

STABLE VECTOR BUNDLES OF RANK TWO ON ENRIQUES SURFACES

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ABSTRACT. We describe the moduli spaces of stable vector bundles of rank 2 on Enriques surfaces. They all have the structure of the fibrations reflecting those of Enriques surfaces.

0. Introduction

Moduli spaces of stable vector bundles of rank 2 on complex surfaces have been studied by several authors. The structures of the moduli spaces of stable bundles on surfaces such as rational surfaces ([2], [12], [9]), ruled surfaces([5], [24]), $K3$ surfaces ([15], [16], [21], [22]), elliptic surfaces ([11], [10], [18]) and some surfaces of general type ([4], [8]) have been described. In this paper, we study the moduli spaces of stable vector bundles of rank two on Enriques surfaces. Every Enriques surface is elliptic. So, we can compare our results with those of R. Friedman, who gave a complete description of the moduli spaces of stable vector bundles of rank two with $c_1 = 0$ and $c_2 \geq 2p_g + 2$ on regular elliptic surfaces ([10]). We could describe the moduli spaces of vector bundles of rank two with general Chern classes on Enriques surfaces as two different types (also showing the existences), so that Friedman's result can be included in these two types. They all have the structure of fibrations reflecting the fibration structure of the original Enriques surfaces.

On the other hand, the universal covering space of an Enriques surface is a $K3$ surface. So, we can compare these moduli spaces with moduli spaces of stable bundles on the corresponding $K3$ surface. In fact, we showed that the moduli space of stable bundles on an Enriques surface is mapped by degree two onto a Lagrangian subvariety in the moduli space of stable bundles, which is a symplectic variety, on the covering $K3$ surface [13]. In this paper, we mention the corresponding components

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of the moduli spaces of stable bundles on the $K3$ surface. The complete description will be described later.

The contents of this paper is as follows. In chapter one, we will cover the preliminary facts and known results related to our work. In the following two chapters, we state the main results based on this classification and in the final chapter, we will make some remarks.

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I. Preliminaries

An Enriques surface is a projective nonsingular surface X with $2K_X \sim 0$ (but $K_X \not\sim 0$), where K_X is the canonical divisor of X and $H^1(X, \mathcal{O}_X) = 0$. The unramified double covering space of X defined by the torsion class K_X is an algebraic $K3$ surface with a fixed point free involution. Every Enriques surface X admits an elliptic fibration over \mathbf{P}^1 and every elliptic fibration $f : X \rightarrow \mathbf{P}^1$ has exactly two multiple fibres F_A and F_B such that $2F_A, 2F_B$ are linearly equivalent to a generic fibre F . K_X (briefly K) $\sim F_A - F_B$. Here F_A and F_B are called half fibres. The map

$$c_1 : \text{Pic}X \rightarrow H^2(X, \mathbf{Z}) = \mathbf{Z}^{10} \oplus \mathbf{Z}_2$$

is an isomorphism and

$$\text{Num}X = H^2(X, \mathbf{Z}) / \text{Tor}(H^2(X, \mathbf{Z})) = \mathbf{Z}^{10}.$$

So, we identify $\text{Pic}X$ with $H^2(X, \mathbf{Z})$ in this paper. On an Enriques surface or a $K3$ surface $L^2 = L \cdot L$ is an even number for any divisor L .

On an Enriques surface X ,

$$\chi(\mathcal{O}_X(L)) = h^0(L) - h^1(L) + h^2(L) = \frac{1}{2}L^2 + 1.$$

DEFINITION. An Enriques surface with a smooth rational curve R ($R^2 = -2$) is called a nodal surface, otherwise it is called an unnodal surface.

REMARK. In the 10 dimensional moduli space of Enriques surfaces, a generic one is unnodal, while the nodal ones form a 9 dimensional subvariety ([6]).

DEFINITION. For any divisor $D > 0$, with $D^2 > 0$, we define

$$\phi(D) = \inf\{D \cdot f \mid f \in \text{Num}X, f^2 = 0, f > 0\}.$$

THEOREM 1. [6] $0 < \phi(D)^2 \leq D^2$.

DEFINITION. Let C be an irreducible curve on an Enriques surface X with $C^2 \geq 2$. Then $|C|$ is called a hyperelliptic linear system if one of the following equivalent conditions is satisfied.

- i) The map ψ_C associated to C is hyperelliptic, that is, $C^2 = 2$ or it is of degree 2 onto a surface of degree $n - 1$ in \mathbf{P}^n .
- ii) $|C|$ has base points.
- iii) $\phi(C) = 1$.
- iv) $C \sim (n - 1)E + F$ or $C \sim nE + R$ for some n , where $|2E|$ and $|2F|$ are elliptic pencils and R is a smooth rational curve such that $E \cdot F = E \cdot R = 1$.

THEOREM 2. [6] Let D be an effective divisor on an Enriques surface X with $D^2 \geq 0$. Then

$$D \sim D' + \sum m_i R_i, \quad m_i \geq 0,$$

where R_i is a smooth rational curve and one of the following cases occurs:

- i) D' is an irreducible curve with $D'^2 > 0$;
- ii) D' is a divisor of canonical type, that is, $D' \sim \sum n_i D_i$ is an effective divisor with irreducible components D_i such that $K \cdot D_i = D' \cdot D_i = 0$ for all i ;
- iii) $D' \sim 2E + R$, where $|2E|$ is a genus 1 pencil and R is a smooth rational curve with $E \cdot R = 1$.

REMARK. [6] There exists w , an element of the Weyl group of X generated by the reflections with the smooth rational curves, such that $D' = w(D)$. On an unnodal Enriques surface, every effective divisor D with $D^2 > 0$ is ample.

THEOREM 3. [3] Let D be a divisor with $D^2 \geq 0$ and $D \approx 0$, K . Then D is effective or $-D$ is effective. If D is effective, then $D + K$ is also effective.

THEOREM 4. [6] For every elliptic pencil $|2E|$ on an Enriques surface X , there exists an elliptic pencil $|2F|$ such that $E \cdot F = 1$.

DEFINITION. A vector bundle E is called (semi-)stable with respect to an ample divisor H , if for any subsheaf F , where $0 < \text{rank}(F) < \text{rank}(E)$,

$$(c_1(F) \cdot H)/\text{rank}(F)(\leq) < (c_1(E) \cdot H)/\text{rank}(E).$$

REMARK. If the moduli space of stable bundles, M , exists, then it is a quasi-projective variety. By the deformation theory, we can compute the dimension of the tangent space $T_E M$ at $E \in M$ and the dimension of M at E . If the rank = 2,

$$\dim T_E M = 4c_2 - c_1^2 - 3 + h^2(\text{End}E),$$

where $h^2(\text{End}E) = 0$ if $E \not\cong E(K)$ and 1 if $E \cong E(K)$ and

$$\dim_E M \geq 4c_2 - c_1^2 - 3.$$

THEOREM 5. [13] *If E is a singular point in the moduli space of bundles of rank 2 on an Enriques surface X , then $E \cong \pi_* L$, where L is a line bundle on its covering K3 surface \bar{X} and π is a natural projection map from \bar{X} to X .*

We determined the possible types of Chern classes of rank 2 stable bundles (mod Picard groups) on the Enriques surfaces.

THEOREM 6. [14] *we can find a divisor $D(V, n)$ depending only on $V \in \text{Pic}X$ and $n \in \mathbf{Z}$ such that for any rank 2 vector bundle E with $c_1(E) = V$ and $c_2(E) = n$ on an Enriques surface X , one of the following holds.*

- (A) $\frac{1}{2}c_1(E(D))^2 = c_2(E(D)) - 1,$
- (B) $\frac{1}{2}c_1(E(D))^2 = c_2(E(D)).$

REMARK. In the case (A), we have $2c_2 \geq \dim M \geq 2c_2 - 1$ and in the case (B), we have $2c_2 - 2 \geq \dim M \geq 2c_2 - 3$. So, we have only to consider $c_2 \geq 0$ in the case (A) and $c_2 \geq 1$ in the case (B). However, in the two main theorems in this paper we assume that $c_2 \geq 2$ in the case of (A) and $c_2 \geq 3$ in the case of (B) on unnodal Enriques surfaces. In these cases c_1 is ample. The cases $c_2 = 0, 1$ in (A) and the cases $c_2 = 1, 2$ in (B) will be treated differently. We know from Riemann-Roch theorem that any stable bundle E with $c_1(E)$ ample satisfying the conditions (A)(or (B)) has a section. Indeed,

$$\chi(E) = h^0(E) - h^1(E) + h^2(E) = 1(2),$$

so that $h^0(E) \geq 1(2)$, since $h^2(E) = 0$.

Let us fix the notations.

X is an Enriques surface and its universal covering space, which is a $K3$ surface, is denoted by \bar{X} and the quotient map from \bar{X} to X is π . Let $M_{X,H}(r, D, c_2)$ be the moduli space of stable vector bundles with respect to an ample divisor H , where r is the rank of the bundle, D is the determinant bundle and c_2 is the second Chern class. Let $M_{\bar{X},\pi^*H}(r, \pi^*D, 2c_2)$ be the corresponding moduli space of stable bundles with respect to π^*H on \bar{X} . We denote by i the involution on \bar{X} compatible to π and by i^* the induced involution on $M_{\bar{X}}$.

II. $\frac{1}{2}c_1^2 + 1 = c_2$

First we state the main theorem of this chapter.

THEOREM I. (1) Let $M_{X,H}(2, H, c_2)$ (briefly M) be the moduli space of stable vector bundles on an unnodal Enriques surface X of rank 2 with determinant H , the second Chern class c_2 which satisfy $\frac{1}{2}H^2 + 1 = c_2 \geq 2$ and are stable with respect to H . Then M is nonempty and there is a rational map f from M onto an open set U in $P = |H + K_X|$, the linear system of $H + K_X$, where the fiber over a curve $C \in U$ is a nonempty Zariski open subset of the symmetric power of C , $\text{Sym}^{c_2}(C)$.

(2) M has at most finitely many isolated singularities and the bundle E corresponding to a singular point is π_*L , where L is a line bundle on the universal covering $K3$ surface \bar{X} with $L^2 = -2$.

Proof. First assume that M is non-empty. Then, $\dim M \geq 2c_2 - 1$. For $E \in M$, there exist an effective divisor D , I_Z , the ideal sheaf of a zero scheme Z and an exact sequence,

$$0 \rightarrow O_X(D) \rightarrow E \rightarrow I_Z(H - D) \rightarrow 0,$$

since $h^0(E) > 0$. Since E is stable with respect to H , we have

$$D \cdot H < \frac{1}{2}H^2.$$

Here $D \cdot (H - D) + \text{deg}Z = c_2$, which implies

$$D \cdot H \leq D^2 + c_2 - \text{deg}Z.$$

By the Hodge Index theorem, we obtain

$$D^2H^2 \leq (D \cdot H)^2 \leq \frac{1}{2}H^2D^2 + \frac{1}{2}H^2(c_2 - \text{deg}Z).$$

(Note that this implies $D^2H^2 \leq \frac{1}{2}D^2H^2 + \frac{1}{2}H^2c_2$, so that $D^2 \leq c_2$.)
 Now we can divide into two cases;

$$\text{i) } \deg Z \geq c_2, \text{ ii) } \deg Z < c_2.$$

i) Case 1; $\deg Z \geq c_2$;

In this case we get

$$D^2H^2 \leq (D \cdot H)^2 \leq \frac{1}{2}D^2H^2.$$

The only possibility is that

$$D^2 = 0, \quad D \cdot H = 0.$$

Here H is ample and D is effective. So,

$$D = 0, \quad \deg Z = c_2.$$

Let $M_0 \subset M$ be the collection of bundles which has an isolated zero section and let

$$M_0^i = \{E \in M_0 \mid h^0(E(K)) = i + 1\}.$$

Then M_0^i is empty for $i < 0$, since $h^0(E(K)) \geq 1$ for any $E \in M$. From the exact sequence,

$$0 \rightarrow O_X(K) \rightarrow E(K) \rightarrow I_Z(H + K) \rightarrow 0,$$

we have

$$h^0(E(K)) = h^0(I_Z(H + K)) = h^1(I_Z(H + K)) = i + 1,$$

since $h^0(K) = h^1(K) = \chi(I_Z(H + K)) = h^2(I_Z(H + K)) = 0$. We compute the dimension of M_0^i for $i > 0$. Let $S^i = \{Z \mid h^0(I_Z(H + K)) = i + 1\} \subset \text{Sym}^{c_2}(X)$. To every bundle $E \in M_0^i$ we can associate a cycle $Z \in S^i$ and E corresponds to a non-trivial element in

$$\text{Ext}(I_Z(H), O_X) \cong \text{Ext}(I_Z(H + K), O_X(K)) \cong H^1(I_Z(H + K))^*.$$

So, we have

$$\dim M_0^i \leq \dim S^i + i.$$

Now we compute the dimension of $S^i (i > 0)$. Let $G = \text{Grass}(2, h^0(H + K))$ and let

$$S = \{(V, Z) \mid V \subset H^0(I_Z(H + K))\} \subset G \times \text{Sym}^{c_2}(X).$$

Then, $\dim S = \dim G$ and we have the second projection map

$$f : S \rightarrow \text{Sym}^{c_2}(X).$$

The image is the union of $S^i (i > 0)$ and $f^{-1}(Z) = \{(V, Z) | V \subset H^0(I_Z(H + K))\}$ for $Z \in S^i$. So, we have

$$\dim S^i \leq \dim G - \dim \text{Grass}(2, i + 1) = 2c_2 - 2 - 2i,$$

since $\chi(H + K) = h^0(H + K) = c_2$ for an ample divisor H . This implies that

$$\dim M_0^i \leq 2c_2 - 2 - i < 2c_2 - 1 \leq \dim M,$$

if $i > 0$.

ii) Case 2; $\deg Z < c_2$;

In this case $D > 0$, so that we obtain an exact sequence with a zero scheme Z ,

$$0 \rightarrow O_X \rightarrow E(-D) \rightarrow I_Z(H - 2D) \rightarrow 0.$$

Note that $H \cdot D - D^2 > 0$ and $\deg Z = c_2 - (H \cdot D - D^2)$.

From $\chi(I_Z(H - 2D + K)) = -(H \cdot D - D^2) < 0$, we have

$$\begin{aligned} \dim \text{Ext}(O_X, I_Z(H_2D)) &= h^1(I_Z(H - 2D + K)) \\ &= h^0(I_Z(H - 2D + K)) + (H \cdot D - D^2) \\ &= h^0(E(-D + K)) + H \cdot D - D^2 \end{aligned}$$

since $h^2(I_Z(H - 2D + K)) = h^2(O_X(H - 2D + K)) = h^0(O_X(2D - H)) = 0$ due to $2D \cdot H < H^2$. Let $M_D \subset M$ be the collection of the bundles E , where $E(-D)$ has an isolated zero section and let

$$M_D^i = \{E \in M_D | h^0(E(-D + K)) = i\}.$$

Then

$$\begin{aligned} \dim M_D^0 &\leq \dim \text{Sym}^{c_2 - (H \cdot D - D^2)}(X) + H \cdot D - D^2 - 1 \\ &\leq 2(c_2 - (H \cdot D - D^2)) + (H \cdot D - D^2 - 1) \\ &< 2c_2 - 1 \leq \dim M. \end{aligned}$$

Now we want to have a bound of $h^0(H - 2D + K)$. From the exact sequence,

$$0 \rightarrow O_X(-2D + K) \rightarrow O_X(H - 2D + K) \rightarrow O_C(H - 2D + K) \rightarrow 0,$$

where C is a smooth curve in $|H|$, we have

$$h^0(O_X(H - 2D + K)) \leq h^0(O_C(H - 2D + K)) \leq c_2 - H \cdot D,$$

by the Clifford's theorem.

(Note that $2 \text{genus}(C) - 2 = H^2 > C \cdot (H - 2D + K) > 0$.) So,

$$\begin{aligned} \dim M_D^1 &\leq h^0(H - 2D + K) - 1 + c_2 - (H \cdot D - D^2) + (H \cdot D - D^2) \\ &\leq 2c_2 - 1 - H \cdot D < 2c_2 - 1 \leq \dim M. \end{aligned}$$

$$\begin{aligned} \dim M_D^i (i \geq 2) &\leq 2(h^0(H - 2D + K) - 2) - 2(i - 2) \\ &\quad + (H \cdot D - D^2) + i - 1 \\ &\leq 2c_2 - 1 - i - D^2 < 2c_2 - 1 \leq \dim M \end{aligned}$$

with the same argument as before. Note that $D^2 \geq 0$ for any effective divisor D since any effective divisor is generated by the curves of the arithmetic genus 1 on an unnodal Enriques surface ([6]).

So, M_0 is open and dense in M and M_0^0 is open and dense in M_0 , so that we conclude that M_0^0 is open and dense in M .

Conversely, each zero cycle Z of degree c_2 consisting of distinct points on a smooth curve $C \in |H + K|$ with $h^0(I_Z(H)) = 0$ and $h^0(I_Z(H + K)) = 1$ determines a unique vector bundle $E(Z, C)$,

$$0 \rightarrow O_X \rightarrow E(Z, C) \rightarrow I_Z(H) \rightarrow 0,$$

since $\dim \text{Ext}(I_Z(H), O_X) = 1$ (See [21] and [22]). We claim that E is stable. Suppose that E is not stable. Then there exists an exact sequence,

$$0 \rightarrow O_X(L) \rightarrow E \rightarrow I_W(H - L) \rightarrow 0,$$

such that $L \cdot H \geq \frac{1}{2}H^2 = c_2 - 1$ and W is a 0-dimensional scheme. So, we have

$$L \cdot (H - L) = c_2 - \deg W \leq c_2,$$

which implies

$$L^2 \geq 0,$$

so that L is effective. (Note that L^2 is even.)

From the condition that $h^0(I_Z(H)) = 0$, we have $h^0(E) = 1$ and we already know that E has an isolated 0-section. So, the injectivity of $O_X(L) \rightarrow E$ forces L to be 0. Otherwise, E has no isolated 0-section. This gives a contradiction.

Let N be $\{(Z, C) | Z \text{ is a 0-cycle of degree } c_2 \text{ of distinct points on a smooth curve } C \in |H + K|, h^0(I_Z(H)) = 0 \text{ and } h^0(I_Z(H + K)) = 1\} \subset \text{Sym}^{c_2}(X) \times |H + K|$.

Now we can construct a vector bundle Ξ on $N \times X$ such that $\Xi|_{(Z, C)} = E(Z, C)$, exactly following the method of Okonek and Van de Ven [18]. We define $\Gamma = \{((Z, C), x) | (Z, C) \in N, x \in X \text{ and } x \in \text{Supp} Z\} \subset N \times X$ and denote by I_Γ its ideal sheaf. We define on N ,

$$L = \underline{\text{Ext}}_{\pi_1}^1(I_\Gamma \otimes \pi_2^*(O_X(H)), \pi_2^*(O_X)).$$

Since $\dim \text{Ext}^1(I_Z(H), O_X) = 1$ for any $(Z, C) \in N$, we see that L is a line bundle. By [1], [18], there is a spectral sequence,

$$\begin{aligned} & H^p(\underline{\text{Ext}}_{\pi_1}^q(I_\Gamma \otimes \pi_2^*(O_X(H)), \pi_2^*(O_X) \otimes \pi_1^*(L^*))) \\ & \rightarrow \text{Ext}^{p+q}(I_\Gamma \otimes \pi_2^*(O_X(H)), \pi_2^*(O_X) \otimes \pi_1^*(L^*)). \end{aligned}$$

From this sequence we derive

$$\begin{aligned} & \text{Ext}^1(I_\Gamma \otimes \pi_2^*(O_X(H)), \pi_2^*(O_X) \otimes \pi_1^*(L^*)) \\ & = H^0(\underline{\text{Ext}}_{\pi_1}^1(I_\Gamma \otimes \pi_2^*(O_X(H)), \pi_2^*(O_X) \otimes \pi_1^*(L^*))) \\ & = H^0(L \otimes L^*) = H^0(O_N). \end{aligned}$$

Here we used the fact $\text{Hom}(I_Z(H), O_X) = 0$. So, we can associate to $1 \in H^0(O_N)$ (N is connected), an extension

$$0 \rightarrow \pi_2^*(O_X) \rightarrow \Xi \rightarrow I_\Gamma \otimes \pi_2^*(O_X(H) \otimes \pi_1^*(L)) \rightarrow 0$$

on $N \times X$, with Ξ locally free, and $\Xi|_{(Z,C)}$ is isomorphic to the bundle $E(Z, C)$. Then by the universal property of a coarse moduli space M , the bundle Ξ induces a morphism $g : N \rightarrow M$, such that it is injective and the image is in M_0^0 . By the Zariski's main theorem, N is biregular to its image. $M_0^0 - g(N) = M_1 \cup M_2$, where $M_1 = \{E \in M_0 | h^0(E) \geq 2\}$ and $M_2 = \{E \in M_0 | \text{the unique curve } |I_Z(H + K)| \text{ is singular}\}$, where E is the non-trivial extension of $I_Z(H)$ by O_X . Then, $\dim M_1 \leq \dim |H| + \dim |H + K| < 2c_2 - 1$ since $h^0(I_Z(H)) = h^0(E) - 1 \geq 1$ and $\dim M_2 \leq \dim |H + K| - 1 + c_2 < 2c_2 - 1$. So, $g(N)$ is open and dense in M_0^0 . Since N is of dimension $2c_2 - 1$, connected and irreducible, M is also of dimension $2c_2 - 1$, connected and irreducible. (Note that M_0^0 is open and dense in M .)

Let E be a singular point in M . Then, $E \cong E(K)$ and $E \cong \pi_* L$ for some line bundle L on the universal covering space \bar{X} ([20]). $\pi^* E = L \oplus i^* L$. We can get $2c_1(E)^2 = 2L^2 + 2L \cdot i^* L$, $2c_2(E) = L \cdot i^* L$. So, we obtain $L^2 = c_1(E)^2 - 2c_2(E) = -2$, $(L - i^* L)^2 = -4c_2 - 4$ and $L - i^* L \in \pi^* H^\perp$ which is a negative definite lattice. So, there can be only finitely many decompositions of $\pi^* c_1(E)$ into a direct sum of a divisor L and its involution $i^* L$. So, if $c_2 \geq 2$, then $\dim M$ must be $2c_2 - 1 > 0$, since for a generic $E \in M$, $E \not\cong E(K)$ and $h^2(\text{End} E) = 0$, so that M is generically smooth. (In fact, $\dim N = 2c_2 - 1$.) \square

REMARK 1. We can see that there is no singular point in $g(N)$. In fact, any E in the image of N in M_0^0 is uniquely determined by an effective zero cycle Z of degree $c_2(E)$ ($= \text{genus}(C)$) on a smooth curve C . Then, $O_C(Z + K)$ has an effective divisor Z' which determines E' , since the degree of $Z + K$ on C is $c_2 = \text{genus}(C)$. Here $\pi^*(O_C(Z))$

is linearly equivalent to $\pi^*(O_C(Z + K))$ on $\pi^{-1}(C)$, so that $\pi^*E \cong \pi^*E'$ ([21]). Then, $E \cong E'$ or $E \cong E'(K)$, since $h^0(\pi^*(E^* \otimes E')) = h^0(E^* \otimes E') + h^0(E^* \otimes E'(K))$. However E is not isomorphic to E' , since $O_C(Z) \not\cong O_C(Z + K)$ if $\text{genus}(C) \geq 2$. So, we can conclude that $E(K) \cong E' \in g(N)$ and E is not a singular point.

REMARK 2. Let $M_{\overline{X}}(2, \pi^*H, \pi^*c_2)$ be the moduli space of stable bundles of rank 2 with determinant π^*H and the second Chern class π^*c_2 which are stable with respect to π^*H . Then $M_{\overline{X}}$ has a component \overline{M} which has a fibration structure over $|\pi^*H|$, where the fibre over a curve D in $|\pi^*H|$ is the $\text{Jac}(D)$. Then M_0^0 is mapped into \overline{M} by π^* . We can illustrate it by the following diagram

$$\begin{array}{ccc} M_0^0 & \xrightarrow{\pi^*} & \overline{M} \\ f \downarrow & & h \downarrow \\ |H + K| & \xrightarrow{\pi^{-1}} & |\pi^*H|, \end{array}$$

where $f^{-1}(C) = \text{Sym}^{c_2}(C)$ for $C \in |H + K|$ and $h^{-1}(D) = \text{Jac}(D)$ for $D \in |\pi^*H|$. Here $\pi^*(\text{Sym}^{c_2}(C)) = \text{Jac}(\pi^{-1}(C))^{i^*}$, the fixed locus in $\text{Jac}(\pi^{-1}(C))$ by i^* . (Note that $\text{Sym}^{c_2}(C)$ is birational to $\text{Jac}(C)$.) The details will be described in a later paper.

REMARK 3. In theorem I, we assumed that $c_1^2 \geq 2$. However, $c_1^2 \geq -2$ is necessary for the existence of stable bundles. If $c_1^2 = -2$ and $c_2 = 0$, then there exists a stable bundle if and only if $c_1 = N + 2S + K$, where N is a nodal class and S is a divisor ([13]). If $c_1^2 = 0$, then there exists no stable bundle with $c_1 = 0$ and $c_2 = 1$, but there exists a stable bundle with $c_1 = f$ and $c_2 = 1$, where $|2f|$ gives an elliptic system. We will show this as an example.

EXAMPLE. Let M be the moduli space with $c_1 = F_A$ and $c_2 = 1$. Then, we can show easily that M is birational to F_B and vice versa following the methods in ([18]).

III. $\frac{1}{2}c_1^2 = c_2$

Here we state the main theorem of this chapter.

THEOREM II. (1) Let $M_{X,H}(2, H, c_2)$ (briefly M) be the moduli space of stable vector bundles on an unnodal Enriques surface X of rank 2 with determinant H with the second Chern class c_2 which satisfy $\frac{1}{2}H^2 = c_2 \geq 3$ and are stable with respect to H . Then, M is non-empty.

(a) If $|H|$ is not hyperelliptic, then there is a rational map f from M' , a component of M , onto an open set $U \in \mathbf{P} = |H|$, the linear system of H , where the fiber over a curve $C \in U$ is a non-empty open subset of $W_{c_2}^1(C)$ (special effective divisors of degree c_2 on C).

(b) If $|H|$ is hyperelliptic, then M has a structure described in theorem I in chapter 2 ($\frac{1}{2}H^2 + 1 = c_2$).

(2) M has at most finitely many isolated singularities and the bundle E corresponding to a singular point is π_*L , where L is a line bundle on the universal covering K3 surface \bar{X} with $L^2 = 0$.

Proof. First assume that M is non-empty. Then, $\dim M \geq 2c_2 - 3$. For $E \in M$ we have $h^0(E) \geq 2$. As in the previous case, we have an exact sequence,

$$0 \rightarrow O_X(D) \rightarrow E \rightarrow I_Z(H - D) \rightarrow 0,$$

where D is an effective divisor and I_Z is an ideal sheaf of a zero scheme Z . In the same way as in the previous proof, we can divide into two cases,

$$\text{i) } \deg Z \geq c_2, \text{ ii) } \deg Z < c_2.$$

i) Case 1: $\deg Z \geq c_2$

In this case $D = 0$ and $\deg Z = c_2$ as we discussed in the proof of the theorem I. Let $M_0 \subset M$ be the collection of vector bundles which have an isolated zero section and let $M_0^i = \{E \in M_0 | h^0(E(K)) = i + 2\}$. Then M_0^i is non-empty for $i \geq 0$ (empty for $i < 0$) and for each $E \in M_0^i$ we have an exact sequence

$$0 \rightarrow O_X \rightarrow E \rightarrow I_Z(H) \rightarrow 0$$

and from this and $\chi(I_Z(H + K)) = 1$, we get

$$h^0(E(K)) = h^0(I_Z(H + K)) = h^1(I_Z(H + K)) + 1 = i + 2.$$

As in the previous argument,

$$\begin{aligned} \dim M_0^i &\leq \dim \text{Grass}(2, h^0(H + K)) - \dim \text{Grass}(2, h^0(I_Z(H + K))) \\ &\quad + i - (h^0(E) - 1) \\ &\leq 2(c_2 + 1 - 2) - 2(i + 2 - 2) + i - 1 = 2c_2 - 3 - i. \end{aligned}$$

So, $\dim M_0^i < 2c_2 - 3 \leq \dim M$ for $i > 0$.

ii) Case 2: $\deg Z < c_2$

In this case $D > 0$ and $H \cdot D - D^2 = c_2 - \deg Z > 0$ and we obtain an exact sequence,

$$0 \rightarrow O_X \rightarrow E(-D) \rightarrow I_Z(H - 2D) \rightarrow 0.$$

Let $M_D \subset M$ be the collection of the bundles E , where $E(-D)$ has an isolated zero section. Now we divide into two cases again, a) $\deg Z = c_2 - 1$ and b) $\deg Z < c_2 - 1$.

ii-a) $\deg Z = c_2 - 1$

In this case $H \cdot D - D^2 = (H - D) \cdot D = 1$. We claim that D is a half fiber such that $H \cdot D = 1$. So, these cases happen if and only if $|H|$ is hyperelliptic. To prove this claim, first let us compute $(H - D)^2$.

$$(H - D)^2 = H^2 - 2H \cdot D + D^2 > 0,$$

since $H^2 - 2H \cdot D > 0$ and $D^2 \geq 0$. The first inequality comes from the stability of E and the second one comes from that D is an effective divisor on an unnodal Enriques surface. So, $H - D$ is ample and D is a half fiber, since D is a curve of arithmetic genus 1. Then, $H \cdot D = (H - D) \cdot D + D^2 = 1$, so that $|H|$ is a hyperelliptic system. Note that

$$\frac{1}{2}c_1^2(E(-D)) = \frac{1}{2}(H - 2D)^2 = \deg Z - 1 = c_2(E(-D)) - 1.$$

We know that a generic element in M_D is stable with respect to $H' = H - 2f$, which is ample, by the result of Qin ([25]). So, we can conclude that $M_{X,H}(2, H, c_2)$ is birational to $M_{X,H'}(2, H', c_2 - 1)$ if $|H|$ is hyperelliptic.

ii-b) $\deg Z < c_2 - 1$

Let $M_D^i = \{E \in M_D | h^0(E(-D + K)) = i\}$ for $i \geq 0$. Then, for $E \in M_D^i$ we have

$$\begin{aligned} h^0(I_Z(H - 2D + K)) &= i, \\ h^1(I_Z(H - 2D + K)) &= i - 1 + (H \cdot D - D^2). \end{aligned}$$

Note that $H \cdot D - D^2 \geq 2$. As in the previous case,

$$\begin{aligned} \dim M_D^0 &\leq 2(c_2 - (H \cdot D - D^2)) + (H \cdot D - D^2 - 2) \\ &= 2c_2 - (H \cdot D - D^2) - 2 \\ &\leq 2c_2 - 4 < \dim M. \end{aligned}$$

Since $h^0(H - 2D + K) \leq c_2 + 1 - H \cdot D$ by the Clifford's theorem as before,

$$\begin{aligned} \dim M_D^1 &\leq (h^0(H - 2D + K) - 1) + (c_2 - (H \cdot D - D^2)) \\ &\quad + (H \cdot D - D^2 - 1) \\ &\leq c_2 - (H \cdot D) + c_2 - 1 \\ &= 2c_2 - 1 - (H \cdot D). \end{aligned}$$

If $H \cdot D \geq 3$, then $\dim M_D^1 < \dim M$. Suppose that $H \cdot D = 2$. Then $H - 2D$ is ample if $H^2 \geq 10$, since $(H - 2D)^2 = 10 - 4H \cdot D + 4D^2 > 0$, so that $h^0(H - 2D + K) = \chi(O_X(H - 2D + K)) = c_2 - 2(H \cdot D - D^2) + 1$. So, in this case also, $\dim M_0^1 < \dim M$. If $H^2 = 8$, then $\dim M_D^1 < \dim M$ since $h^0(H - 2D + K) \leq 2$ ([6]). If $H^2 = 4$ or 6 , then M_D^1 is empty, since $h^0(H - 2D + K) = 0$ ([6]). For $i \geq 2$, we have

$$\begin{aligned} \dim M_0^i &\leq 2(h^0(H - 2D + K) - 2) - 2(i - 2) + (i - 2 + H \cdot D - D^2) \\ &\leq 2c_2 - i - (H \cdot D - D^2) < \dim M. \end{aligned}$$

So, M_0^0 is open and dense in M if $|H|$ is not hyperelliptic and if $|H|$ is hyperelliptic, M is birational to the moduli space in the previous case $\frac{1}{2}c_1^2 + 1 = c_2$.

Conversely, if Z is a divisor on a smooth curve $C \in |H|$ of degree $c_2 = \text{genus}(C) - 1$ with

$$\begin{aligned} h^0(C, O_C(Z)) &= h^0(C, O_C(K_C - Z)) = h^0(I_Z(H + K)) = 2, \\ h^0(I_Z(H)) &= 1, \end{aligned}$$

then we can construct a locally free sheaf $E(Z, C)$, uniquely determined by $H^1(I_Z(H + K))$, whose dimension is 1

$$0 \rightarrow O_X \rightarrow E(Z, X) \rightarrow I_Z(H) \rightarrow 0,$$

as in the theorem I. We claim that E is semi-stable and stable for a generic E . To prove that E is semi-stable, we can use just the same method that we used in the previous theorem. Suppose E is not stable. Then there exists a divisor L such that $L \cdot H = \frac{1}{2}H^2$ and there is an exact sequence,

$$0 \rightarrow O_X(L) \rightarrow E \rightarrow I_W(H - L) \rightarrow 0.$$

We have

$$\text{deg}W = c_2 - H \cdot L + L^2 \geq 0,$$

so that,

$$L^2 \geq 0.$$

However, $L^2 < 2$, not to contradict that E has an isolated zero section. (Note that $h^0(E) = 2$.) Then,

$$L^2 = D^2 = 0, \quad L \cdot D = \frac{1}{2}H^2 = c_2, \quad \text{deg}W = 0,$$

where $D = H - L$. Hence if E is not stable, E is an extension,

$$0 \rightarrow O_X(L) \rightarrow E \rightarrow O_X(D) \rightarrow 0,$$

with the above condition. However there are only finitely many such decompositions $(H = L + D)$, since H is ample and $(L - D) \cdot H = 0$, $(L - D)^2 = -H^2$ and the intersection form on H^\perp is negative definite. Note that also the dimension of $\text{Ext}^1(O_X(D), O_X(L)) - 1 = h^1(L - D) - 1 = c_2 - 2$ which is smaller than $\dim |H| + \rho(c_2 + 1, 1, c_2) = 2c_2 - 3$ for $c_2 > 1$, where $\rho(g, r, d) = g - (r + 1)(g - d + r)$ is the Brill-Noether number.

If $|H|$ is hyperelliptic, then $E(Z, C)$ is not locally free if $c_2 > 2$. This claim comes from the fact that $E(Z, C)$ is locally free if and only if $|Z|$ is fixed point free on C .

Let N be $\{(Z, C) \mid Z \text{ is a 0-cycle of degree } c_2 \text{ on a smooth curve } C \in |H| \text{ with } h^0(C, O_C(Z)) = 2 \text{ and } h^0(I_Z(H)) = 1\} \subset \text{Sym}^{c_2} X \times |H|$.

Then we can construct a vector bundle Ξ on $N \times X$ such that $\Xi|_{(Z,C) \times X} = E(Z, C)$ as in the previous case. By the universal property of the coarse moduli space M , the bundle Ξ induces a morphism,

$$g : N \rightarrow M.$$

We will show that $g((Z, C)) = g((Z', C'))$ if and only if $C = C'$ and Z is linearly equivalent to Z' on C . First we show that $O_C(Z) \cong O_C(Z')$ implies $E(Z, C) \cong E(Z', C)$. We follow the method of Tyurin exactly [21]. The curve $C \supset Z$ defines a one dimensional space of sections spanned by $s \in H^0(I_Z(C))$ which lifts uniquely to a one-dimensional subspace spanned by $\bar{s} \in H^0(E(Z, C))$. Consider the linear span $W = \langle H^1(I_Z(H + K), \bar{s}) \subset H^0(E(Z, C))$ and the canonical homomorphism,

$$0 \rightarrow W \otimes O_X \rightarrow E(Z, C) \rightarrow O_C(K_C - Z) \rightarrow 0.$$

Dualizing this, we get an exact sequence,

$$0 \rightarrow E(Z, C)^* \rightarrow W^* \otimes O_X \rightarrow O_C(Z) \rightarrow 0$$

and an isomorphism

$$W^* \cong H^0(O_C(Z)),$$

since $h^0(E(Z, C)^*) = h^1(E(Z, C)^*) = h^1(E(Z, C)(K)) = 0$. To the hyperplane $H^1(I_Z(H + K)) \subset W$ there corresponds the one dimensional space of sections spanned by $s \in H^0(O_C(Z)) \cong W^*$ such that $(s)_0 = Z$. Next we consider a section $s' \in H^0(O_C(Z))$ such that $(s')_0 = Z'$ and the corresponding hyperplane $V' \subset W$. The canonical map yields the exact triple

$$0 \rightarrow V' \otimes O_X \rightarrow E(Z, C) \rightarrow I_{Z'}(H) \rightarrow 0$$

and this shows that $E(Z, C) \cong E(Z', C)$, since $\dim \text{Ext}(I_{Z'}(H), O_X) = 1$.

Next we show the converse. The rank two bundles $E(Z, C) \cong E(Z', C') \cong E$ has two non-proportional sections s and s' with zeros,

$$(s)_0 = Z, (s')_0 = Z'.$$

The restriction of the canonical map $H^0(E) \otimes O_X \rightarrow E$ to the linear span $\langle s, s' \rangle$ yields the following exact triple

$$0 \rightarrow \langle s, s' \rangle \otimes O_X \rightarrow E \rightarrow L \rightarrow 0,$$

$$\text{Supp}L = C = C' \in |H|,$$

since $h^0(I_Z(H)) = h^0(I_{Z'}(H)) = 1$. Then $L \cong O_C(Z) \cong O_C(Z')$.

So, this induces a map from $N' = \{(L, C) | L \in \text{Pic}^{c_2}(C), h^0(C, L) = 2, C \text{ a smooth curve in } |H|, h^0(I_Z(H)) = 1, \text{ where } Z \text{ is a zero section of } L\}$ to M . This is injective and the image is in M_0^0 as an open and dense subset in the closure M' in M as was shown in theorem I. So, this completes the first part of the theorem.

For the second part we use the same method as in the previous theorem to conclude that E corresponding to a singular point is π_*L with $L^2 = 0$. So, $\dim M = \dim N$ must be $2c_2 - 3 > 0$. This implies that for a generic curve $C \in |H|$, $\dim W_{c_2}^1(C) = \rho(c_2 + 1, 1, c_2) = c_2 - 3$. \square

REMARK 1. We do not know whether or not M is irreducible, even though we believe it.

REMARK 2. The corresponding component in the moduli space of stable bundles on \bar{X} is $M_{\bar{X}}^0(2, \pi^*H, \pi^*c_2)$. (See chapter I.) Then M_0^0 is mapped to $M_{\bar{X}}^0$ by π^* . The image of the bundle $E(Z, C) \in g(N)$ by π^* is in the branch locus if and only if $O_C(Z)$ is a theta characteristic on $C(O_C(Z) \cong O_C(K_C - Z))$. We expect that the intersection of the image of the moduli space by pull back map with the fixed locus of involution in [21] is of $\dim \frac{1}{2}(\dim M_{\bar{X}} + 1)$.

REMARK 3. In theorem II, we assumed that $H^2 \geq 6$, so that $c_2 \geq 3$. However, M is non-empty only if $H^2 \geq 2(c_2 \geq 1)$. In the case $H^2 = 2, c_2 = 1$, this corresponds to the case of the exceptional bundle. For $H^2 = 4, c_2 = 2$ and $H^2 = 6, c_2 = 3$, we will explain as examples.

EXAMPLE 1. ($H^2 = 4, c_2 = 2$) If $|H|$ is non-hyperelliptic, then M is just the locus of hyperelliptic curves in $|H| = \mathbf{P}^2$, since to each hyperelliptic curve C we can associate a bundle uniquely determined by the hyperelliptic system on C . The image of M by π^* is one of the two elliptic curves in $M_{\bar{X}}$ which determine the double covering of $M_{\bar{X}}$ over

\mathbf{P}^2 . The moduli space of bundles with determinant $H + K$ are mapped to the other elliptic curve by π^* .

If $|H|$ is hyperelliptic, then M is birational to some moduli space of bundles with $c_1^2 = 0$ and $c_2 = 1$. (See the remark 3 in chapter II.)

EXAMPLE 2. ($H^2 = 6$, $c_2 = 3$) Every curve of genus 4 is trigonal. It is known ([19]) that a non-hyperelliptic curve of genus 4 has exactly two g_3^1 's, D and $K - D$ or one g_3^1 , $D(\sim K - D)$. So, if $|H|$ is not hyperelliptic, then M is birational to a double covering of $\mathbf{P}^3 = |H|$ branched over the locus of curves which has a theta characteristic of g_3^1 . If $|H|$ is hyperelliptic, then it can be transformed to another moduli space of bundles with $c_1^2 = 2$ and $c_2 = 2$.

IV. Remarks

(1) Qin's result ([25]) shows that the birational type of the moduli space of stable bundles of rank 2 on an Enriques surface is independent of the choice of an ample divisor. In the case $c_1 = 0$, the moduli space can be described in two different ways. The first one is to transform to the cases (A) or (B) we described in the previous two chapters. The second one is to follow the description of Friedman's result. (He assumed that every fiber is irreducible, so that this corresponds to an unnodal Enriques surface.)

(2) In both theorems, we assumed that X is unnodal. However, for a non-ample divisor H on a nodal Enriques surface, our method does not work automatically. There might be a transformation to another moduli space of bundles whose determinant is ample with possibly different rank using exceptional bundles of even rank ([23]).

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