J. Korean Math. Soc. **43** (2006), No. 4, pp. 859–869

A SELF-NORMALIZED LIL FOR CONDITIONALLY TRIMMED SUMS AND CONDITIONALLY CENSORED SUMS

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Abstract. Let $\{X, X_n; n \geq 1\}$ be a sequence of *i.i.d.* random variables which belong to the attraction of the normal law, and $X_n^{(1)}, \ldots, X_n^{(n)}$ be an arrangement of $X_1, \ldots, X_n$ in decreasing order of magnitude, i.e., $|X_n^{(1)}| \geq \cdots \geq |X_n^{(n)}|$. Suppose that $\{r_n\}$ is a sequence of constants satisfying some mild conditions and $d'(t_{n_k})$ is an appropriate truncation level, where $n_k = [\beta^k]$ and β is any constant larger than one. Then we show that the conditionally trimmed sums obeys the self-normalized law of the iterated logarithm (LIL). Moreover, the self-normalized LIL for conditionally censored sums is also discussed.

1. Introduction and main results

Let $\{X, X_n; n \geq 1\}$ be a sequence of *i.i.d.* random variables. Put $S_n = \sum_{i=1}^n X_i$ and let $X_n^{(1)}, \ldots, X_n^{(n)}$ be an arrangement of $X_1, \ldots, X_n$ in decreasing order of magnitude, i.e., let $|X_n^{(1)}| \geq \cdots \geq |X_n^{(n)}|$ and $\{r_n\}$ be a sequence of constants satisfying the conditions

$$(1.1) \qquad \begin{cases} r_n \uparrow \infty, \\ \frac{n}{r_n} \uparrow \infty, \\ \liminf\limits_{n\to\infty} \dfrac{r_n}{\log\log n} = \eta > 0. \end{cases}$$

In order to define the centering factors and normalized factors, we give some notations. Let

$$h(t) = t^{-2}\mathsf{E}X^2 I\{|X| \leq t\}, b = \inf\{t \geq 1 : h(t) > 0\}.$$

Received June 7, 2005. Revised October 13, 2005.

2000 Mathematics Subject Classification: Primary 60F15; Secondary 60G50.

Key words and phrases: self-normalized, law of the iterated logarithm, trimmed sums, censored sums, *i.i.d.* random variables.

This project is supported by NSFC (10131040) and SRFDP (2002335090).

Then define for $s > 0$

$$d(s) = \inf\{t \geq b+1 : h(t) \leq \frac{1}{s}\}, \tag{1.2}$$

obviously, $h(d(s)) = 1/s$.

Given $\beta > 1$, let $n_k = [\beta^k]$, where $[x]$ stands for the integer part of x. Denote the k^{th} block of integers by

$$I(k) = [n_k, n_{k+1}).$$

Then we define the centering factors

$$\delta_n(\beta) = n\mathsf{E}XI\{|X| \leq d(t_{n_k})\},\ n \in I(k),$$

and the normalized factors

$$\gamma_n = \sqrt{r_n \log\log n}\ d(t_n),$$

where $t_n = n/r_n$. For any r $(0 \leq r \leq n)$, the conditionally trimmed sum is given by

$$^{(r)}S_n(\beta) = S_n - \sum_{j=1}^{[r]} X_n^{(j)} I\{|X_n^{(j)}| > d(t_{n_k})\},\ n \in I(k). \tag{1.3}$$

Under the above notations and conditions, Hahn, Kuelbs and Weiner ([9]) established the following LIL.

THEOREM A. *Suppose that* $\mathsf{E}X^2 = \infty$ *and* r_n *is a sequence of positive constants satisfying* (1.1). *Then for any* $\beta > 1$ *and* $\eta > 0$, *there exists a sequence of constants* $\{\xi_n\}$ *such that*

$$\lim_{n\to\infty} \frac{\xi_n r_n}{n} = 0 \tag{1.4}$$

and

$$\limsup_{n\to\infty} \frac{|^{(\xi_n r_n)}S_n(\beta) - \delta_n(\beta)|}{\gamma_n} = C \text{ a.s.,} \tag{1.5}$$

where $0 < C < \infty$ *is some constant and one particular choice of* ξ_n *satisfying* (1.4) *and* (1.5) *is*

$$\xi_n = \max\{e^2\beta^2 n_k P(|X| > d(t_{n_k})), 2\log\log n_k\}/r_n,\ n \in I(k).$$

Hahn, Kuelbs and Weiner ([9]) also established an analogue result for conditionally censored sums. To state their result, we give the definition of the conditionally censored sums. Define $a(t)$ to be a function such that if $t \geq 1/\mathsf{P}(X \neq 0)$, then

$$t\mathsf{E}\big((X^2 \wedge a^2(t))/a^2(t)\big) = 1.$$

The r_n is defined as above. Then we define the normalized factors

$$\Gamma_n = \sqrt{r_n \log\log n}\; a(t_n).$$

For any $\beta > 1$ and $n \in I(k)$, we define the centering factors

$$\Lambda_n(\beta) = n\mathsf{E}\big(|X| \wedge a(t_{n_k})\big)\mathrm{sgn}(X),$$

and for each r $(0 \le r \le n)$, we define the conditionally censored sum

$$(1.6)\quad S_n^{(r)}(\beta) = \sum_{j=[r]+1}^{n} X_n^{(j)} + \sum_{j=1}^{[r]} \big(|X_n^{(j)} \wedge a(t_{n_k})|\big)\mathrm{sgn}(X_n^{(j)}),\ \ n \in I(k).$$

By these notations and definitions, Hahn, Kuelbs and Weiner ([9]) proved the following result.

THEOREM B. *Suppose that $EX^2 = \infty$ and $\{r_n\}$ is a sequence of constants satisfying* (1.1). *Then for any $\beta > 1$ and $\eta > 0$, whenever*

$$0 < \lambda_n = o(t_n)$$

and

$$0 < \lambda = \liminf_n \lambda_n > (\beta e^2 \vee \frac{1}{\eta}),$$

there exists a positive constant C such that

$$(1.7)\qquad \limsup_{n\to\infty} \frac{|S_n^{(\lambda_n r_n)}(\beta) - \Lambda_n(\beta)|}{\Gamma_n} = C \ \text{a.s.}$$

It is well-known that the so-called self-normalized limit theorem put a totally new countenance upon classical limit theorems. We refer to Bentkus et al. ([1]) for Berry-Esseen inequalities, Giné et al. ([5]) for the necessary and sufficient condition for the asymptotic normality, Griffin et al. ([7]) for law of the iterated logarithm, Csörgő et al. ([2]) for studentized increments, Lin ([11]) for Chung-type law of the iterated logarithm, Csörgő et al. ([3]) for Donsker's theorem. For a survey on recent developments in this area, we refer to Shao ([13]). The aim of this paper is to establish the self-normalized LIL for conditionally trimmed sums and censored sums. Denote $V_n^2 = \sum_{i=1}^n X_i^2$. We state our results as follows.

THEOREM 1.1. *Let $\{X, X_n; n \ge 1\}$ be a sequence of i.i.d. random variables which belong to the attraction of the normal law and let r_n be* *defined as* (1.1). *Suppose that* $\log\log r_n = o(\log\log n)$. *In addition, for any* $a > 0$, *we redefine*

$$d'(s) = \inf\{t \geq b+1 : h(t) \leq \frac{(\log\log s)^a}{s}\},$$
$$\delta_n'(\beta) = nEXI\{|X| \leq d'(t_{n_k})\},$$
$${}^{(r)}S_n'(\beta) = S_n - \sum_{j=1}^{[r]} X_n^{(j)} I\{|X_n^{(j)}| > d'(t_{n_k})\},\ n \in I(k).$$

Then for any $\beta > 1$ *and* $\eta > 0$, *there exists a sequence of constants* $\{\xi_n'\}$ *such that*

$$\lim_{n\to\infty} \frac{\xi_n' r_n}{n} = 0$$

and

$$\limsup_{n\to\infty} \frac{|{}^{(\xi_n' r_n)}S_n'(\beta) - \delta_n'(\beta)|}{\sqrt{2V_n^2 \log\log n}} = 1 \text{ a.s.,} \tag{1.8}$$

where ξ_n' *can be taken as*

$$\xi_n' = \max\{e^2\beta^2 n_k P(|X| > d'(t_{n_k})), 2\log\log n_k\}/r_n,\ n \in I(k).$$

THEOREM 1.2. *Let* $\{X, X_n; n \geq 1\}$ *be a sequence of i.i.d. random variables which belong to the attraction of the normal law and let* r_n *be defined as* (1.1). *Suppose that* $\log\log r_n = o(\log\log n)$. *If* $t \geq 1/P(X \neq 0)$, *we redefine* $a'(t)$ *to be a function satisfying*

$$tE\big((X^2 \wedge a'^2(t))/(a'^2(t)(\log\log t)^a)\big) = 1,$$

where a *is any constant larger than one. And we also redefine*

$$\Lambda_n'(\beta) = nE\big(|X| \wedge a'(t_{n_k})\big)\operatorname{sgn}(X),$$
$$S_n^{(r)'}(\beta) = \sum_{j=[r]+1}^{n} X_n^{(j)} + \sum_{j=1}^{[r]} \big(|X_n^{(j)} \wedge a'(t_{n_k})|\big)\operatorname{sgn}(X_n^{(j)}),\ n \in I(k).$$

Then for any $\beta > 1$ *and* $\eta > 0$, *whenever*

$$0 < \lambda_n = o(t_n)$$

and

$$0 < \lambda = \liminf_n \lambda_n > (\beta e^2 \vee \frac{1}{\eta}),$$

we have

$$\limsup_{n\to\infty} \frac{|S_n^{(\lambda_n r_n)'}(\beta) - \Lambda_n'(\beta)|}{\sqrt{2V_n^2 \log\log n}} = 1 \text{ a.s.} \tag{1.9}$$

REMARK. Compared with Theorem A and Theorem B, our results are valid not only for a class of sequences of i.i.d. random variables without the variance, but for any sequence of i.i.d. random variables with the finite variance. At the same time, we can determine the constant C in Theorem A and Theorem B by replacing the centering and normalized factors by other appropriate ones, respectively.

2. The proofs of Theorem 1.1 and Theorem 1.2

We state the well-known Kolmogorov exponential inequality (cf., e.g. Stout ([14])), since it is the main tool for proving the results above.

LEMMA 2.1. *Let $X_1, \ldots, X_n$ be independent random variables with $EX_i = 0$ and $EX_i^2 < \infty, i = 1, \ldots, n$. Put $\sigma_n^2 = E(X_1 + \cdots + X_n)^2$ and suppose that*

$$|X_j| \le c\sigma_n \text{ a.s. } (1 \le j \le n). \tag{2.1}$$

Then for any $0 < x < 1/c$, we have

$$P(|\sum_{j=1}^n X_j| \ge \sigma_n x) \le 2\exp\{-\frac{x^2}{2}(1 - \frac{xc}{2})\}. \tag{2.2}$$

And, for any $\delta > 0$, there exists $\varepsilon > 0$ and $M > 0$ such that for any x satisfying $cx \le \varepsilon$ and $x > M$, we have

$$P(|\sum_{j=1}^n X_j| \ge \sigma_n x) \ge 2\exp\{-\frac{x^2}{2}(1 + \delta)\}. \tag{2.3}$$

The proof of Theorem 1.1. For $n \in I(k)$, we define

$$\begin{cases} u_j = u_j(k) = X_j I\{|X_j| \le d'(t_{n_k})\} \\ z_j = X_j - u_j \end{cases}, 1 \le j < n_{k+1},$$

and

$$U_n = \sum_{j=1}^n u_j,\ Z_n = \sum_{j=1}^n z_j.$$

Clearly, $S_n = U_n + Z_n$. By noting that $\delta_n'(\beta) = \mathsf{E}U_n$, we have
(2.4)

$$^{(\xi_n' r_n)}S_n'(\beta) - \delta_n'(\beta) = \big(U_n - \mathsf{E}U_n\big) + Z_n - \sum_{j=1}^{[\xi_n' r_n]} X_n^{(j)} I\{|X_n^{(j)}| > d'(t_{n_k})\}.$$

Denote

$$N_k = n_{k+1} - 1,\ M_k = \min_{n\in I(k)} [\xi_n' r_n],\ p_{n_k}' = \mathsf{P}(|X| > d'(t_{n_k})).$$

Along the lines of the proof in Hahn et al. ([9]), we have

$$\begin{aligned}
&\mathsf{P}(\max_{n\in I(k)} |Z_n - \sum_{j=1}^{[\xi_n' r_n]} X_n^{(j)} I\{|X_n^{(j)}| > d'(t_{n_k})\}| > 0)\\
\le\ & \mathsf{P}(\text{at least } (M_k+1) \text{ of } X_j \text{ satisfy } |X_j| > d'(t_{n_k}),\ 1 \le j < n_{k+1})\\
\le\ & \sum_{j=M_{k+1}}^{N_k} C_n^j p_{n_k}'^{\,j} (1 - p_{n_k}')^{N_k - j}\\
=\ & N_k C_{N_k-1}^{M_k} \int_0^{p_{n_k}'} t^{M_k} (1-t)^{N_k - M_k - 1} dt \text{ (by Feller ([4]), p.173)}\\
\le\ & 2(n_{k+1} p_{n_k}' e / M_k)^{M_k} M_k^{-1/2} (N_k/(N_k - M_k))^{1/2}\\
& \qquad \text{(by Stirling formula).}
\end{aligned}$$

Take

$$\xi_n' = \max\{e^2\beta^2 n_k p_{n_k}', 2\log\log n_k\}/r_n. \tag{2.5}$$

Then for any $\varepsilon > 0$, we have

$$\frac{n_{k+1} p_{n_k}' e}{M_k} \le \frac{n_{k+1} p_{n_k}' e}{[e^2\beta^2 n_k p_{n_k}']} \le \frac{1}{e^{1-\varepsilon}}.$$

Thus, it is easy to see that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathsf{P}(\max_{n\in I(k)} |Z_n - \sum_{j=1}^{[\xi_n' r_n]} X_n^{(j)} I\{|X_n^{(j)}| > d'(t_{n_k})\}| > 0)\\
\le\ & \sum_{k=1}^{\infty} \exp\{-2(1-\varepsilon)\log\log n_k\} < \infty. \qquad (2.6)
\end{aligned}$$

By the Borel-Cantelli lemma, we have

$$\lim_{k\to\infty} \max_{n\in I(k)} |Z_n - \sum_{j=1}^{[\xi_n' r_n]} X_n^{(j)} I\{|X_n^{(j)}| > d'(t_{n_k})\}| = 0 \text{ a.s.} \tag{2.7}$$

Next we will show

$$\limsup_{k\to\infty} \frac{|U_n - \mathsf{E}U_n|}{\sqrt{2V_n^2 \log\log n}} \le 1 \text{ a.s.} \tag{2.8}$$

By the definition of $d'(s)$ and the fact $V_n^2 \ge \sum_{j=1}^n X_j^2 I\{|X_j| \le d'(t_{n_k})\}$ and noting that $\{X_n; n \ge 1\}$ is a sequence of random variables which belong to the attraction of the normal law, we conclude that for $\varepsilon_1 = \varepsilon/2$, we have

$$\begin{aligned} V_n^2 &\ge n(1-\varepsilon_1)^2 \mathsf{E}X_1^2 I\{|X_1| \le d'(t_{n_k})\} \text{ a.s.} \\ &= n(1-\varepsilon_1)^2 \frac{d'^2(t_{n_k})(\log\log t_{n_k})^a}{t_{n_k}} \end{aligned} \tag{2.9}$$

as k large enough. On the other hand,

$$\begin{aligned} &\max_{n\in I(k)} \frac{\mathsf{E}(U_n - \mathsf{E}U_n)^2}{2V_n^2 \log\log n} \\ \le\ & \frac{n_{k+1}\mathsf{E}X_1^2 I\{|X_1| \le d'(t_{n_k})\}}{2n_k(1-\varepsilon_1)^2 \mathsf{E}X_1^2 I\{|X_1| \le d'(t_{n_k})\} \log\log n_k} \to 0 \end{aligned}$$

as $k \to \infty$. So by the Ottaviani inequality, it is enough to prove

$$\sum_{k=1}^{\infty} \mathsf{P}(|U_{N_k} - \mathsf{E}U_{N_k}| \ge (1+\varepsilon)\sqrt{2V_{N_k}^2 \log\log N_k}) < \infty \tag{2.10}$$

for showing (2.8). From (2.9) we have

$$V_{N_k}^2 \ge \beta(1-\varepsilon_1)^2 r_{n_k} d'^2(t_{n_k})(\log\log t_{n_k})^a \text{ a.s.}$$

Thus it is sufficient to prove

$$\sum_{k=1}^{\infty} \mathsf{P}(|U_{N_k} - \mathsf{E}U_{N_k}| \ge (1+\varepsilon)(1-\varepsilon_1)\sqrt{2\beta r_{n_k}(\log\log N_k)^{(a+1)}} d'(t_{n_k})) < \infty \tag{2.11}$$

for showing (2.10). Note that

$$\begin{aligned}\mathsf{Var}\, U_{N_k} &= N_k \mathsf{E}\big(XI\{|X| < d^{'}(t_{n_k})\} - \mathsf{E}XI\{|X| < d^{'}(t_{n_k})\}\big)^2 \\ &\sim N_k \cdot \frac{d^{'2}(t_{n_k})(\log\log t_{n_k})^a}{t_{n_k}} \\ &\sim \beta r_{n_k} d^{'2}(t_{n_k})(\log\log t_{n_k})^a\end{aligned}$$

and

$$\begin{aligned}&|u_j - \mathsf{E}u_j| \le 2d^{'}(t_{n_k}) \\ =\ & \sqrt{\beta r_{n_k}(\log\log t_{n_k})^a} d^{'}(t_{n_k}) \cdot \frac{2}{\sqrt{\beta r_{n_k}(\log\log t_{n_k})^a}}.\end{aligned}$$

By taking $x = (1+\varepsilon)(1-\varepsilon_1)\sqrt{2\log\log n_k}$ and $c = 2/\sqrt{\beta r_{n_k}(\log\log t_{n_k})^a}$ in Lemma 2.1 and obversing that $\log\log N_k \sim \log\log n_k$, and the condition $\log\log r_n = o(\log\log n)$ implies that $\log\log t_{n_k} \sim \log\log n_k$, we have

$$\begin{aligned}&\sum_{k=1}^{\infty} \mathsf{P}\Big(|U_{N_k} - \mathsf{E}U_{N_k}| \ge (1+\varepsilon)(1-\varepsilon_1) \\ &\quad \times \sqrt{2\beta r_{n_k}(\log\log N_k)^{(a+1)}} d^{'}(t_{n_k}) \Big) \\ \le\ & \sum_{k=1}^{\infty} \exp\{-(1+\varepsilon)^2(1-\varepsilon_1)^2 \\ &\quad \times \log\log N_k (1 - \frac{(1+\varepsilon)(1-\varepsilon_1)\sqrt{2\log\log N_k}}{\sqrt{\beta r_{n_k}(\log\log N_k)^a}})\} \\ \le\ & C\sum_{k=1}^{\infty} \exp\{-(1+\varepsilon/2)\log\log N_k\} \\ <\ & \infty.\end{aligned}$$

By the Borel-Cantelli lemma again, we conclude that (2.11) holds, consequently, (2.8) holds.

Next, we will prove

$$\limsup_{k\to\infty} \frac{|U_n - \mathsf{E}U_n|}{\sqrt{2V_n^2 \log\log n}} \ge 1 \text{ a.s.} \tag{2.12}$$

Along the lines of the proof of (3.16) in Griffin and Kuelbs ([7]), we have

$$\limsup_{k\to\infty} \frac{V_{m_k}^2}{\mathsf{Var}\, U_{m_k}} \le 1+\varepsilon \text{ a.s.},$$

where we take $m_k = \left[\beta^{[k^{1+\varepsilon^2}]^2}\right]$, that is to say

$$\limsup_{k\to\infty} \frac{t_{m_k} V_{m_k}^2}{m_k d'^2(t_{n_k})(\log\log t_{n_k})^a} \le 1+\varepsilon \text{ a.s.} \tag{2.13}$$

For any $n \in I([k^{1+\varepsilon^2}]^2) = [m_k, \beta m_k)$, we have

$$U_n = \sum_{j=1}^{n} (u_j([k^{1+\varepsilon^2}]^2) - \mathsf{E}u_j([k^{1+\varepsilon^2}]^2)).$$

Next we will show that

$$\sum_{k=1}^{\infty} \mathsf{P}\Big(\frac{|(U_{m_k} - U_{m_{k-1}}) - \mathsf{E}(U_{m_k} - U_{m_{k-1}})|}{\sqrt{2V_{m_k}^2 \log\log m_k}} \ge 1-\varepsilon\Big) = \infty \tag{2.14}$$

and

$$\limsup_{k\to\infty} \frac{|\sum_{j=1}^{m_{k-1}} (u_j([k^{1+\varepsilon^2}]^2) - \mathsf{E}u_j([k^{1+\varepsilon^2}]^2))|}{\sqrt{2V_{m_k}^2 \log\log m_k}} = 0 \text{ a.s.} \tag{2.15}$$

Consider (2.15). By (2.8) and the proof of Proposition 5.2 in Griffin et al. ([7]), we have

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{|\sum_{j=1}^{m_{k-1}} (u_j([k^{1+\varepsilon^2}]^2) - \mathsf{E}u_j([k^{1+\varepsilon^2}]^2))|}{\sqrt{2V_{m_k}^2 \log\log m_k}} \\
= &\limsup_{k\to\infty} \frac{|\sum_{j=1}^{m_{k-1}} (u_j([k^{1+\varepsilon^2}]^2) - \mathsf{E}u_j([k^{1+\varepsilon^2}]^2)|}{\sqrt{2V_{m_{k-1}}^2 \log\log m_{k-1}}} \\
&\cdot \frac{\sqrt{V_{m_{k-1}}^2 \log\log m_{k-1}}}{\sqrt{V_{m_k}^2 \log\log m_k}} \\
= &\ 0 \text{ a.s.}
\end{aligned} \tag{2.16}$$

Consider (2.14). We have

$$\begin{aligned}
\mathsf{Var}\, U_{m_k - m_{k-1}} &\sim (m_k - m_{k-1}) \cdot \frac{d'^2(t_{m_k})(\log\log t_{m_k})^a}{t_{m_k}} \\
&\sim r_{m_k} d'^2(t_{m_k})(\log\log t_{m_k})^a
\end{aligned}$$

as $k \to \infty$ and

$$\begin{aligned} & |u_j([k^{1+\varepsilon^2}]^2) - \mathsf{E}u_j([k^{1+\varepsilon^2}]^2)| \le 2d'(t_{m_k}) \\ = & \sqrt{r_{m_k}(\log\log t_{m_k})^a} d'(t_{m_k}) \cdot \frac{2}{\sqrt{r_{m_k}(\log\log t_{m_k})^a}}. \end{aligned}$$

Then by (2.13) and the stationary of the sequence of random variables, we only need to prove

$$(2.17) \qquad \sum_{k=1}^{\infty} \mathsf{P}\Big(\frac{|U_{m_k-m_{k-1}} - \mathsf{E}U_{m_k-m_{k-1}}|}{\sqrt{2(1+\varepsilon)r_{m_k}(\log\log m_k)^{(a+1)}d'(t_{m_k})}} \ge 1-\varepsilon\Big) = \infty$$

for showing (2.14). However,

$$\frac{2}{\sqrt{r_{m_k}(\log\log t_{m_k})^a}} \cdot (1-\varepsilon)\sqrt{2(1+\varepsilon)\log\log m_k} \to 0$$

as $k \to \infty$ by (1.1). Taking $\delta = \varepsilon$ in (2.3) in Lemma 2.1, we have

$$\begin{aligned} & \sum_{k=1}^{\infty} \mathsf{P}\Big(\frac{|U_{m_k-m_{k-1}} - \mathsf{E}U_{m_k-m_{k-1}}|}{\sqrt{2(1+\varepsilon)r_{m_k}(\log\log m_k)^{(a+1)}d'(t_{m_k})}} \ge 1-\varepsilon\Big) \\ (2.18) \quad \ge & \ 2\sum_{k=1}^{\infty} \exp\{-(1-\varepsilon)^2(1+\varepsilon)\log\log m_k \cdot (1+\delta)\} \\ = & \ 2\sum_{k=1}^{\infty} k^{-(1-\varepsilon^2)^2} = \infty. \end{aligned}$$

By the Borel-Cantelli lemma, (2.17) holds, consequently, (2.12) holds. (2.7), (2.8) and (2.12) imply (1.8). The proof is completed. □

The proof of Theorem 1.2. The proof is similar to that of Theorem 1.1, except for some difference of notations, so we omit the proof here. □

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