J. Korean Math. Soc. **43** (2006), No. 4, pp. 815–828

ON THE RATE OF COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF ARRAYS OF RANDOM ELEMENTS

Soo Hak Sung and Andrei I. Volodin

Abstract. Let $\{V_{nk}, k \geq 1, n \geq 1\}$ be an array of rowwise independent random elements which are stochastically dominated by a random variable X with $E|X|^{\frac{\alpha}{\gamma}+\theta}\log^{\rho}(|X|) < \infty$ for some $\rho > 0, \alpha > 0, \gamma > 0, \theta > 0$ such that $\theta+\alpha/\gamma < 2$. Let $\{a_{nk}, k \geq 1, n \geq 1\}$ be an array of suitable constants. A complete convergence result is obtained for the weighted sums of the form $\sum_{k=1}^{\infty} a_{nk}V_{nk}$.

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [5] as follows. A sequence $\{U_n, n \geq 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \epsilon) < \infty \quad \text{for all } \epsilon > 0.$$

The classical Hsu-Robbins-Erdős theorem (Erdős [3, 4]) states that, for a sequence $\{X_n, n \geq 1\}$ of independent and identically distributed random variables, $\sum_{k=1}^{n} X_k/n$ converges completely to EX_1 if and only if the variance of X_1 is finite. Baum and Katz [2] obtained an elegant generalization of Hsu-Robbins-Erdős theorem, namely they proved that, for $r \geq 1$ and $1 \leq t < 2r \leq 2t$, $\sum_{n=1}^{\infty} n^{r-2}P(|\sum_{k=1}^{n}(X_k - EX_1)| > n^{r/t}\epsilon) < \infty$ for all $\epsilon > 0$ if and only if $E|X_1|^t < \infty$.

Many authors extended the above results to Banach space valued random elements, for example, see Ahmed et al. [1], Hu et al. [6, 7],

Received March 2, 2005. Revised June 10, 2005.
2000 Mathematics Subject Classification: Primary 60B12, 60F05, 60F15.
Key words and phrases: arrays of random elements, rowwise independence, weighted sums, complete convergence, rate of convergence, convergence in probability.
This work was supported by the Korea Research Foundation Grant (KRF-2003-041-C00034).

Kuczmaszewska and Szynal [8], Sung [9], and Wang et al. [11]. A sequence of Banach space valued random elements is said to converge completely to the 0 element in the Banach space if the corresponding sequence of norms converges completely to 0.

Hu, Rosalsky, Szynal and Volodin [7] presented a general result (cf. Theorem 1 below) establishing complete convergence for the row sums of an array of rowwise independent but not necessarily identically distributed Banach space valued random elements. Their result also specified the corresponding rate of convergence. The Hu, Rosalsky, Szynal and Volodin [7] result unifies and extends previously obtained results in the literature in that many of them (for example, results of Hsu and Robbins [5], Hu et al. [6], Kuczmaszewska and Szynal [8], Sung [9], Volodin et al. [10], and Wang et al. [11]) follow from it.

In the following we assume that $\{V_{nk}, k \geq 1, n \geq 1\}$ is an array of rowwise independent random elements in a real separable Banach space and $\{a_{nk}, k \geq 1, n \geq 1\}$ is an array of constants. Denote

$$S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk}.$$

In the next theorem the weights a_{nk} are built into the array (that is, $a_{nk} = 1$ for all k and n).

THEOREM 1. (Hu et al. [7]) *Let $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that*

$$(1) \qquad \sum_{n=1}^{\infty} c_n \sum_{k=1}^{\infty} P(||V_{nk}|| > \epsilon) < \infty \text{ for all } \varepsilon > 0,$$

$$(2) \qquad \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{\infty} E||V_{nk}||^q \right)^J < \infty \text{ for some } 0 < q \leq 2 \text{ and } J \geq 2,$$

$$(3) \qquad \sum_{k=1}^{\infty} V_{nk} \xrightarrow{P} 0,$$

and

$$(4) \qquad \text{if } \liminf_{n\to\infty} c_n = 0, \text{ then } \sum_{k=1}^{\infty} P(||V_{nk}|| > \delta) = o(1) \text{ for some } \delta > 0.$$

Then

$$\sum_{n=1}^{\infty} c_n P(||S_n|| > \epsilon) < \infty \text{ for all } \varepsilon > 0.$$

It is implicitly assumed in Theorem 1 that the series S_n converges a.s.

The article Hu et al. [6] is devoted to presenting applications of Theorem 1 to obtain new complete convergence results. Theorem 2 generalizes results of Hsu and Robbins [5], Kuczmaszewska and Szynal [8], Sung [9], Wang et al. [11] in three directions, namely:

(i) Banach space valued random elements instead of random variables are considered.

(ii) An array rather than a sequence is considered.

(iii) The rate of convergence is obtained.

THEOREM 2. (Hu et al. [6]). *Suppose that the array* $\{V_{nk}, k \geq 1, n \geq 1\}$ *is stochastically dominated by a random variable* X. *That is,*

$$\begin{aligned} & P(||V_{nk}|| > x) \\ \leq \; & CP(|X| > x) \text{ for all } x > 0 \text{ and for all } k \geq 1 \text{ and } n \geq 1, \end{aligned}$$

where C *is a positive constant. Assume that*

$$\begin{aligned} \sup_{k\geq 1} |a_{nk}| &= O(n^{-\gamma}) \text{ for some } \gamma > 0, \text{ and} \\ \sum_{k=1}^{\infty} |a_{nk}| &= O(n^{\alpha}) \text{ for some } \alpha \in [0, \gamma). \end{aligned}$$

If

$$E|X|^{1+(1+\alpha+\beta)/\gamma} < \infty \text{ for some } \beta \in (-1, \gamma - \alpha - 1], \text{ and } S_n \xrightarrow{P} 0,$$

then

$$\sum_{n=1}^{\infty} n^{\beta} P(||S_n|| > \epsilon) < \infty \text{ for all } \varepsilon > 0.$$

The proof of Theorem 2 is rather complicated once it uses the Stieltjes integral techniques, summation by parts lemma and so on. The initial objective of an investigation resulted in the paper Ahmed et al. [1] was only to find a simpler proof. But it appears that they were able to establish a more general result and with simpler proof. The result presented in Theorem 3 below is more general than the main result of Hu et al. [6], since rates of convergence for moving averages can be established, which cannot be proved using Theorem 2.

THEOREM 3. (Ahmed et al. [1]) *Suppose that the array* $\{V_{nk}, k \geq 1, n \geq 1\}$ *is stochastically dominated by a random variable* X. *Assume* *that*

$$\sup_{k\geq 1}|a_{nk}| = O(n^{-\gamma}) \text{ for some } \gamma > 0,$$

and

$$\sum_{k=1}^{\infty}|a_{nk}| = O(n^{\alpha}) \text{ for some } \alpha < \gamma.$$

Let β *be such that* $\alpha+\beta \neq -1$ *and fix* $\delta > 0$ *such that* $\frac{\alpha}{\gamma}+1 < \delta \leq 2$. *If*

$$E|X|^{\nu} < \infty, \text{ where } \nu = \max\{1+\frac{1+\alpha+\beta}{\gamma}, \delta\},$$

and

$$S_n \xrightarrow{P} 0,$$

then

$$\sum_{n=1}^{\infty} n^{\beta}P(||S_n|| > \epsilon) < \infty \text{ for all } \varepsilon > 0.$$

Theorem 3 was slightly generalized in Volodin et al. [10] as follows.

THEOREM 4. (Volodin et al. [10]) *Suppose that the array* $\{V_{nk}, k \geq 1, n \geq 1\}$ *is stochastically dominated by a random variable* X. *Assume that*

$$\sup_{k\geq 1}|a_{nk}| = O(n^{-\gamma}) \text{ for some } \gamma > 0, \text{ and} \sum_{k=1}^{\infty}|a_{nk}|^{\theta} = O(n^{\alpha})$$

for some $0 < \theta \leq 2$ *and any* α *such that* $\theta + \frac{\alpha}{\gamma} < 2$. *Let* β *be such that* $\alpha+\beta \neq -1$ *and fix* $\delta > \theta$ *such that* $\frac{\alpha}{\gamma}+\theta < \delta \leq 2$. *If*

$$E|X|^{\nu} < \infty, \text{ where } \nu = \max\{\theta+\frac{1+\alpha+\beta}{\gamma}, \delta\}, \text{ and } S_n \xrightarrow{P} 0,$$

then

$$\sum_{n=1}^{\infty} n^{\beta}P(||S_n|| > \epsilon) < \infty \text{ for all } \varepsilon > 0.$$

If $\beta < -1$, then the conclusions of Theorems 3 and 4 are immediate and hence Theorems 3 and 4 are of interest only for $\beta \geq -1$. In particular, the case $\beta = -1$ is of special interest. Ahmed et al. [1] conjectured that when $\beta = -1$, the assumption $E|X|^{\nu} < \infty$ can be replaced by $E|X|^{\frac{\alpha}{\gamma}+1}\log^{\rho}(|X|) < \infty$ $(\rho > 0)$ in Theorem 3. In the context of Theorem 4 this conjecture should be rewritten as: when $\beta = -1$, the assumption $E|X|^{\nu} < \infty$ can be replaced by the strictly weaker assumption $E|X|^{\frac{\alpha}{\gamma}+\theta} \log^{\rho}(|X|) < \infty$ $(\rho > 0)$. In this paper we give the positive answer on this conjecture.

It proves convenient to define $\log(x) = \max\{1, \ln(x)\}$, where $\ln(x)$ denotes the natural logarithm. The symbol C denotes a positive constant which is not necessarily the same one in each appearance, the symbol $[x]$ denotes the greatest integer in x, and for a finite set A the symbol $\#A$ denotes the number of elements in the set A.

2. Preliminaries

In this section, we present three lemmas which will be used to prove our main result.

LEMMA 1. *Let $\{a_{nk}, k \geq 1, n \geq 1\}$ be an array of constants such that for some $\theta > 0$, some α, and any $n \geq 1$*

$$\sum_{k=1}^{\infty} |a_{nk}|^{\theta} \leq n^{\alpha}.$$

Let $\{\phi(j), j \geq 1\}$ be an increasing sequence of positive numbers and

$$I_{nj} = \left\{k \middle| \frac{1}{n^{\gamma}\phi(j+1)} < |a_{nk}| \leq \frac{1}{n^{\gamma}\phi(j)}\right\}, \; j \geq 1, \; n \geq 1,$$

where γ is a constant. Then for any $m \geq 1$

$$\sum_{j=1}^{m} \sharp I_{nj} \leq n^{\alpha+\gamma\theta}\phi^{\theta}(m+1).$$

Proof. Really,

$$\sum_{j=1}^{m} \sharp I_{nj} = \sum_{j=1}^{m} \sum_{k \in I_{nj}} |a_{nk}|^{\theta} \frac{1}{|a_{nk}|^{\theta}}$$
$$\leq n^{\gamma\theta} \sum_{j=1}^{m} \phi^{\theta}(j+1) \sum_{k \in I_{nj}} |a_{nk}|^{\theta} \leq n^{\alpha+\gamma\theta}\phi^{\theta}(m+1).$$

□

LEMMA 2. *Let $\{V_{nk}, k \geq 1, n \geq 1\}$ be an array of random elements which are stochastically dominated by a random variable X. Let*

$\{a_{nk}, k \geq 1, n \geq 1\}$ be an array of constants such that

$$\sup_{k\geq 1} |a_{nk}| \leq n^{-\gamma} \quad \textit{for some } \gamma > 0$$

and

$$\sum_{k=1}^{\infty} |a_{nk}|^{\theta} \leq n^{\alpha} \quad \textit{for some } \theta > 0 \textit{ and some } \alpha.$$

Then for any $\epsilon > 0$ and all $n \geq 1$

$$\sum_{k=1}^{\infty} P(\|a_{nk}V_{nk}\| > \epsilon) \leq Cn^{\alpha} \sum_{k=n}^{\infty} k^{\gamma\theta} P(k < \left|\frac{X}{\epsilon}\right|^{1/\gamma} \leq k+1).$$

Proof. In Lemma 1 consider $\phi(j) = j^{\gamma}, j \geq 1$. Then

$$I_{nj} = \left\{k| \ \frac{1}{(n(j+1))^{\gamma}} < |a_{nk}| \leq \frac{1}{(nj)^{\gamma}}\right\} \quad \text{for } j \geq 1 \text{ and } n \geq 1$$

and

$$\sum_{j=1}^{m} \sharp I_{nj} \leq n^{\alpha+\gamma\theta}(m+1)^{\gamma\theta}.$$

Mention that the condition $\sup_{k\geq 1} |a_{nk}| \leq n^{-\gamma}$ ensures us that, for any $n \geq 1$, $\cup_{j\geq 1} I_{nj} = \{k|a_{nk} \neq 0\}$. It follows that

$$\begin{aligned}
\sum_{k=1}^{\infty} P(\|a_{nk}V_{nk}\| > \epsilon) &= \sum_{j=1}^{\infty} \sum_{k\in I_{nj}} P(\|a_{nk}V_{nk}\| > \epsilon) \\
&\leq \sum_{j=1}^{\infty} \sum_{k\in I_{nj}} P(\|V_{nk}\| > \epsilon(nj)^{\gamma}) \\
&\leq C \sum_{j=1}^{\infty} \sharp I_{nj} P(|\frac{X}{\epsilon}| > (nj)^{\gamma}) \\
&= C \sum_{j=1}^{\infty} \sharp I_{nj} \sum_{k=nj}^{\infty} P(k < |\frac{X}{\epsilon}|^{1/\gamma} \leq k+1) \\
&= C \sum_{k=n}^{\infty} P(k < |\frac{X}{\epsilon}|^{1/\gamma} \leq k+1) \sum_{j=1}^{[\frac{k}{n}]} \sharp I_{nj}
\end{aligned}$$

$$\leq C\sum_{k=n}^{\infty}P(k<|\frac{X}{\epsilon}|^{1/\gamma}\leq k+1)n^{\alpha+\gamma\theta}([\frac{k}{n}]+1)^{\gamma\theta}$$
$$\leq C2^{\gamma\theta}n^{\alpha}\sum_{k=n}^{\infty}P(k<|\frac{X}{\epsilon}|^{1/\gamma}\leq k+1)k^{\gamma\theta}.$$

□

LEMMA 3. *Let all the conditions of Lemma 2 be satisfied and $\alpha \geq 0$. Then for all $\epsilon > 0$*

$$\sum_{k=1}^{\infty}P(\|a_{nk}V_{nk}\|>\epsilon)\leq CE\left|\frac{X}{\epsilon}\right|^{\frac{\alpha}{\gamma}+\theta}I(|X|>\epsilon n^{\gamma}).$$

Proof. By Lemma 2

$$\sum_{k=1}^{\infty}P(\|a_{nk}V_{nk}\|>\epsilon)$$
$$\leq Cn^{\alpha}\sum_{k=n}^{\infty}P\Big(k<\left|\frac{X}{\epsilon}\right|^{1/\gamma}\leq k+1\Big)k^{\gamma\theta}$$
$$\leq C\sum_{k=n}^{\infty}P\Big(k<\left|\frac{X}{\epsilon}\right|^{1/\gamma}\leq k+1\Big)k^{\gamma\theta+\alpha}\ (\text{since } \alpha\geq 0)$$
$$\leq CE|\frac{X}{\epsilon}|^{\frac{\alpha}{\gamma}+\theta}I(|X|>\epsilon n^{\gamma}).$$

□

3. Main result

In this section, we state and prove our main result.

THEOREM 5. *Let $\{V_{nk}, k\geq 1, n\geq 1\}$ be an array of rowwise independent random elements which are stochastically dominated by a random variable X. Let $\{a_{nk}, k\geq 1, n\geq 1\}$ be an array of constants such that*

$$\sup_{k\geq 1}|a_{nk}|=O(n^{-\gamma})\quad \text{for some } \gamma>0$$

and

$$\sum_{k=1}^{\infty}|a_{nk}|^{\theta}=O(n^{\alpha})\quad \text{for some } \alpha>0 \text{ and } \theta>0 \text{ such that } \theta+\frac{\alpha}{\gamma}<2.$$

Assume that

$$S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk} \xrightarrow{P} 0.$$

If $E|X|^{\frac{\alpha}{\gamma}+\theta} \log^{\rho}(|X|) < \infty$ *for some* $\rho > 0$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_n\| > \epsilon) < \infty \quad \textit{for all } \epsilon > 0.$$

Proof. Without loss of generality, we may assume that $\sup_{k\geq 1} |a_{nk}| \leq n^{-\gamma}$ and $\sum_{k=1}^{\infty} |a_{nk}|^{\theta} \leq n^{\alpha}$. For any $n \geq 1$ let

$$V_{nk}^{(1)} = V_{nk} I(\|a_{nk} V_{nk}\| \leq 1) \text{ and}$$

$$V_{nk}^{(2)} = V_{nk} I(\|a_{nk} V_{nk}\| > 1), \quad 1 \leq k < \infty.$$

Then

$$S_n = \sum_{k=1}^{\infty} a_{nk} V_{nk} = \sum_{p=1}^{2} \sum_{k=1}^{\infty} a_{nk} V_{nk}^{(p)} = \sum_{p=1}^{2} S_n^{(p)}, \text{ say.}$$

To prove the theorem, it suffices to show that for $p = 1$ and 2:

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_n^{(p)}\| > \epsilon) < \infty \quad \text{for all } \epsilon > 0.$$

To do this, we apply Theorem 1 with $c_n = 1/n$ to the random elements $a_{nk} V_{nk}^{(p)}, p = 1, 2$.

Then

$$\max_{p=1,2} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} P(\|a_{nk} V_{nk}^{(p)}\| > \epsilon)$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} P(\|a_{nk} V_{nk}\| > \epsilon)$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha-1}\sum_{k=n}^{\infty} k^{\gamma\theta}P(k < |\frac{X}{\epsilon}|^{1/\gamma} \leq k+1) \text{ (by Lemma 2)}$$
$$= C\sum_{k=1}^{\infty} k^{\gamma\theta}P(k < |\frac{X}{\epsilon}|^{1/\gamma} \leq k+1)\sum_{n=1}^{k} n^{\alpha-1}$$
$$\leq C\sum_{k=1}^{\infty} k^{\alpha+\gamma\theta}P(k < |\frac{X}{\epsilon}|^{1/\gamma} \leq k+1) \text{ (since } \alpha > 0)$$
$$\leq CE|\frac{X}{\epsilon}|^{\frac{\alpha}{\gamma}+\theta} < \infty.$$

Hence (1) is satisfied for both series.

By Lemma 3, we have for any $\epsilon > 0$

$$\begin{aligned}P(\|\sum_{k=1}^{\infty} a_{nk}V_{nk}I(\|a_{nk}V_{nk}\| > 1)\| > \epsilon) &\leq P(\cup_{k=1}^{\infty}\|a_{nk}V_{nk}\| > 1)\\ &\leq \sum_{k=1}^{\infty} P(\|a_{nk}V_{nk}\| > 1)\\ &\leq CE|X|^{\frac{\alpha}{\gamma}+\theta}I(|X| > n^{\gamma}) = o(1),\end{aligned}$$

since $E|X|^{\frac{\alpha}{\gamma}+\theta} < \infty$. Hence $S_n^{(2)} \xrightarrow{P} 0$. By the hypothesis $S_n \xrightarrow{P} 0$ and $S_n^{(1)} = S_n - S_n^{(2)}$, we have $S_n^{(1)} \xrightarrow{P} 0$. We conclude that condition (3) from Theorem 1 is satisfied for both series.

The condition (4) from Theorem 1 with $\delta = 1$ is obviously satisfied for the first series, since

$$\sum_{k=1}^{\infty} P(\|a_{nk}V_{nk}^{(1)}\| > 1) = 0.$$

For the second series we have by Lemma 3 that

$$\begin{aligned}\sum_{k=1}^{\infty} P(\|a_{nk}V_{nk}^{(2)}\| > 1) &= \sum_{k=1}^{\infty} P(\|a_{nk}V_{nk}\| > 1)\\ &\leq CE|X|^{\frac{\alpha}{\gamma}+\theta}I(|X| > n^{\gamma}) = o(1).\end{aligned}$$

Hence, condition (4) is satisfied for both series.

Finally, we check condition (2) from Theorem 1 for both series. To do this, we introduce the following notations. For $t > 0$ such that

$0 < t\theta < \rho$ and any $n \geq 1$, let

$$A_n = \left\{ k|\ |a_{nk}| \leq \frac{1}{n^\gamma \log^t(n)} \right\}, \quad B_n = \left\{ k|\ \frac{1}{n^\gamma \log^t(n)} < |a_{nk}| \leq \frac{1}{n^\gamma} \right\}.$$

Next, for any $n \geq 1$ let

$$a_{nk}^{(1)} = \begin{cases} a_{nk} & \text{if } k \in A_n \\ 0 & \text{otherwise} \end{cases}, \qquad a_{nk}^{(2)} = \begin{cases} a_{nk} & \text{if } k \in B_n \\ 0 & \text{otherwise.} \end{cases}$$

Let $r = \frac{\alpha}{\gamma} + \theta < 2$. First we mention that

$$\begin{aligned} \max_{p=1,2} \sum_{k=1}^{\infty} E\|a_{nk}^{(1)} V_{nk}^{(p)}\|^r &\leq \sum_{k \in A_n} E\|a_{nk} V_{nk}\|^r \\ &\leq C \left(\frac{1}{n^\gamma \log^t(n)} \right)^{\alpha/\gamma} E|X|^{\alpha/\gamma+\theta} \sum_{k=1}^{\infty} |a_{nk}|^\theta \\ &\leq C \log^{-t\alpha/\gamma}(n) E|X|^{\alpha/\gamma+\theta}. \end{aligned}$$

Hence we have that

$$\sum_{k=1}^{\infty} E\|a_{nk}^{(1)} V_{nk}^{(1)}\|^2 \leq \sum_{k=1}^{\infty} E\|a_{nk}^{(1)} V_{nk}^{(1)}\|^r \leq C \log^{-t\alpha/\gamma}(n) E|X|^{\alpha/\gamma+\theta} \tag{5}$$

and

$$\sum_{k=1}^{\infty} E\|a_{nk}^{(1)} V_{nk}^{(2)}\|^r \leq C \log^{-t\alpha/\gamma}(n) E|X|^{\alpha/\gamma+\theta}. \tag{6}$$

In order to verify condition (2) for other cases, put

$$B_{nj} = \left\{ k|\ \frac{1}{n^\gamma \log^t(j+1)} < |a_{nk}| \leq \frac{1}{n^\gamma \log^t(j)} \right\}.$$

Then $\{B_{nj}, 1 \leq j \leq n-1\}$ are disjoint, $\cup_{j=1}^{n-1} B_{nj} = B_n$, and by Lemma 1 with $\phi(j) = \log^t(j)$, $\sharp B_n \leq n^{\alpha+\gamma\theta} \log^{t\theta}(n)$. We can estimate

$$\begin{aligned} \sum_{k=1}^{\infty} E\|a_{nk}^{(2)} V_{nk}^{(2)}\|^r &= \sum_{k \in B_n} |a_{nk}|^r E\|V_{nk}\|^r I\left(\|V_{nk}\| > \frac{1}{|a_{nk}|} \right) \\ &\leq C \sum_{k \in B_n} |a_{nk}|^r E|X|^r I(|X| > n^\gamma) \end{aligned}$$

$$
\begin{aligned}
&\le CE|X|^r I(|X| > n^\gamma)(n^{-\gamma})^r \sharp B_n \\
(7) \qquad &\le CE|X|^r I(|X| > n^\gamma)\log^{t\theta}(n) \\
&\le CE|X|^r \log^\rho(|X|) I(|X| > n^\gamma)\log^{t\theta-\rho}(n) \\
&\le CE|X|^r \log^\rho(|X|)\log^{t\theta-\rho}(n).
\end{aligned}
$$

Next, we estimate the remaining part in the following way

$$
\begin{aligned}
&\sum_{k=1}^{\infty} E\|a_{nk}^{(2)}V_{nk}^{(1)}\|^2 \\
&= \sum_{j=1}^{n-1}\sum_{k\in B_{nj}} E\|a_{nk}V_{nk}\|^2 I(\|a_{nk}V_{nk}\| \le 1) \\
&\le \sum_{j=1}^{n-1}\sum_{k\in B_{nj}} \frac{1}{(n^\gamma \log^t(j))^2} E\|V_{nk}\|^2 I(\|V_{nk}\| \le n^\gamma \log^t(j+1)) \\
&\le C\sum_{j=1}^{n-1} \sharp B_{nj} n^{-2\gamma}\log^{-2t}(j) EX^2 I(|X| \le n^\gamma \log^t(j+1)) \\
&\quad + C\sum_{j=1}^{n-1} \sharp B_{nj} P(|X| > n^\gamma \log^t(j+1)) \\
&= I_1 + I_2, \text{ say.}
\end{aligned}
$$

Here we used the fact that if a random variable Y is stochastically dominated by a random variable X, then for all $s > 0$ and $b > 0$

$$E|Y|^s I(|Y| \le b) \le CE|X|^s I(|X| \le b) + Cb^s P(|X| > b).$$

Let $\mu = 2 - \frac{\alpha}{\gamma} - \theta > 0$. Since $\frac{x^\mu}{\log^\rho(x)} \uparrow$ as $x \to \infty$,

$$\frac{x^\mu}{\log^\rho(x)} \le \frac{(n^\gamma \log^t(j+1))^\mu}{\log^\rho(n^\gamma \log^t(j+1))} \le C\frac{n^{\gamma\mu}\log^{t\mu}(j)}{\log^\rho(n)},$$

if $x \le n^\gamma \log^t(j+1)$. Then

$$
\begin{aligned}
I_1 &\le C\sum_{j=1}^{n-1} \sharp B_{nj} n^{-2\gamma}\log^{-2t}(j) E|X|^r \log^\rho(|X|)\frac{n^{\gamma\mu}\log^{t\mu}(j)}{\log^\rho(n)} \\
&= C\frac{n^{\gamma\mu-2\gamma}}{\log^\rho(n)} E|X|^r \log^\rho(|X|)\sum_{j=1}^{n-1}\log^{t\mu-2t}(j)\sharp B_{nj}
\end{aligned}
$$

$$\le C\frac{n^{\gamma\mu-2\gamma}}{\log^\rho(n)}E|X|^r\log^\rho(|X|)\sum_{j=1}^{n-1}\sharp B_{nj} \text{ (since } t\mu-2t<0)$$
$$\le CE|X|^r\log^\rho(|X|)\log^{t\theta-\rho}(n).$$

Clearly, I_2 is dominated by

$$CP(|X|>n^\gamma)\sum_{j=1}^{n-1}\sharp B_{nj}\le CP(|X|>n^\gamma)n^{\alpha+\gamma\theta}\log^{t\theta}(n)$$
$$\le CE|X|^r\log^\rho(|X|)\log^{t\theta-\rho}(n).$$

Hence

$$\sum_{k=1}^{\infty}E\|a_{nk}^{(2)}V_{nk}^{(1)}\|^2\le CE|X|^{\alpha/\gamma+\theta}\log^\rho(|X|)\log^{t\theta-\rho}(n). \tag{8}$$

Take J such that $J(\rho-t\theta)\ge 2$ and $Jt\alpha/\gamma\ge 2$. We have by (5) and (8) that

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{\infty}E\|a_{nk}V_{nk}^{(1)}\|^2\right)^J$$
$$=\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{\infty}E\|a_{nk}^{(1)}V_{nk}^{(1)}\|^2+\sum_{k=1}^{\infty}E\|a_{nk}^{(2)}V_{nk}^{(1)}\|^2\right)^J$$
$$\le\sum_{n=1}^{\infty}\frac{1}{n}\left(CE|X|^{\alpha/\gamma+\theta}\log^{-t\alpha/\gamma}(n)+CE|X|^{\alpha/\gamma+\theta}\log^\rho(|X|)\log^{t\theta-\rho}(n)\right)^J$$
$$\le C\sum_{n=1}^{\infty}\frac{1}{n\log^2(n)},$$

since $E|X|^{\alpha/\gamma+\theta}\log^\rho(|X|)<\infty$. Hence for the first series condition (2) from Theorem 1 is satisfied with $q=2$.

Next, by (6) and (7) we have that

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{\infty}E\|a_{nk}V_{nk}^{(2)}\|^r\right)^J$$
$$=\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{\infty}E\|a_{nk}^{(1)}V_{nk}^{(2)}\|^r+\sum_{k=1}^{\infty}E\|a_{nk}^{(2)}V_{nk}^{(2)}\|^r\right)^J$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} \left(CE|X|^{\alpha/\gamma+\theta} \log^{-t\alpha/\gamma}(n) + CE|X|^{\alpha/\gamma+\theta} \log^{\rho}(|X|) \log^{t\theta-\rho}(n) \right)^{J}$$

$$\leq C \sum_{n=1}^{\infty} \frac{1}{n \log^2(n)}.$$

Hence for the second series condition (2) from Theorem 1 is satisfied with $q = r$.

Therefore all conditions from Theorem 1 are satisfied for both series, and so the proof is complete. □

ACKNOWLEDGEMENTS. We would like to thank the referee for his careful reading of this paper and for his helpful comments.

References

[1] S. E. Ahmed, R. G. Antonini, and A. I. Volodin, *On the rate of complete convergence for weighted sums of arrays of Banach space valued random elements with application to moving average processes*, Statist. Probab. Lett. **58** (2002), no. 2, 185–194.

[2] L. E. Baum and M. Katz, *Convergence rates in the law of large numbers*, Trans. Amer. Math. Soc. **120** (1965), 108–123.

[3] P. Erdős, *On a theorem of Hsu and Robbins*, Ann. Math. Statistics **20** (1949), 286–291.

[4] ———, *Remark on my paper "On a theorem of Hsu and Robbins"*, Ann. Math. Statistics **21** (1950), 138.

[5] P. L. Hsu and H. Robbins, *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 25–31.

[6] T.-C. Hu, D. Li, A. Rosalsky, and A. I. Volodin, *On the rate of complete convergence for weighted sums of arrays of Banach space valued random elements*, Theory Probab. Appl. **47** (2003), 455–468.

[7] T.-C. Hu, A. Rosalsky, D. Szynal and A. I. Volodin, *On complete convergence for arrays of rowwise independent random elements in Banach spaces*, Stochastic Anal. Appl. **17** (1999), no. 6, 963–992.

[8] A. Kuczmaszewska and D. Szynal, *On complete convergence in a Banach space*, Internat. J. Math. Math. Sci. **17** (1994), no. 1, 1–14.

[9] S. H. Sung, *Complete convergence for weighted sums of arrays of rowwise independent B-valued random variables*, Stochastic Anal. Appl. **15** (1997), no. 2, 255–267.

[10] A. Volodin, R. G. Antonini and T.-C. Hu, *A note on the rate of complete convergence for weighted sums of arrays of Banach space valued random elements*, Lobachevskii J. Math. (electronic) **15** (2004), 21–33.

[11] X. Wang, M. B. Rao and X. Yang, *Convergence rates on strong laws of large numbers for arrays of rowwise independent elements*, Stochastic Anal. Appl. **11** (1993), no. 1, 115–132.

Soo Hak Sung
Department of Applied Mathematics
Pai Chai University
Taejon 302-735, Korea
E-mail: sungsh@pcu.ac.kr

Andrei I. Volodin
Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan, S4S 0A2, Canada
E-mail: andrei@math.uregina.ca