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TWO APPROACHES FOR STOCHASTIC INTEREST RATE OPTION MODEL

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ABSTRACT. We present two approaches of the stochastic interest rate European option pricing model. One is a bond numeraire approach which is applicable to a nonzero value asset. In this approach, we assume log-normality of returns of the asset normalized by a bond whose maturity is the same as the expiration date of an option instead that of an asset itself. Another one is the expectation hypothesis approach for value zero asset which has futures-style margining. Bond numeraire approach allows us to calculate volatilities implied in options even though stochastic interest rate is considered.

1. Introduction

Since the seminal paper by Black-Scholes [3], option pricing models have been dramatically developed. Almost all option pricing models have been extended by ruling out restrictions imposed by Black-Scholes. In order to price options by Black-Scholes formula, two market data are needed, i.e., a risk-free interest rate and an underlying asset. Merton [10] and Amin and Jarrow [1] derive close-form formulas of a European option for index and currency respectively, by assuming a stochastic interest rate instead of a constant interest rate. As time varying volatility models, Heston [6] and GARCH (see an e.g. [12]) models are most well known. Meanwhile Heston assumes the stochastic volatility of returns of an underlying asset which satisfies Ornstein-Uhlenbeck process, the volatility of GARCH models is deterministic. In addition, Carr, Geman, Madan and Yor [4] develop an option pricing model with stochastic volatility Levy process.

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Even though there are fancy models as above, Black-Schole option formula still seems to work well in financial industries. Probably parsimony of the model is the main reason since there is only one parameter to be estimated in Black-Scholes model. In this paper we establish a simple model with only one parameter like Black-Scholes model. Nonetheless our model incorporates stochastic interest rate model. The basic idea is to normalize underlying asset by a risk free bond price with the same maturity as that of option. Our model is more realistic than Black-Scholes model since the interest rate used in our model is the yield of the bond matching maturity with the expiration of an option. It is obvious that an yield of the bond with one year time to maturity is different from that of the bond with one month time to maturity. If an option price is calculated in this way, then the model which people applies is not Black-Scholes formula but other stochastic interest rate option price formula. Moreover, in such a case, implied volatility has different meaning from the volatility people usually think. As we will show later, implied volatility does not represent the volatility of returns itself, but that of excess log returns. This is a unique feature of approach one for our model. Our model enables to calculate an implied volatility of an option in this way (see Kim, Park and Hyun [8] for empirical implication and test). This approach works well for non-zero value assets and the idea of approach one is originally developed by Margrabe [9]. The other approach is for value zero asset which has futures-style margining. Futures has non-zero price but zero value because of daily settlement. The approach one can not be applied to such an asset since we are unable to make a self-financing and risk-free portfolio with three assets such as futures, option, and bond with the same maturity as that of an option. Hence we need a stronger assumption for preference and assume a local expectation hypothesis for bonds with any maturities. This argument is an line with the methodology of Ramaswamy and Sundaresan [11] for a futures option pricing.

The remainder of the paper is organized as follows. Section 2 describes our option pricing model under stochastic interest rates. In section 3 we propose the other approach for value zero asset like futures.

2. Bond numeraire approach

Under a continuous time economy with the complete and frictionless market, we evaluate a European call option with strike price K expiring at time T. We assume no dividend for simplicity. Let $S(t)$ be the price of an underlying asset at time t. At time t, $B(t,T)$ is denoted by a price of the risk-free zero coupon bond with a payoff of \$1 at the maturity T which is the same date with the option's expiration date. Sometimes $B(\tau)$ is used to denote $B(t,T)$, where $\tau = T - t$ is time to maturity. We introduce the underlying asset price normalized by the bond price, $F(t,T)$ defined by $S(t)/B(t,T)$. It is known that $F(t,T)$ is a theoretical price of a forward price of the underlying asset. As in bond price, both $F(t,T)$ and $F(\tau)$ are used for the normalized underlying asset price. Different from the Black-Scholes model, we assume that the fractional change of the normalized underlying asset price follows one factor diffusion process, i.e.,

$$\frac{dF(\tau)}{F(\tau)} = \mu dt + \sigma dW(t), \tag{2.1}$$

where $dW(t)$ is a Wiener process, μ is the instantaneous expected rate and σ is the standard deviation of the fractional change of the normalized underlying asset price.

If $S(t)$ and $B(t,T)$ are homogenous of degree 1, the option price $C(t,T,S(t),B(t,T))$ becomes

$$\frac{C(t,T,S(t),B(t,T))}{B(t,T)} = C\left(t,T,\frac{S(t)}{B(t,T)},1\right).$$

For example, if $S(t)$ and $B(t,T)$ are both log-normal processes, then they are homogenous of degree 1 and Merton model is one special case. Since T is fixed for each option, $C(t,T,S(t),B(t,T))$ can be rephrased in terms of two variables τ and $F(\tau)$ as long as $S(t)$ and $B(\tau)$ are homogenous of degree 1. Let us define $V(\tau,F(\tau))$ as a new option price normalized by the numeraire, $B(t,T)$, i.e.,

$$V(\tau,F(\tau)) = \frac{C(t,T,S(t),B(t,T))}{B(t,T)}.$$

Let the fractional change of the normalized option price satisfy the following:

$$\frac{dV(\tau,F)}{V(\tau,F)} = \alpha dt + \delta dW(t), \tag{2.2}$$

where $dW(t)$ is a Wiener process, α is the instantaneous expected rate and δ is the standard deviation of the fractional change of the normalized option price. Similar to Merton's method [10], we can make a risk-free self-financing portfolio Π_t by investing w_1 amount for the normalized option and w_2 amount for the normalized underlying asset. By equations

(2.1) and (2.2), the change of instantaneous dollar return to the portfolio Π_t is

$$\frac{d\Pi_t}{\Pi_t} = w_1\frac{dV}{V} + w_2\frac{dF}{F} = (w_1\alpha + w_2\mu)dt + (w_1\delta + w_2\sigma)dW(t).$$

Since the portfolio is risk-free and self-financing, the next two simultaneous equations are hold.

$$w_1\alpha + w_2\mu = 0, \quad w_1\sigma + w_2\delta = 0.$$

In order to exist a nontrivial solution of the simultaneous equations above, the determinant of the coefficient matrix should be zero and the equation holds below.

$$\alpha/\mu = \delta/\sigma. \tag{2.3}$$

This equation implies that the ratio of expected rate to the standard deviation of the normalized asset's return equals to that of the normalized option's returns. On the other hand, by applying Ito's lemma to the normalized option $V(\tau, F)$, we have

$$\begin{aligned} dV &= V_F dF - V_\tau dt + \frac{1}{2}V_{FF}(dF)^2 \\ &= (\mu F V_F - V_\tau + \frac{1}{2}\sigma^2 F^2 V_{FF})dt + \sigma F V_F dW(t). \end{aligned}$$

Comparing the equation (2.2) with the equation above, we can find a restriction between coefficients of $V(\tau, F)$ as follows:

$$\alpha = \frac{\mu F V_F - V_\tau + \frac{1}{2}\sigma^2 F^2 V_{FF}}{V} \quad \text{and} \quad \delta = \frac{\sigma F V_F}{V}. \tag{2.4}$$

Combining equation (2.4) with a restriction (2.3), the following partial differential equation is satisfied.

$$\frac{1}{2}\sigma^2 F^2 V_{FF} = V_\tau. \tag{2.5}$$

This is a simple heat equation whose solution is easily derived. Note that at expiration date, $S(T) = F(T,T)$ and so the payoff of the option will be

$$\max[S - K] = \max[F - K, 0], \tag{2.6}$$

which is the terminal condition for equation (2.5). To guarantee the uniqueness of the solution we assume the following regularity condition,

$$\lim_{F\to\infty} \frac{\partial V}{\partial F} = 1. \tag{2.7}$$

When F approaches infinity, the price of underlying asset S approaches infinity also since a bond price is bounded below 0. Hence equation (2.7) means that the change of an option price is same as that of an underlying asset price when the normalized underlying asset price increases without a bound. This implies that finding the option price is equivalent to solving the equations (2.5), (2.6), and (2.7). Therefore we can obtain the option pricing formula in the Theorem below. We omit the precise proof since it is similar to the proof of Theorem 3.1 in next section.

THEOREM 2.1. *Suppose that the fractional change of the normalized price of an underlying asset is satisfied by (2.1) and (2.7). Assume that an underlying asset price $S(t)$ and a bond price $B(t,T)$ with the same maturity as that of option are homogeneous of degree 1. Then a European call option price with a strike price K and an expiration date T is given by*

$$C(t,T,S(t),B(t,T)) = S(t)N(d_1) - KB(t,T)N(d_2),$$

where $N(\cdot)$ is the cumulative normal distribution function and

$$d_1 = \frac{\ln(S(t)/KB(t,T)) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}; \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Proof. Applying the well known method solving a heat equation satisfying the boundary conditions (2.6) and (2.7), the solution of (2.5) is as follows:

$$V(\tau, F(\tau)) = F(\tau)N(d_1) - KN(d_2),$$

where $N(\cdot)$ is the cumulative normal distribution function and d_1 and d_2 are given by

$$d_1 = \frac{\ln(F(\tau)/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Replacing $F(\tau)$ and $V(\tau, F)$ by $S(t)/\ B(t,T)$ and $C(t,T,S(t),B(t,T))/\ B(t,T)$ respectively, the formula in the theorem is obtained. □

Note that the formula in the theorem is similar to that of Black-Scholes for stock index option. Moreover, the formula is exactly the same as that of Merton [10] for option pricing which assumes stochastic interest rate. However, the meaning is different. In the case of Black-Scholes option formula, volatility is the standard deviation of returns of underlying asset itself. As for Merton's case, the volatility is the mixture of standard deviations and the covariance of returns of an underlying asset and interest rate. On the other hand, the volatility in equation (2.1) is the standard deviation of returns for forward prices of an underlying asset and its meaning is clear below. Taking logarithm with $F(t,T)$,

$$d\ln F(t,T) = \ln F(t+1,T) - \ln F(t,T) = \ln\frac{S(t+1)}{S(t)} - \ln\frac{B(t+1,T)}{B(t,T)}.$$

Hence the difference of $\ln F(t,T)$ is a kind of premium which is the excess log return. Since volatilities of $\frac{dF(t,T)}{F(t,T)}$ and $d\ln F(t,T)$ are same by Ito's lemma, the volatility defined in equation (2.1) is that of a excess log return as we already point out in Introduction. As mentioned above, Merton model [10] is a special case of our result. When we impose the assumption on log-normality for bond price dynamics, Merton's formula on a stock index option holds. Next Lemma shows that Merton's assumptions induce homogeneity of degree 1 in bond price and underlying asset price.

LEMMA 2.1. *Suppose that returns of an stock index and risk-free domestic bond prices satisfy log-normal processes as follows:*

$$\frac{dS(t)}{S(t)} = \mu_1 dt + \sigma_1 dW_1(t),\ \frac{dB(t,T)}{B(t,T)} = \mu_2 dt + \sigma_2 dW_2(t),$$

where $\langle dW_1(t), dW_2(t)\rangle = \rho dt$. Then the stock price is homogeneous of degree 1 with the bond price in calculating option price.

Proof. Recall Merton's [10] solution for an index option given by

$$C(t,T,S(t),B(t,T)) = S(t)N(d_1) - KB(t,T)N(d_2), \tag{2.8}$$

where $N(\cdot)$ is the cumulative normal distribution function, d_1 and d_2 are given by

$$d_1 = \frac{\ln(S(t)/KB(t,T)) + \frac{1}{2}\Sigma^2(T-t)}{\Sigma\sqrt{T-t}},\quad d_2 = d_1 - \Sigma\sqrt{T-t}$$

and

$$\Sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

Dividing both sides of the formula (2.8) by $B(t,T)$, the option price is

$$\frac{C(t,T,S(t),B(t,T))}{B(t,T)} = \frac{S(t)}{B(t,T)}N(d_1) - KN(d_2). \tag{2.9}$$

Equation (2.9) and the definition of $N(d_1)$ show that the option price depends only on the ratio $S(t)/B(t,T)$. This fact implies that even if

$S(t)$ and $B(t,T)$ move in distinct ways, the option price will not change as long as the ratio does not. Hence we have

$$\frac{C(t,T,S(t),B(t,T))}{B(t,T)} = C(t,T,\frac{S(t)}{B(t,T)},1).$$

□

The next Corollary shows that Merton model is nested in our model.

COROLLARY 2.2. *Under the same assumptions as in Lemma 2.1, the option price formula is given by*

$$C(t,T,S(t),B(t,T)) = S(t)N(d_1) - KB(t,T)N(d_2),$$

where $N(\cdot)$ is the cumulative normal distribution function, d_1 and d_2 are given by

$$d_1 = \frac{\ln(S(t)/KB(t,T)) + \frac{1}{2}\Sigma^2(T-t)}{\Sigma\sqrt{T-t}}, \quad d_2 = d_1 - \Sigma\sqrt{T-t}$$

and

$$\Sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \tag{2.10}$$

Proof. The assertion will be easily obtained by Theorem 2.2 and Lemma 2.1 since the normalized asset price process $F(t,T)$ is calculated as follows:

$$\frac{dF(t,T)}{F(t,T)} = \frac{d(\frac{S(t)}{B(t,T)})}{\frac{S(t)}{B(t,T)}}$$

$$= \Big(\mu_1 - \mu_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 - \rho\sigma_1\sigma_2)dt + (\sigma_1 dW_1(t) - \sigma_2 dW_2(t)\Big).$$

Hence the standard deviation of $F(t,T)$ is Σ given by (2.10). □

COROLLARY 2.3. *Suppose that returns of a stock index price follow a log normal process as in the Lemma 2.1. Moreover suppose that an risk-free interest rate is constant and given by r. Then Black-Scholes formula is obtained, i.e.,*

$$C(t,T,S(t),r) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2),$$

where d_1 and d_2 are given by

$$d_1 = \frac{\ln(S(t)/K) + \frac{1}{2}(r+\sigma_1^2)(T-t)}{\sigma_1\sqrt{T-t}}, \quad d_2 = d_1 - \sigma_1\sqrt{T-t},$$

Proof. It is obvious that Black-Scholes is a special case when $\mu_2 = r$ and $\sigma_2 = 0$ in Corollary 2.2. □

The theorem will hold when stock continuously pays dividend and the dividend rate is known. The formula is similar to Merton's dividend-paying stock option formula. It is straightforward to apply the method above to a European spot currency option. For currency assets, denote $S(t)$ by exchange spot rate of two currencies. Let $B_f(t,T)$ and $B_d(t,T)$ be the price of the risk-free foreign and domestic zero coupon bonds with a payoff of 1 with their own currencies at the maturity T. Then the forward rate of a spot rate is given by

$$\widehat{F}(t,T) = \frac{S(t)B_f(t,T)}{B_d(t,T)}. \tag{2.11}$$

Making a self-financing risk-free portfolio similar to the argument above, we can obtain a closed form formula for a European currency spot call option price. The following theorem is stated without proof.

THEOREM 2.4. *Suppose that the forward price defined by* (2.11) *follows lognormal process, i.e.,*

$$\frac{d\widehat{F}(t,T)}{\widehat{F}(t,T)} = \widehat{\mu}dt + \widehat{\sigma}dW(t) \tag{2.12}$$

and the regularity condition holds similar to (2.7) *as follows:*

$$\lim_{\widehat{F}\to\infty} \frac{\partial V}{\partial \widehat{F}} = 1.$$

Also assume that $S(t)B_f(t,T)$ *and* $B_d(t,T)$ *are homogeneous of degree 1. Then a European currency spot call option price is given by*

(2.13)

$$C(t,T,S(t)B_f(t,T),B_d(t,T)) = S(t)B_f(t,T)N(d_1) - KB_d(t,T)N(d_2),$$

where $N(\cdot)$ *is the cumulative normal distribution function and*

$$d_1 = \frac{\ln(S(t)B_f(t,T)/KB_d(t,T)) + \frac{1}{2}\widehat{\sigma}^2(T-t)}{\widehat{\sigma}\sqrt{T-t}}, \quad d_2 = d_1 - \widehat{\sigma}\sqrt{T-t}.$$

Sometimes equation (2.12) is called UIP (uncovered interest rate parity) deviation. UIP means that the expected value of return of spot rate is equal to the spread between domestic and foreign interest rates. So equation (2.12) implies that UIP does not hold and stochastically moves. This assumption can be supported by many empirical documents (see Hyun, Kim and Rhee [7]). It is obvious to see that the formula (2.13) is the stochastic interest rate version of Garman-Kohlhagen currency option pricing formula [5]. When the term structure of foreign and domestic interest rates are flat, we have the Corollary below. The proof is similar to that of stock option price and omitted.

COROLLARY 2.5. *Assume that returns of spot exchange rate evolves as log-normal process and the volatility is given by σ_S. Also we assume that the foreign as well as the domestic interest rate are constant and given by r_f and r_d, respectively. Then the Garman-Kohlhagen currency call option formula holds, i.e.,*

$$C(t,T,S(t),r_f,r_d) = e^{-r_f(T-t)}S(t)N(d_1) - Ke^{-r_d(T-t)}N(d_2),$$

where $N(\cdot)$ is the cumulative normal distribution function and

$$d_1 = \frac{ln(S/K) - r_d + r_f + \frac{1}{2}\sigma_S^2(T-t)}{\sigma_S\sqrt{T-t}}, \quad d_2 = d_1 - \sigma_S\sqrt{T-t}.$$

3. Expectation hypothesis approach

In contrast with a spot asset, futures is daily settled so that futures value is always zero. According to futures price, cash inflow or outflow occurs for both a holder and a seller for futures option. Such cash inflow or outflow is called margin. This situation makes futures value zero and allows us to invest amount of daily margin to a bond. This is the main reason why bond numeraire approach can not be applied. As in bond numeraire approach, if we consider only two assets which are normalized futures and normalized option by bond, it is unable to make a self-financing risk-free portfolio. In order to make a self-financing portfolio, cash inflow or outflow should be daily invested to bonds. The maturities of these bonds are not necessarily same as the expiration date of the futures option. It is supposed to be rolled over everyday since settlement happens everyday. If interest rate would be constant, futures option would be treated in a similar way of stock or currency option as in the previous section.

Hence we need to impose a certain type of preference for bonds as in Ramaswamy and Sundaresan [11] which develop futures option price. Ramaswamy and Sundaresan [11] defines a process for a stock index which is the underlying asset of futures. However, we directly define a process of futures since futures is traded in a market with its own price. Also direct handle of futures will make error small. Therefore, we assume that the local expectation hypothesis for bonds with any maturities s

holds, i.e.,

$$E\left(\frac{dB(t,s)}{B(t,s)}\right) = rdt, \tag{3.1}$$

where r is a expected rate of returns of bonds. This assumption implies that the expected instantaneous holding period return on any default free bond is equal to the prevailing risk-free rate. Let $F(t, T_F)$ be a futures price at t whose delivery date of an underlying asset is T_F. Even though we mention futures option only, the idea seems to work on any option whose underlying is an asset having future-style margin. As in Section 2, we assume a continuous time economy with complete and frictionless market. In order to evaluate a European futures call option with strike price K expiring at time $T(< T_F)$, we assume that the fractional change of futures price follows log-normal diffusion process, i.e.,

$$\frac{dF(t,T_F)}{F(t,T_F)} = \mu_F dt + \sigma_F dW_1(t), \tag{3.2}$$

where $dW_1(t)$ is a Wiener process, μ_F is the instantaneous expected rate and σ_F is the standard deviation of the fractional change of the normalized underlying asset price. We also assume that the fractional change of the bond price which gives \$1 at maturity T, follows log-normal diffusion process, i.e.,

$$\frac{dB(t,T)}{B(t,T)} = rdt + \delta dW_2(t), \tag{3.3}$$

where $dW_2(t)$ is a Wiener process, r is the instantaneous expected rate and δ is the standard deviation of the fractional change of the normalized bond price. Allowing for the possibility of habitat and other term structure effects, it is not assumed that dW_1 is perfectly correlated with dW_2, i.e.,

$$\langle dW_1(t), dW_2(t)\rangle = \rho dt, \tag{3.4}$$

where ρ may be less than 1 for $T_F \neq T$. Even though the volatility term structure can be considered, we assume that volatility is constant for simplicity. Then the stochastic interest rate version of option pricing formula for futures option can be obtained.

THEOREM 3.1. *Suppose that the fractional change of the futures price of an underlying asset is satisfied by (3.1)–(3.4). Then a European futures call option price with the strike price K and expiration date T is given by*

$$C(t,T,F(t,T_F),B(t,T)) = F(t,T_F)e^{-r(T-t)}N(d_1) - KB(t,T)N(d_2),$$

where $N(\cdot)$ *is the cumulative normal distribution function and*

$$d_1 = \frac{\ln(F(t,T_F)/KB(t,T)) + (\frac{1}{2}\Sigma_F^2 - r)(T-t)}{\Sigma_F\sqrt{T-t}}, \quad d_2 = d_1 - \Sigma_F\sqrt{T-t}$$

with $\Sigma_F = \sqrt{\sigma_F^2 + \delta^2 - 2\rho\delta\sigma_F}$.

Proof. By mimicking the argument of proof of Theorem 2.1, we can make a self-financing risk-free portfolio Π_t with futures, option and bonds with various maturities. Different from Theorem 2.1, the current value of the portfolio is composed of option and bond only since the value of future is currently zero. Let $w_i (i = 1,2,3)$ be weights invested in futures $F(t,T_F)$, option $C(t,T,F(t,T_F),B(t,T))$ and bond $B(t,T)$ in order. Then a expected change of the portfolio is

$$E(d\Pi_t) = E(w_1 dF + w_2 dC + w_3 dB) = r(w_2 V + w_3 B)dt.$$

Choosing the weights w_i to make the portfolio risk-free, we have a parabolic equation below.

$$\text{(3.5a)} \quad \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 C}{\partial F^2} + \frac{1}{2}\delta^2 B^2 \frac{\partial^2 C}{\partial B^2} + \rho\delta\sigma_F FB \frac{\partial^2 C}{\partial F \partial B} + rB\frac{\partial C}{\partial B} + \frac{\partial C}{\partial t} = rC;$$

$$\text{(3.5b)} \quad C(t,T,0,B(t,T)) = 0;$$

$$\text{(3.5c)} \quad C(T,T,F(T,T_F),1) = \max(0, F(T,T_F) - K).$$

Since futures and bond evolve as log-normal processes, two asset prices are homogeneous of degree 1. So by the change of variables

$$x = \frac{F(t,T_F)}{KB(t,T)} \quad \text{and} \quad h(t,x) = \frac{C(t,T,F(t,T_F),B(t,T))}{KB(t,T)},$$

the equation (3.5) becomes

$$\text{(3.6a)} \quad \frac{1}{2}\Sigma_F^2 x^2 \frac{\partial^2 h}{\partial x^2} - rx\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} = 0;$$

$$\text{(3.6b)} \quad h(t,0) = 0;$$

$$\text{(3.6c)} \quad h(0,x) = \max(0, x-1),$$

where $\Sigma_F^2 = \sigma_F^2 + \delta^2 - 2\rho\delta\sigma_F$. Applying the change of variables to equation (3.6a) once more as below

$$y = \log x, \quad t = T - \frac{\tau}{\frac{1}{2}\Sigma_F^2} \quad \text{and} \quad H(\tau,y) = h(t,x),$$

we have a simple constant coefficient parabolic equation as follows:

$$\frac{\partial^2 H}{\partial y^2} - (1 + \frac{2r}{\Sigma_F^2})\frac{\partial H}{\partial y} = \frac{\partial H}{\partial \tau}; \tag{3.7a}$$

$$H(-\infty, 0) = 0; \tag{3.7b}$$

$$H(0, y) = \max(0, e^y - 1). \tag{3.7c}$$

To solve equation (3.7), we try to find a trial solution. Let $H(\tau, y) = e^{ay+b\tau}u(\tau, y)$ be a trial solution. If we choose

$$a = \frac{1}{2}(1 + \frac{2r}{\Sigma_F^2}) \quad \text{and} \quad b = -\frac{1}{4}(1 + \frac{2r}{\Sigma_F^2})^2,$$

then $u(\tau, y)$ satisfies a simple heat equation as follows:

$$u_\tau = u_{yy} \quad \text{for} \quad -\infty < y < \infty, \quad \tau > 0,$$

$$u(0, y) = \max(0, e^{(1-a)y} - e^{-ay}). \tag{3.8}$$

We know that the solution is

$$\begin{aligned} u(\tau, y) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u(0, s) e^{-\frac{(y-s)^2}{4\tau}} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(0, y + \sqrt{2\tau}x) e^{-\frac{x^2}{2}} dx. \end{aligned}$$

The second equality holds by change of variable $s = y - \sqrt{2\tau}x$. By putting initial condition (3.8) back, we have

$$u(\tau, y) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{y}{\sqrt{2\tau}}}^{\infty} (e^{(1-a)(y+\sqrt{2\tau}x)} - e^{-a(y+\sqrt{2\tau}x)}) e^{-\frac{x^2}{2}} dx.$$

Finally simple calculation of two integrals allows us to get a close form of $u(\tau, y)$ which is stated in the theorem by tracing change of variables back. □

The formula looks puzzling since two different bond prices come up. As mentioned above, futures option is not only relatively priced but absolutely because of preference. The first term, $e^{-r(T-t)}$ is shown because of cash flow by daily settlement of futures and the second term $B(t, T)$ comes from relative pricing of option with respect to bond and underlying asset. When the risk-free interest rates are constant, the formula for futures option price by Black [2] can be achieved.

COROLLARY 3.2. *When the risk-free interest rates are constant and given by r, the futures option price formula is*

$$C(t, F(t)) = e^{-r(T-t)}(F(t)N(d_1) - KN(d_2)),$$

where $N(\cdot)$ is the cumulative normal distribution function and

$$d_1 = \frac{\ln(F(t)/K) + (r + \frac{1}{2}\sigma_F^2)(T-t)}{\sigma_F\sqrt{T-t}}, \quad d_2 = d_1 - \sigma_F\sqrt{T-t}.$$

Proof. When the interest rate is constant and given by r, note that $B(t,T) = e^{-r(T-t)}$, $\ln B(t,T) = -r(T-t)$ and $\delta = 0$. The assertion holds. □

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