \{t$: each row of φ_i has an element outside $\mathfrak{m}^t$ for all $i >$ depth $A.\}$ (see Definition 1.1 for crs and drs). The conjecture in [9] asserts that $\mathrm{col}(A) = \mathrm{crs}(A)$ and $\mathrm{row}(A) = \mathrm{drs}(A)$ for local Cohen-Macaulay rings with infinite residue field.

EXAMPLE 2.5. Let $R = k[\![t^e, t^{e+1}, t^{(e-1)e-1}]\!]$ for $e \geq 4$. R is an one dimensional Cohen-Macaulay local ring and it is known that the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ of R is not Cohen-Macaulay. It is shown ([10, Theorem 2.6]) that $\mathrm{col}(R) = 2 = \mathrm{crs}(R)$, and $\mathrm{row}(R) = \ell\ell(R) = \mathrm{drs}(R) = e-1$, i.e., the conjecture in [9] holds. However, the condition $(*)$ fails. To show this, we first claim that if $(x)\mathfrak{m}^r = \mathfrak{m}^{r+1}$ and $\mathfrak{m}^r \subseteq (x)$ for some system of parameters x of R and a positive integer r, then $\mathfrak{m}^r \subseteq (y)$ for any system of parameters (y) of R whenever $(y)\mathfrak{m}^r = \mathfrak{m}^{r+1}$. Indeed, since $(x)\mathfrak{m}^r = (y)\mathfrak{m}^r$, we know, for any $a \in \mathfrak{m}^r$, $ax = by$ for some $b \in \mathfrak{m}^r$. Since $b \in \mathfrak{m}^r \subseteq (x)$ and x is a non zero divisor, we have $a \in (y)$. Therefore, by the facts that $(t^e)\mathfrak{m}^{e-1} = \mathfrak{m}^e$ but $\mathfrak{m}^{e-1} \subseteq (t^e)$, and $(x)\mathfrak{m}^{e-1} = \mathfrak{m}^e$ for any system of parameters (x) (see [10, Proposition 2.4]), we know that the condition $(*)$ fails.

References

[1] M. Auslander and R. O. Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations,* Mém. Soc. Math. France **38** (1989), 5–37.

[2] M. Auslander, *Minimal Cohen-Macaulay approximations,* preprint.

[3] S. Ding, *Cohen-Macaulay approximations and multiplicity,* J. Algebra **153** (1992), no. 2, 271–288.

[4] ———, *A note on the index of Cohen-Macaulay local rings,* Comm. Algebra **21** (1993), no. 1, 53–71.

[5] ———, *The associated graded ring and the index of a Gorenstein local ring,* Proc. Amer. Math. Soc. **120** (1994), no. 4, 1029–1033.

[6] M. Hashimoto and A. Shida, *Some remarks on index and generalized Loewy length of a Gorenstein local ring,* J. Algebra **187** (1997), no. 1, 150–162.

[7] J. Herzog, *On the index of a homogeneous Gorenstein ring,* Contem. Math. **159** (1994), 95–102.

[8] J. Koh and K. Lee, *Some restrictions on the maps in the minimal resolutions,* J. Algebra **202** (1998), 671–689.

[9] ______, *New homological invariants of local rings,* J. Algebra **235** (2001), 431–452.

[10] K. Lee, *Computation of numerical invariants* col*(-),* row*(-) for a ring* $k[\![t^e, t^{e+1}, t^{(e-1)e-1}]\!]$, J. Korean Math. Soc. **37** (2000), no. 4, 521–530.

[11] ______, *Column invariants over Cohen-Macaulay local rings,* Comm. Algebra **33** (2005), no. 5, 1303–1314.

[12] J. D. Sally, *Super-Regular Sequences,* Pacific J. Math. **84** (1979), no. 2, 465–481.

[13] A. Shida, *On indices of local rings along local homomorphisms,* Comm. Algebra **23** (1995), no. 12, 4535–4541.

[14] P. Valabrega and G. Valla, *Form rings and regular sequences,* Nagoya Math. J. **72** (1978), 93–101.

[15] Y. Yoshino, *On the higher delta invariants of a Gorenstein local ring,* Proc. Amer. Math. Soc. **124** (1996), no. 9, 2641–2647.

Jee Koh
Department of Mathematics
Indiana University
Bloomington, Indiana 47405, U.S.A.
E-mail: kohj@indiana.edu

Kisuk Lee
Department of Mathematics
Sookmyung Women's University
Seoul 140-742, Korea
E-mail: kilee@sookmyung.ac.kr