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EXISTENCE OF QUASI-STATIONARY STOKES FLOW IN A DIHEDRAL DOMAIN

BUM JA JIN

ABSTRACT. We study quasi-stationary Stokes flow in a dihedral domain arising from a study of a free boundary problem of viscous fluid in a container. We construct an exact solution of quasi-stationary Stokes equations and derive its estimates with norm in a weighted Sobolev spaces.

1. Introduction

Let us consider a two-dimensional dihedral domain $d_{\theta_0} = \{(r\cos\theta, r\sin\theta) : r \in (0,\infty), \theta \in (0,\theta_0)\}$ surrounded by two straight lines $\gamma_0 = \{(x_1, 0) : x_1 \in (0,\infty)\}$ and $\gamma_{\theta_0} = \{(r\cos\theta_0,\ r\sin\theta_0) : r \in (0,\infty)\}$.

We consider a quasitationary Stokes equations

$$\left.\begin{array}{rcl} -\Delta\mathbf{v} + \nabla p & = & \mathbf{f} \\ \mathrm{div}\mathbf{v} & = & f_4 \end{array}\right\} \text{ in } d_{\theta_0} \times (0,T) \tag{1.1}$$

$$\mathbf{v} = 0 \quad \text{on } \gamma_{\theta_0} \times (0,T) \tag{1.2}$$

$$\left.\begin{array}{rcl} \partial_1 v_2 + \partial_2 v_1 & = & b_0 \\ 2\partial_2 v_2 - p - \alpha\partial_1^2 \int_0^t v_2 ds & = & b_1 \end{array}\right\} \text{ on } \gamma_0 \times (0,T). \tag{1.3}$$

Suppose $\mathbf{u}$ has the following initial condition;

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \text{ in } d_{\theta_0}.$$

The above equation is a simplified representation of the dynamics of incompressible viscous fluid in Lagrangian coordinate system, flowing in a container, partially contacting with open air, with contact angle θ_0. We refer the paper of Sollonikov [17] for the reasoning that we

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have adopted the terminology "quasi-stationary". In [17], he considered quasistationary approximation of the Navier-Stokes equations

$$\epsilon(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) - \Delta\mathbf{u} + \nabla p = 0,\ \operatorname{div}\mathbf{u} = 0,$$

where $\epsilon > 0$. He refers there to a paper by M. Günter and G. Prokert [2], where the model consists in the elimination of the terms $\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}$ by setting $\epsilon = 0$ and the surface tension plays an important role in evolution.

In this paper, We construct a unique solution $\mathbf{u}$ of (1.1)-(1.3) and derive the estimates of solutions in a weighted Sobolev space $L^2(0,T; W^{k+2}_\mu(d_{\theta_0}))$.

It is motivated by the paper [18] of V. A. Solonnikov and E. Frolova, where the following heat equation with mixed boundary condition is mainly considered:

$$-\Delta u = f, \quad u(\mathbf{x},0) = 0, \text{ in } d_{\theta_0}\times(0,T)$$
$$u = \phi_1, \text{ on } \gamma_{\theta_0}\times(0,T), \quad k\partial_t u + \frac{\partial v}{\partial x_2} + z\frac{\partial v}{\partial r} = \phi_0, \text{ on } \gamma_0\times(0,T).$$

During the process of the construction of the solution of the problem (1.1)-(1.3), it is necessary to solve the following difference equation

$$M(\sigma+1) = \omega(\sigma)M(\sigma),$$

where $\omega(\sigma) = \frac{\alpha(\sigma+1)}{4}\frac{\sigma\sin 2\theta_0 - \sin 2\sigma\theta_0}{\cos^2\sigma\theta_0 + \sigma^2\sin^2\theta_0}$.

For the parabolic problem in [18], the form of $\omega(\sigma)$ is simple, since all the zeros and poles of $\omega(\sigma)$ is real and appears periodically. In our case, the form of $\omega(\sigma)$ is rather complicated since $\omega(\sigma)$ has complex zeros and poles as well as real zeros and poles and they are not periodic. At first sight, this gives an obstacle to follow the same reasoning as in [18] for the construction and estimates of M. Nevertheless, we can adopt a similar reasoning as in [18] with little modification for the construction of the well-defined solution and derive similar estimates as in [18].

Free boundary problem in a container has strong singularity at the contact point, where the wall of the container and the free upper surface meet, because of the non-smoothness of the fluid region at the contact point and the incompatibility of the boundary conditions on the rigid wall and the free upper surface.

For the steady case, the free boundary problem has been well studied by solving Stokes equation in a dihedral domain with mixed boundary condition. Refer [6] of D. H. Sattinger's, [7], [8] of J. Socolowsky's, [9], [10], [12], [11] of V. A. Solonnikov's. For the problem of moving contact point, we refer [5] of V. V. Puknachov and V. A. Solonnikov's, [13]-[16]

of V. A. Solonnikov's. All of the above are considering stationary flow, hence deeply related with elliptic problem in a Lipschitz domain with corner point as in [3].

As far as I know, there has not been considered non-steady free boundary problem of viscous incompressible fluid with contact angle yet, even for the non-steady Stokes equations. Our study on the equations (1.1)-(1.3) will be the first step to the study of the original non-steady free boundary problem of contact point and we hope that the solvability and the estimates of unique solution of (1.1)-(1.3) in this paper gives key idea for the solvability of the non-steady free boundary problem of contact point.

2. Notations and statement of main theorem

We define $H_\mu^k(d_{\theta_0})$ as Hilbert spaces of functions $u(x)$, $x \in d_{\theta_0}$ with finite norm

$$\|u\|_{H_\mu^k(d_{\theta_0})} = \Big(\sum_{|j|\le k} |x|^{2(\mu-k+|j|)} |D^j u(x)|^2 dx \Big)^{1/2},$$

where $j=(j_1,j_2)$, $|j|=j_1+j_2$, $D^j = \frac{\partial^{j_1}}{\partial x_1^{j_1}} \frac{\partial^{j_2}}{\partial x_2^{j_2}}$, and we define $H_\mu^{k+1/2}(\gamma_0)$ as the spaces of traces of functions in $H_\mu^{k+1}(d_{\theta_0})$, with finite norm

$$\|v\|_{H_\mu^{k+1/2}(\gamma_0)} = \inf_{u \in H_\mu^{k+1}(d_{\theta_0}),\ u|_{\gamma_0}=v} \|u\|_{H_\mu^{k+1}(d_{\theta_0})}.$$

It is well known that the norm of $v \in H_\mu^{k+1/2}(\gamma_0)$ is equivalently defined by

$$\|v\|_{H_\mu^{k+1/2}(\gamma_0)} = \Big(\sum_{i=0}^{k} \int_0^\infty |D_r^i v|^2 r^{2\mu-2k+2i-1} dr + \|v\|_{L_\mu^{k+1/2}(\gamma_0)} \Big)^{1/2},$$

where $r=|x|$, $x\in\gamma_0$ and

$$\|v\|_{L_\mu^{k+1/2}(\gamma_0)} = \Big(\int_0^\infty r^{2\mu} dr \int_0^r |D_r^k v(r+\rho) - D_r^k v(r)|^2 \frac{d\rho}{\rho^2} \Big)^{1/2}.$$

For the more explanation of Hilbert space and its traces with weight, we refer [3] of V. A. Kondratiev. Here the derivatives are understood in the generalized Sobolev sense.

We consider a quasi-stationary Stokes equations in a dihedral domain d_{θ_0}:

$$\left.\begin{array}{rcl} -\Delta \mathbf{v} + \nabla p &=& \mathbf{f} \\ \operatorname{div} \mathbf{v} &=& f_4 \end{array}\right\} \text{ in } d_{\theta_0} \tag{2.1}$$

$$\mathbf{v} = 0 \qquad \text{on } \gamma_{\theta_0} \tag{2.2}$$

$$\left.\begin{array}{rcl} \partial_1 v_2 + \partial_2 v_1 &=& b_0 \\ 2\partial_2 v_2 - p - \alpha \partial_1^2 \int_0^t v_2 ds &=& b_1 \end{array}\right\} \text{ on } \gamma_0. \tag{2.3}$$

We state our main theorem.

THEOREM 2.0.1. *Let k be a nonnegative integer. We consider a real number μ in an interval $(k+1, k+1+\epsilon_0)$, where $\epsilon_0 = \min\{\mathrm{Re}(\sigma) > 0;\ \sin 2\sigma\theta_0 = \sigma \sin 2\theta_0 \text{ or } \cos^2 \sigma\theta_0 = \sigma^2 \sin^2 \theta_0\}$.*

Suppose that $\mathbf{f}, f_4 \in L^2(0,T; H^k_\mu(d_{\theta_0}))$ and $b_0, b_1 \in L^2(0,T; H^{k+1/2}_\mu (d_{\theta_0}))$ are given. Then, there is unique solution $\mathbf{v}$, p satisfying (2.1)-(2.3). Moreover, the following inequality holds:

$$\int_0^T \|\mathbf{u}\|^2_{H^{k+2}_\mu(d_{\theta_0})} + \|p\|^2_{H^{k+1}_\mu(d_{\theta_0})} + \|\int_0^t u_2 ds\|^2_{H^{k+5/2}_\mu(\gamma_0)} dt$$
$$< c\int_0^T \|\mathbf{f}\|^2_{H^k_\mu(d_{\theta_0})} + \|f_4\|^2_{H^{k+1}_\mu(d_{\theta_0})} + \|\mathbf{b}_0\|^2_{H^{k+1/2}_\mu(\gamma_0)} + \|\mathbf{b}_1\|^2_{H^{k+1/2}_\mu(\gamma_0)} dt.$$

For simplicity, we assume $\mathbf{f} = 0$, $f_4 = 0$ and $b_0 = 0$. We rewrite the equations (2.1)-(2.3) by adopting a polar coordinate system:

(2.4)
$$\left.\begin{array}{rcl} -\frac{1}{r}\partial_r(r\partial_r v_r) - \frac{1}{r^2}\partial_\theta^2 v_r + \frac{1}{r^2} v_r + \frac{2}{r^2}\partial_\theta v_\theta + \partial_r p &=& 0 \\ -\frac{1}{r}\partial_r(r\partial_r v_\theta) - \frac{1}{r^2}\partial_\theta^2 v_\theta + \frac{1}{r^2} v_\theta - \frac{2}{r^2}\partial_\theta v_r + \frac{1}{r}\partial_\theta p &=& 0 \\ \partial_r v_r + \frac{1}{r} v_r + \frac{1}{r}\partial_\theta v_\theta &=& 0 \end{array}\right\} \text{ in } d_{\theta_0}$$

$$v_r = v_\theta = 0 \quad \text{on } \gamma_{\theta_0} \tag{2.5}$$

$$\left.\begin{array}{rcl} \partial_r v_\theta + \frac{1}{r}\partial_\theta v_r - \frac{1}{r} v_\theta &=& 0 \\ 2(\frac{1}{r} v_r + \frac{1}{r}\partial_\theta v_\theta) - p - \alpha\partial_r^2 \int_0^t v_\theta ds &=& b_1 \end{array}\right\} \text{ on } \gamma_0. \tag{2.6}$$

REMARK 2.0.2. Suppose that $\mathbf{f}(\cdot,t) \in H^k_\mu(d_{\theta_0})$, $f_4(\cdot,t) \in H^{k+1}_\mu(d_{\theta_0})$ and $b_0(\cdot,t) \in H^{k+1/2}_\mu(\gamma_0)$ for each fixed $t > 0$ and that at least one of $\mathbf{f}$, f_4 and b_0 are not zero. Then we can construct $\mathbf{V}$ and P satisfying the Stokes equation

$$-\Delta \mathbf{V} + \nabla P = \mathbf{f}, \operatorname{div} \mathbf{V} = f_4 \text{ in } d_{\theta_0}$$
$$V = 0 \text{ on } \gamma_{\theta_0}$$
$$\partial_1 V_2 + \partial_2 V_1 = b_0,\ V_2 = 0 \text{ on } \gamma_0.$$

We refer [12] for the existence $\mathbf{V}(\cdot,t) \in H_\mu^{k+2}(d_{\theta_0})$ and $P(\cdot,t) \in H_\mu^{k+1}(d_{\theta_0})$, $\mu \neq \mathrm{Re}(\sigma)+k+1$ for any σ satisfying $\sin 2\theta_0\sigma = \sigma \sin 2\theta_0$. Moreover, $\mathbf{V}$ and P satisfy

$$\begin{aligned}
&\|\mathbf{V}(\cdot,t)\|^2_{H_\mu^{k+2}(d_{\theta_0})} + \|P(\cdot,t)\|^2_{H_\mu^{k+1}(d_{\theta_0})} \\
< \; & c\|\mathbf{f}(\cdot,t)\|^2_{H_\mu^{k}(d_{\theta_0})} + c\|f_4(\cdot,t)\|^2_{H_\mu^{k+1}(d_{\theta_0})} + c\|\mathbf{b}_0(\cdot,t)\|^2_{H_\mu^{k+1/2}(\gamma_0)}.
\end{aligned}$$

By integrating over $(0,T)$, we obtain

$$\begin{aligned}
&\int_0^T \|\mathbf{V}\|^2_{H_\mu^{k+2}(d_{\theta_0})} + \|P\|^2_{H_\mu^{k+1}(d_{\theta_0})} dt \\
< \; & c\int_0^T \|\mathbf{f}\|^2_{H_\mu^{k}(d_{\theta_0})} + \|f_4\|^2_{H_\mu^{k+1}(d_{\theta_0})} + \|\mathbf{b}_0\|^2_{H_\mu^{k+1/2}(\gamma_0)} dt.
\end{aligned}$$

Now we construct $\mathbf{v}$ and p satisfying (2.4)-(2.6) with $b_1\cdot$ and b_2 replaced by $\hat{b}_1 = b_1 - 2\partial_2 V_2 + P + \alpha\partial_1^2\int_0^t V_2 ds$. $V_2 = 0$ on γ_0 implies $\alpha\partial_1^2\int_0^t V_2 ds = 0$ on γ_0. Hence, if $b_1 \in L^2(0,T;H_\mu^{k+1/2}(\gamma_0))$, then $\hat{b}_1 \in L^2(0,T;H_\mu^{k+1/2}(\gamma_0))$ with

$$\begin{aligned}
&\|\hat{b}_1\|_{H_\mu^{k+1/2}(\gamma_0)} \\
&\leq c\|b_1\|_{H_\mu^{k+1/2}(\gamma_0)} + c\|\mathbf{V}\|_{H_\mu^{k+3/2}(\gamma_0)} + c\|P\|_{H_\mu^{k+1/2}(\gamma_0)}.
\end{aligned}$$

By taking Laplace transform to (2.4)-(2.6) in terms of t variable, we have the following stationary Stokes system:

$$\left.\begin{aligned}
-\tfrac{1}{r}\partial_r(r\partial_r v_r) - \tfrac{1}{r^2}\partial_\theta^2 v_r + \tfrac{1}{r^2}v_r + \tfrac{2}{r^2}\partial_\theta v_\theta + \partial_r p &= 0 \\
-\tfrac{1}{r}\partial_r(r\partial_r v_\theta) - \tfrac{1}{r^2}\partial_\theta^2 v_\theta + \tfrac{1}{r^2}v_\theta - \tfrac{2}{r^2}\partial_\theta v_r + \tfrac{1}{r}\partial_\theta p &= 0 \\
\partial_r v_r + \tfrac{1}{r}v_r + \tfrac{1}{r}\partial_\theta v_\theta &= 0
\end{aligned}\right\} \text{ in } d_{\theta_0} \tag{2.7}$$

$$v_r = v_\theta = 0 \text{ on } \gamma_{\theta_0} \tag{2.8}$$

$$\left.\begin{aligned}
\partial_r v_\theta + \tfrac{1}{r}\partial_\theta v_r - \tfrac{1}{r}v_\theta &= 0 \\
2(\tfrac{1}{r}v_r + \tfrac{1}{r}\partial_\theta v_\theta) - p - \tfrac{\alpha}{\tau}\partial_r^2 v_\theta &= b_1
\end{aligned}\right\} \text{ on } \gamma_0. \tag{2.9}$$

Here, we consider τ as complex number with $\mathrm{Re}(\tau) > 0$.

The main result as in Theorem 2.0.1 can be obtained from the result of the following theorem:

THEOREM 2.0.3. *Let k be a nonnegative integer. We consider a real number μ in an interval $(k+1, k+1+\epsilon_0)$, where* $\epsilon_0 = \min\{\mathrm{Re}(\sigma) > 0;\ \sin 2\sigma\theta_0 = \sigma\sin 2\theta_0$ *or* $\cos^2\sigma\theta_0 = \sigma^2\sin^2\theta_0\}$.

Suppose $b_1 \in H_\mu^{k+1/2}(\gamma_0)$. *Then there is a unique solution* $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta$ *and* p *of* (2.7)-(2.9). *Moreover, the following inequality holds:*

$$\|\mathbf{v}\|^2_{H_\mu^{k+2}(d_{\theta_0})} + \frac{1}{|\tau|^2}\|v_\theta\|^2_{H_\mu^{k+5/2}(\gamma_0)} + \|p\|^2_{H_\mu^{k+1}(d_{\theta_0})} < c\|b_1\|^2_{H_\mu^{k+1/2}(\gamma_0)}.$$

3. Construction of exact solution

In this section, we devote ourselves to construct solutions of the equations (2.7)-(2.9).

Let us define Mellin transform $M(f)$ of a locally integrable function f defined by

$$M(f)(\sigma) = \int_0^\infty r^{\sigma-1}f(r)dr.$$

We consider Mellin transforms of v_r, v_θ, p, b_1, b_2 defining $u(\sigma,\theta) = M(v_r)(\sigma,\theta), w(\sigma,\theta) = M(v_\theta)(\sigma,\theta)$, $q(\sigma,\theta) = M(p)(\sigma+1,\theta)$, $d_1(\sigma) = M(b_1)(\sigma+1)$. Take Mellin transform to (2.7)-(2.9), then we have the following second order differential system in θ:

$$(3.1)\qquad \left.\begin{aligned} -\partial_\theta^2 u + (1-\sigma^2)u + 2\partial_\theta w - (\sigma+1)q &= 0 \\ -\partial_\theta^2 w + (1-\sigma^2)w - 2\partial_\theta u + \partial_\theta q &= 0 \\ \partial_\theta w - (\sigma-1)u &= 0 \end{aligned}\right\} \text{ in } \theta \in (0,\theta_0)$$

$$(3.2)\qquad u = w = 0 \qquad \text{at } \theta = \theta_0$$

$$(3.3)\qquad \left.\begin{aligned} \partial_\theta u - (\sigma+1)w &= 0 \\ \Big[2(u+\partial_\theta w) - q\Big](\sigma,0) - \tfrac{\alpha}{\tau}(\sigma^2-\sigma)w(\sigma-1,0) &= d_1(\sigma) \end{aligned}\right\} \text{ at } \theta = 0.$$

From the ordinary differential system $(3.1)_{1,2,3}$, we have a solution u, w, q expressed by the following form:

$$\begin{aligned} (3.4)\qquad u &= A(\sigma)e^{(\sigma+1)\imath\theta} + B(\sigma)e^{(\sigma-1)\imath\theta} \\ (3.5)\qquad &\quad +C(\sigma)e^{-(\sigma+1)\imath\theta} + D(\sigma)e^{-(\sigma-1)\imath\theta} \\ (3.6)\qquad w &= -\frac{\sigma-1}{\sigma+1}\imath A(\sigma)e^{(\sigma+1)\imath\theta} - \imath B(\sigma)e^{(\sigma-1)\imath\theta} \\ (3.7)\qquad &\quad +\frac{\sigma-1}{\sigma+1}\imath C(\sigma)e^{-(\sigma+1)\imath\theta} + \imath D(\sigma)e^{-(\sigma-1)\imath\theta} \\ (3.8)\qquad q &= \frac{4\sigma}{\sigma+1}A(\sigma)e^{(\sigma-1)i\theta} + \frac{4\sigma}{\sigma+1}C(\sigma)e^{-(\sigma-1)i\theta}. \end{aligned}$$

Apply (3.4)-(3.8) to the boundary conditions (3.2) and $(3.3)_1$, then A, B, D can be represented in terms of C in the following form:

$$A(\sigma) = -\frac{a_1(\sigma)}{a_0(\sigma)}C(\sigma) \tag{3.9}$$

$$B(\sigma) = \frac{\sigma - 1 - \sigma e^{2i\theta_0} - e^{-2i\sigma\theta_0}}{(\sigma+1)a_0(\sigma)}C(\sigma) \tag{3.10}$$

$$D(\sigma) = \frac{\sigma - 1 - \sigma e^{-2i\theta_0} - e^{2i\sigma\theta_0}}{(\sigma+1)a_0(\sigma)}C(\sigma) \tag{3.11}$$

with the following definitions on a_1 and a_2:

$$\begin{aligned} a_0(\sigma) &= -\frac{\sigma}{\sigma+1}e^{2i\theta_0} + \frac{1}{\sigma+1}e^{2\sigma i\theta_0} + 1, \\ a_1(\sigma) &= -\frac{1}{\sigma+1}e^{-2\sigma i\theta_0} + \frac{\sigma}{\sigma+1}e^{-2i\theta_0} - 1. \end{aligned}$$

Applying the boundary condition $(3.3)_2$ to (3.4)-(3.10), we obtain the following ordinary differential equations in terms of C:

$$\frac{a_2(\sigma)}{a_0(\sigma)}C(\sigma) + \frac{1}{\tau}\frac{a_3(\sigma-1)}{a_0(\sigma-1)}C(\sigma-1) = d_1(\sigma),$$

where a_2, a_3 are the followings:

$$\begin{aligned} a_2(\sigma) &= \frac{16\sigma}{(\sigma+1)^2}\left[\sigma^2\sin^2\theta_0 - \cos^2\sigma\theta_0\right] \\ a_3(\sigma) &= \frac{4\alpha\sigma}{(\sigma+1)}\left[\sigma\sin 2\theta_0 - \sin 2\sigma\theta_0\right]. \end{aligned}$$

Set $M(\sigma) = \frac{a_2(\sigma)}{a_0(\sigma)}C(\sigma)$. Then $M(\sigma)$ satisfies the following ordinary differential equations:

$$M(\sigma) + \frac{1}{\tau}\frac{a_3(\sigma-1)}{a_2(\sigma-1)}M(\sigma-1) = d_1(\sigma). \tag{3.12}$$

For simplicity, we assume that d_1 is an entire function with

$$\int_{\mathrm{Re}(\sigma)=\mu-k-1}(1+|\sigma|)^{2k+1}|d_1(\sigma)|^2\frac{d\sigma}{i} < \infty.$$

REMARK 3.0.4. From Plancherel identity, if we define

$$b_1(r) = \frac{1}{2\pi}\int_{\mathrm{Re}(\sigma)=\mu-k-1} r^{-\sigma-1}d_1(\sigma)d\sigma$$

then $b_1 \in H_\mu^{k+1/2}(\gamma_0)$ and

$$\|b_1\|^2_{H_\mu^{k+1/2}(\gamma_0)} = \int_{\mathrm{Re}(\sigma)=\mu-k-1} (1+|\sigma|)^{2k+1}|d_1(\sigma)|^2\frac{d\sigma}{i} < \infty.$$

Moreover, the viceversa holds, too.

Since $C(\sigma) = \frac{a_0(\sigma)}{a_2(\sigma)}M(\sigma)$,

$$A = -\frac{a_1(\sigma)}{a_2(\sigma)}M(\sigma), \tag{3.13}$$

$$B = \frac{\sigma - 1 - \sigma e^{2i\theta_0} - e^{-2i\sigma\theta_0}}{(\sigma+1)a_2(\sigma)}M(\sigma), \tag{3.14}$$

$$D = \frac{\sigma - 1 - \sigma e^{-2i\theta_0} - e^{2i\sigma\theta_0}}{(\sigma+1)a_2(\sigma)}M(\sigma). \tag{3.15}$$

We observe that A, B, D are analytic in a domain in which M is analytic, except zeros of $\sigma(\sigma^2\sin^2\theta_0 + \cos^2\sigma\theta_0)$.

The following inequalities in Lemma 3.0.5 and Lemma 3.0.6 are important to derive all subsequent lemmas. This is based for the estimates in the subsequent lemmas.

LEMMA 3.0.5. *There is unique $M \in L^2(\Lambda)$ analytic and satisfies (3.12) in the strip $0 < \mathrm{Re}(\sigma) < \epsilon_0$, where $\epsilon_0 = \min\{\mathrm{Re}(\sigma) : \mathrm{Re}(\sigma) > 0,\ \sigma\sin 2\theta_0 = \sin 2\sigma\theta_0$ or $\sigma^2\sin^2\theta_0 = \cos^2\sigma\theta_0\}$. Moreover, the following inequality holds:*

$$\begin{aligned}&\int_{\mathrm{Re}(\sigma)=\mu-k-1}(1+|\sigma|)^{2k+1}|M(\sigma)|^2\frac{d\sigma}{i}\\ \le\ & c\int_{\mathrm{Re}(\sigma)=\mu-k-1}(1+|\sigma|)^{2k+1}|d_1(\sigma)|^2\frac{d\sigma}{i}.\end{aligned}$$

Proof. The proof is given in the next section. □

We observe that ϵ_0 depends on θ_0 and from the definition, the strip $0 < \mathrm{Re}(\sigma) < \epsilon_0$ does not contain any zeros of $\sigma(\sigma^2\sin^2\theta_0 - \cos^2\sigma\theta_0)$. Hence, A, B, C, D are also analytic in the strip $0 < \mathrm{Re}(\sigma) < \epsilon_0$.

LEMMA 3.0.6. *The following estimates hold:*

$$\int_0^{\theta_0}\sum_{l\le k}|\sigma|^{2l+2}|(\partial_\theta^{k-l}u)(\sigma,\theta)|^2 d\theta = O(|\sigma|^{2k+1}|M(\sigma)|^2)$$

$$\int_0^{\theta_0} \sum_{l \le k} |\sigma|^{2l+2} |(\partial_\theta^{k-l} w)(\sigma,\theta)|^2 d\theta = O(|\sigma|^{2k+1} |M(\sigma)|^2)$$

$$\int_0^{\theta_0} \sum_{l \le k} |\sigma|^{2l+2} |(\partial_\theta^{k-l} q)(\sigma,\theta)|^2 d\theta = O(|\sigma|^{2k+1} |M(\sigma)|^2).$$

Proof. From (3.13)-(3.15), we have the following asymptotic behavior for A, B , C and D in terms of M .

$$\begin{aligned}
|A(\sigma)| &= O(\frac{e^{2\sigma_2\theta_0}}{e^{2\sigma_2\theta_0} + e^{-2\sigma_2\theta_0}} M(\sigma)) \\
|B(\sigma)| &= O(\frac{e^{2\sigma_2\theta_0}}{e^{2\sigma_2\theta_0} + e^{-2\sigma_2\theta_0}} M(\sigma)) \\
|C(\sigma)| &= O(\frac{e^{-2\sigma_2\theta_0}}{e^{2\sigma_2\theta_0} + e^{-2\sigma_2\theta_0}} M(\sigma)) \\
|D(\sigma)| &= O(\frac{e^{-2\sigma_2\theta_0}}{e^{2\sigma_2\theta_0} + e^{-2\sigma_2\theta_0}} M(\sigma)).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_0^{\theta_0} \sum_{l \le k} |\sigma|^{2l+2} |(\partial_\theta^{k-l} u)(\sigma,\theta)|^2 d\theta \\
\le\ & 2|\sigma|^{2k+2} \Big[|A(\sigma)|^2 \int_0^{\theta_0} |e^{(\sigma+1)i\theta}|^2 d\theta + |B(\sigma)|^2 \int_0^{\theta_0} |e^{(\sigma-1)i\theta}|^2 d\theta \\
&+ |C(\sigma)|^2 \int_0^{\theta_0} |e^{-(\sigma+1)i\theta}|^2 d\theta + 2|D(\sigma)|^2 \int_0^{\theta_0} |e^{-(\sigma-1)i\theta}|^2 d\theta \Big] \\
=\ & O(|\sigma|^{2k+1} |M(\sigma)|^2).
\end{aligned}$$

By the same argument,

$$\int_0^{\theta_0} \sum_{l \le k} |\sigma|^{2l+2} |(\partial_\theta^{k-l} w)(\sigma,\theta)|^2 d\theta = O(|\sigma|^{2k+1} |M(\sigma)|^2)$$

and

$$\int_0^{\theta_0} \sum_{l \le k} |\sigma|^{2l+2} |(\partial_\theta^{k-l} q)(\sigma,\theta)|^2 d\theta = O(|\sigma|^{2k+1} |M(\sigma)|^2).$$

□

Now we define

$$\begin{aligned} v_r(r,\theta) &= \frac{1}{2\pi}\int_{\mathrm{Re}(\sigma)=\mu-k-1} r^{-\sigma}u(\sigma)d\sigma, \\ v_\theta(r,\theta) &= \frac{1}{2\pi}\int_{\mathrm{Re}(\sigma)=\mu-k-1} r^{-\sigma}w(\sigma)d\sigma, \\ p(r,\theta) &= \frac{1}{2\pi}\int_{\mathrm{Re}(\sigma)=\mu-k-1} r^{-\sigma-1}q(\sigma)d\sigma. \end{aligned}$$

From the construction, v_r, v_θ, p satisfies (2.1)-(2.3), and v_r, v_θ, p satisfy the following inequality.

$$\begin{aligned} &\|v_r\|^2_{H^{k+1}_{\mu-1}(d_{\theta_0})} + \|v_\theta\|^2_{H^{k+1}_{\mu-1}(d_{\theta_0})} + \|p\|^2_{H^{k+1}_{\mu}(d_{\theta_0})} \\ \leq \quad & c\int_{\mathrm{Re}(\sigma)=\mu-k-1} \frac{(1+|\sigma|)^{2k+2}}{|\sigma|}|M(\sigma)|^2\frac{d\sigma}{i}. \end{aligned}$$

If $\mathrm{Re}(\sigma) \geq \sigma_0$ for some positive constant σ_0, then $\frac{(1+|\sigma|)^{2k+2}}{|\sigma|} \leq c(1+|\sigma|)^{2k+1}$. Hence, from Lemma 3.0.5, Lemma 3.0.6 and Remark 3.0.4, we have the following lemma.

LEMMA 3.0.7. *We have* $v_r, v_\theta \in H^{k+1}_{\mu-1}, p \in H^{k+1}_{\mu}$ *for any* μ *with* $\mu-k-1 \in (0,\epsilon_0)$. *Moreover,*

$$\|v_r\|_{H^{k+1}_{\mu-1}(d_{\theta_0})} + \|v_\theta\|_{H^{k+1}_{\mu-1}(d_{\theta_0})} + \|p\|_{H^{k+1}_{\mu}(d_{\theta_0})} \leq c\|b_1\|_{H^{k+1/2}_{\mu}(\gamma_0)}.$$

Direct computation shows that

$$\begin{aligned} w|_{\theta=0} &= -\frac{\sigma-1}{\sigma+1}iA(\sigma) - iB(\sigma) + \frac{\sigma-1}{\sigma+1}iC(\sigma) + iD(\sigma) \\ &= \frac{\sigma\sin 2\theta_0 - \sin 2\sigma\theta_0}{\sigma(\sigma^2\sin^2\theta_0 - \cos^2\sigma\theta_0)}M(\sigma). \end{aligned}$$

Set $\phi = w|_{\theta=0}$, then ϕ is analytic for the strip $0 < \mathrm{Re}(\sigma) < \epsilon_0$ and satisfies

$$\phi(\sigma) = O(|\sigma|^{-1}M(\sigma)).$$

Hence it holds that

$$\begin{aligned} &\int_{\mathrm{Re}(\sigma)=\mu-k-1} (1+|\sigma|)^{2k+3}|\phi(\sigma)|^2 d\sigma \\ \leq\ & c\int_{\mathrm{Re}(\sigma)=\mu-k-1} (1+|\sigma|)^{2k+1}|M(\sigma)|^2 d\sigma \\ \leq\ & c\|b_1\|^2_{H^{k+1/2}_{\mu}(\gamma_0)}. \end{aligned}$$

This implies the following.

LEMMA 3.0.8. *$v_\theta|_{\gamma_0} = \frac{1}{2\pi}\int_{\mathrm{Re}(\sigma)=\mu-k-1} r^{-\sigma}\phi(\sigma)d\sigma \in H_\mu^{k+3/2}(\gamma_0)$ for $\mu \in (k+1, k+1+\epsilon_0)$ and it holds*

$$\|v_\theta\|_{H_\mu^{k+3/2}(\gamma_0)} \leq c\|b_1\|_{H_\mu^{k+1/2}(\gamma_0)}.$$

As we have already seen, $v_r, v_\theta \in H_{\mu-1}^{k+1}$, $p \in H_\mu^{k+1}$ satisfies (2.7), (2.8), $(2.9)_1$ with $v_\theta|_{\gamma_0} \in H_\mu^{k+3/2}(\gamma_0)$. Now, we recall Theorem 3.1 in the paper [12] of V. A. Solonnikov (Chapter 2 of L. Stupelis [19]), in order to conclude that $v_r, v_\theta \in H_\mu^{k+2}$ for μ with $\mu-k-1 \in (0,\epsilon_0)$. (In [12], it is stated that if $v_r, v_\theta \in H_{\mu-1}^{k+1}$, $p \in H_\mu^{k+1}$ is a solution of the (stationary) Stokes equation with Dirichlet boundary data for (v_r, v_θ) on γ_{θ_0} and mixed boundary data for $v_\theta \in H_\mu^{k+3/2}$, $\partial_r v_\theta + \frac{1}{r}\partial_\theta v_r - \frac{1}{r}v_\theta \in H_\mu^{k+1/2}$, with μ satisfying $\mu-k-1 \neq \mathrm{Re}(\sigma)$ for any σ satisfying $\sin 2\theta_0\sigma - \sigma\sin 2\theta_0$, then as a matter of fact, $v_r, v_\theta \in H_\mu^{k+2}$.)

3.1. Proof of Lemma 3.0.5

3.1.1. *Existence.* If $\alpha = 0$, then $M(\sigma) = d_1(\sigma)$ and lemma 3.0.5 follows naturally. From now on, we devote ourselves to construct M for the case $\alpha > 0$. Throughout this section, we assume $\alpha > 0$ and $\mathrm{Re}(\tau) > 0$.

Set $\omega(\sigma) = \frac{a_3(\sigma)}{a_2(\sigma)}$. From the definition of a_3 and a_2, $\omega(\sigma)$ is represented by

$$\omega(\sigma) = \frac{\alpha(\sigma+1)}{4}\frac{\sin 2\sigma\theta_0 - \sigma\sin 2\theta_0}{\cos^2\sigma\theta_0 - \sigma^2\sin^2\theta_0}.$$

For fixed $\mathrm{Re}(\sigma) = \sigma_1$, observe that

$$\lim_{\sigma_2\to\pm\infty}\frac{\omega(\sigma)}{\sigma+1} = \frac{\alpha}{4}\lim_{\sigma_2\to\pm\infty}\frac{\sin 2\sigma\theta_0 - \sigma\sin 2\theta_0}{\cos^2\sigma\theta_0 - \sigma^2\sin^2\theta_0} = \pm\frac{\alpha i}{2}.$$

We observe that

$$\begin{aligned}\omega(\sigma) &= \frac{\omega(\sigma)}{\sigma+1}(\sigma+1) = \mathrm{Re}(\frac{\omega(\sigma)}{\sigma+1})(\sigma_1+1) - \mathrm{Im}(\frac{\omega(\sigma)}{\sigma+1})\sigma_2 \\ &\quad +i[\mathrm{Re}(\frac{\omega(\sigma)}{\sigma+1})\sigma_2 + \mathrm{Im}(\frac{\omega(\sigma)}{\sigma+1})(\sigma_1+1)].\end{aligned}$$

$\arg(\omega)$ can be written by

$$\arg(\omega(\sigma)) = \tan^{-1}\frac{\mathrm{Re}(\frac{\omega(\sigma)}{\sigma+1})\sigma_2 + \mathrm{Im}(\frac{\omega(\sigma)}{\sigma+1})(\sigma_1+1)}{\mathrm{Re}(\frac{\omega(\sigma)}{\sigma+1})(\sigma_1+1) - \mathrm{Im}(\frac{\omega(\sigma)}{\sigma+1})\sigma_2}$$

and the right hand side goes to 0 as we send σ_2 to $\pm\infty$. Therefore the following lemma is obtained.

LEMMA 3.1.1. *Let $\sigma = \sigma_1 + \sigma_2 i$. Then there is a positive number $\epsilon_1 < \pi/2$ so that*

$$|\arg(\omega(\sigma))| < \pi/2 - \epsilon_1$$

for large enough $|\sigma_2|$.

Let us define $\mathrm{sgn}(y)$ by $\mathrm{sgn}(y) = 1$ if $y > 0$, $\mathrm{sgn}(y) = -1$ if $y < 0$.

LEMMA 3.1.2. *Suppose $\mathrm{sgn}(\sigma_2) = \mathrm{sgn}(y)$. Then,*

$$|\arg\omega(\sigma + iy) - \arg\omega(\sigma)| \le \frac{c}{|\sigma_2|}$$

for sufficiently large σ_2.

Proof. Recalling $\arg(\sigma_1 + i\sigma_2) = \tan^{-1}\frac{\sigma_2}{\sigma_1}$, $\arg\omega(\sigma)$ is represented by

$$\begin{aligned}
&\arg\omega(\sigma)\\
= \;&\arg(\sigma+1) + \arg(\sin 2\sigma\theta_0 - \sigma\sin 2\theta_0) - \arg(\cos^2\sigma\theta_0 - \sigma^2\sin^2\theta_0)\\
= \;&\tan^{-1}\frac{\sigma_2}{\sigma_1+1} + \tan^{-1}\frac{\cos 2\sigma_1\theta_0\sinh 2\sigma_2\theta_0 - \sigma_2\sin 2\theta_0}{\sin 2\sigma_1\theta_0\cosh 2\sigma_2\theta_0 - \sigma_1\sin 2\theta_0}\\
&+\tan^{-1}\frac{\sin 2\sigma_1\theta_0\sinh 2\sigma_2\theta_0 + 4\sigma_1\sigma_2\sin^2\theta_0}{1+\cos 2\sigma_1\theta_0\cosh 2\sigma_2\theta_0 - 2(\sigma_1^2-\sigma_2^2)\sin^2\theta_0}.
\end{aligned}$$

Hence, $\omega(\sigma+iy) - \omega(\sigma)$ is a sum of i, ii, iii, where

$$\begin{aligned}
i \;=\;& \tan^{-1}\frac{\sigma_2+y}{\sigma_1+1} - \tan^{-1}\frac{\sigma_2}{\sigma_1+1}\\
ii \;=\;& \tan^{-1}\frac{\cos 2\sigma_1\theta_0\sinh 2(\sigma_2+y)\theta_0 - (\sigma_2+y)\sin 2\theta_0}{\sin 2\sigma_1\theta_0\cosh 2(\sigma_2+y)\theta_0 - \sigma_1\sin 2\theta_0}\\
&-\tan^{-1}\frac{\cos 2\sigma_1\theta_0\sinh 2\sigma_2\theta_0 - \sigma_2\sin 2\theta_0}{\sin 2\sigma_1\theta_0\cosh 2\sigma_2\theta_0 - \sigma_1\sin 2\theta_0}\\
iii \;=\;& \tan^{-1}\frac{\sin 2\sigma_1\theta_0\sinh 2(\sigma_2+y)\theta_0 + 4\sigma_1(\sigma_2+y)\sin^2\theta_0}{1+\cos 2\sigma_1\theta_0\cosh 2(\sigma_2+y)\theta_0 - 2(\sigma_1^2-(\sigma_2+y)^2)\sin^2\theta_0}\\
&-\tan^{-1}\frac{\sin 2\sigma_1\theta_0\sinh 2\sigma_2\theta_0 + 4\sigma_1\sigma_2\sin^2\theta_0}{1+\cos 2\sigma_1\theta_0\cosh 2\sigma_2\theta_0 - 2(\sigma_1^2-\sigma_2^2)\sin^2\theta_0}.
\end{aligned}$$

Recall the well known identity $\tan^{-1}c_1 - \tan^{-1}c_2 = \tan^{-1}\frac{c_1-c_2}{1+c_1c_2}$. Then i, ii, iii are represented by

$$i = \tan^{-1} I, \quad ii = \tan^{-1}\frac{II}{II'}, \quad iii = \tan^{-1}\frac{III}{III'},$$

where

$$
\begin{aligned}
I &= \frac{(\sigma_1+1)y}{(\sigma_1+1)^2+\sigma_2(\sigma_2+y)} \\
II &= \cos 2\sigma_1\theta_0 \sin 2\sigma_1\theta_0 \sinh 2y\theta_0 + \sigma_1 y \sin^2 2\theta_0 \\
&\quad -\sigma_1 \sin 2\theta_0 \cos 2\sigma_1\theta_0 \Big[\sinh 2(\sigma_2+y)\theta_0 - \sinh 2\sigma_2\theta_0\Big] \\
&\quad - \sin 2\theta_0 \sin 2\sigma_1\theta_0 \Big[(\sigma_2+y)\cosh 2\sigma_2\theta_0 - \sigma_2 \cosh 2(\sigma_2+y)\theta_0\Big] \\
II' &= \Big[\sin 2\sigma_1\theta_0 \cosh 2(\sigma_2+y)\theta_0 - \sigma_1 \sin 2\theta_0\Big] \\
&\qquad \times \Big[\sin 2\sigma_1\theta_0 \cosh 2\sigma_2\theta_0 - \sigma_1 \sin 2\theta_0\Big] \\
&\quad + \Big[\cos 2\sigma_1\theta_0 \sinh 2(\sigma_2+y)\theta_0 - (\sigma_2+y)\sin 2\theta_0\Big] \\
&\qquad \times \Big[\cos 2\sigma_1\theta_0 \sinh 2\sigma_2\theta_0 - \sigma_2 \sin 2\theta_0\Big] \\
III &= \sin 2\sigma_1\theta_0 \Big[\sinh 2(\sigma_2+y)\theta_0 - \sinh 2\sigma_2\theta_0\Big] \\
&\quad +4\sigma_1 y \sin^2\theta_0 + \sin 2\sigma_1\theta_0 \cos 2\sigma_1\theta_0 \sinh 2y\theta_0 \\
&\quad +4\sigma_1 \sin^2\theta_0 \cos 2\sigma_1\theta_0 \\
&\quad \times \Big[(\sigma_2+y)\cosh 2\sigma_2\theta_0 - \sigma_2 \cosh 2(\sigma_2+y)\theta_0\Big] \\
&\quad -2\sin^2\theta_0 \sin 2\sigma_1\theta_0 \\
&\quad \times \Big[(\sigma_1^2-\sigma_2^2)\sinh 2(\sigma_2+y)\theta_0 - (\sigma_1^2-(\sigma_2+y)^2)\sinh 2\sigma_2\theta_0\Big] \\
&\quad -8\sigma_1 \sin^4\theta_0 \Big[(\sigma_1^2-\sigma_2^2)(\sigma_2+y) - (\sigma_1^2-(\sigma_2+y)^2)\sigma_2\Big] \\
III' &= \Big[1+\cos 2\sigma_1\theta_0 \cosh 2(\sigma_2+y)\theta_0 - 2(\sigma_1^2-(\sigma_2+y)^2)\sin^2\theta_0\Big] \\
&\qquad \times \Big[1+\cos 2\sigma_1\theta_0 \cosh 2\sigma_2\theta_0 - 2(\sigma_1^2-\sigma_2^2)\sin^2\theta_0\Big] \\
&\quad + \Big[\sin 2\sigma_1\theta_0 \sinh 2(\sigma_2+y)\theta_0 + 4\sigma_1(\sigma_2+y)\sin^2\theta_0\Big] \\
&\qquad \times \Big[\sin 2\sigma_1\theta_0 \sinh 2\sigma_2\theta_0 + 4\sigma_1\sigma_2 \sin^2\theta_0\Big].
\end{aligned}
$$

Now, it is not difficult to observe that for large enough $|\sigma_2|$,

$$|I|, |\frac{II}{II'}|, |\frac{III}{III'}| < \frac{c}{|\sigma_2|}.$$

Recalling $\tan^{-1} x = O(|x|)$ for $x \in (-1,1)$, we conclude that $|\arg\omega(\sigma+iy) - \arg\omega(\sigma)| \le \frac{c}{|\sigma_2|}$ for large enough $|\sigma_2|$. □

We construct $M_0(\sigma)$ satisfying that

$$M_0(\sigma+1) = -\frac{\omega(\sigma)}{\tau} M_0(\sigma) \tag{3.16}$$

and then construct $M_1(\sigma)$ satisfying that

$$M_1(\sigma+1) - M_1(\sigma) = \frac{d_1(\sigma+1)}{M_0(\sigma+1)}. \tag{3.17}$$

Setting $M = M_0 M_1$, it is easy to check that M satisfies (3.12).

LEMMA 3.1.3. *There is $K_0(\sigma)$ which is analytic and does not vanish in the certain strip $0 < \mathrm{Re}(\sigma) < 1 + \epsilon_0$, satisfies the recursive equation $K_0(\sigma+1) = \omega(\sigma) K_0(\sigma)$ in the strip $0 < \mathrm{Re}(\sigma) < \epsilon_0$ and has the following asymptotic behaviors;*

$$\begin{aligned} \ln K_0(\sigma) &= (\sigma - 1/2)\ln \omega(\sigma) + c\sigma + \ln(\sigma+1) + r_1(\sigma), \\ (\ln K_0(\sigma))' &= \ln \omega(\sigma) + r_2(\sigma), \end{aligned}$$

for some $r_1(\sigma) = O(1)$ and $r_2(\sigma) = O(1)$ as $|\sigma| \to \infty$. Here $\epsilon_0 = \min\{\mathrm{Re}(\sigma) : \mathrm{Re}(\sigma) > 0,\ \sigma \sin 2\theta_0 = \sin 2\sigma\theta_0$ or $\sigma^2 \sin^2\theta_0 = \cos^2 \sigma\theta_0\}$.

Proof. The construction of such K_0 is given in the Appendix. □

Set $M_0(\sigma) = \frac{e^{i\pi\sigma}}{\tau^{\sigma-1/2}} K_0(\sigma)$. Then we observe that $M_0(\sigma)$ satisfies (3.16).

Now we define $M_1(\sigma)$ dy

$$M_1(\sigma) = \frac{1}{2i} \int_{L_\epsilon} \frac{d_1(\sigma+\zeta+1)}{M_0(\sigma+\zeta+1)} [\cot \pi\zeta + i] d\zeta, \tag{3.18}$$

where L_ϵ is an infinite path described by $L_\epsilon = \{-1 + yi : \ |y| \geq \epsilon\} \cup \{\epsilon(e^{i\theta} - 1) : \ \theta \in (-\pi/2, \pi/2)\}$.

Combining Lemma 3.1.3 and Lemma 3.1.1, it is easy to observe the following asymptotic behaviors.

LEMMA 3.1.4.

$$\lim_{y \to -\infty} \left| \frac{d_1(\sigma+iy+1)}{M_0(\sigma+iy+1)} \right| = 0, \quad \lim_{y \to +\infty} \left| \frac{d_1(\sigma+iy+1)}{M_0(\sigma+iy+1)} \right| e^{-2\pi|y|} = 0.$$

Referring Proposition 6.1 of [18], we conclude that M_1 is well defined analytic function in the strip $0 < \mathrm{Re}(\sigma) < 1 + \epsilon_0$, satisfying the recursive algebraic equation (3.17) in the strip $0 < \mathrm{Re}(\sigma) < \epsilon_0$.

By using the Lemma 3.1.3, Lemma 3.1.1, Lemma 3.1.2, the following asymptotic behaviors can be derived. For the details of the proof of Lemma 3.1.5 and Lemma 3.1.6, we refer to the proof of Lemma 7.1 and Lemma 7.2 of [18].

LEMMA 3.1.5. *If $y \in (-\infty, \infty)$ and the complex $\sigma = \sigma_1 + i\sigma_2$ lies in the strip $0 < \sigma_1 < 1 + \epsilon_0$, then*

$$\left|\frac{K_0(\sigma)}{K_0(\sigma+iy)}\right| \le ce^{(\pi/2-\epsilon_1)|y|},$$

where $\epsilon_1 > 0$ and c does not depend on σ_2.

LEMMA 3.1.6. *For all $\sigma = \sigma_1 + i\sigma_2$ in the strip $0 < \sigma_1 < 1 + \epsilon_0$ and for all $y \in (-1, 1)$ the function $B(y,\sigma) = \frac{K_0(\sigma)}{K_0(\sigma+iy)}\tau^{iy}\left(\frac{1+|\sigma|}{1+|\sigma+iy|}\right)^{k+1/2} - \left|\frac{\tau}{\omega(\sigma+iy)}\right|^{i(\sigma_2+y)}\left|\frac{\tau}{\omega(\sigma)}\right|^{-i\sigma_2}$ admits the estimate*

$$|B(y,\sigma)| \le c|y|$$

for some constant c independent of σ.

Now we consider $M(\sigma) = M_0(\sigma)M_1(\sigma) = \frac{1}{2i}\int_{L_\epsilon} \frac{K_0(\sigma)}{K_0(\sigma+\zeta+1)} d_1(\sigma+\zeta+1)[\cot \pi\zeta + i]d\zeta$. By change of variables and Residue theorem, M can be written equivalently by

$$M(\sigma) = \frac{1}{2}d_1(\sigma) + ip.v. \int_{-\infty}^{\infty} \frac{K_0(\sigma)}{K_0(\sigma+iy)} \frac{\tau^{iy}}{e^{-\pi y} - e^{\pi y}} d_1(\sigma+iy)dy.$$

Set $\tilde{M}(\sigma) = ip.v. \int_{-\infty}^{\infty} \frac{K_0(\sigma)}{K_0(\sigma+iy)} \frac{\tau^{iy}}{e^{-\pi y}-e^{\pi y}} d_1(\sigma+iy)dy$. We would like to show that $(1+|\sigma|)^{k+1/2}\tilde{M}(\sigma) \in L^2(\Lambda)$, $\Lambda = \{\sigma;\ \mathrm{Re}(\sigma) = \mu - k - 1\}$. $(1+|\sigma|)^{k+1/2}\tilde{M}(\sigma)$ can be divided by the following I, II and III

$$\begin{aligned}
I &= \int_{|y|\ge 1} d_1(\sigma+iy)(1+|\sigma|)^{k+\frac{1}{2}} \frac{K_0(\sigma)}{K_0(\sigma+iy)} \frac{\tau^{iy}}{e^{-\pi y} - e^{\pi y}} dy \\
II &= ip.v. \int_{-1}^{1} d_1(\sigma+iy)(1+|\sigma+iy|)^{k+1/2} B(y,\sigma) \frac{1}{e^{-\pi y} - e^{\pi y}} dy \\
III &= \left|\frac{\tau}{\omega(\sigma)}\right|^{-i\sigma_2} ip.v. \int_{-1}^{1} d_1(\sigma+iy)(1+|\sigma+iy|)^{k+1/2} \\
&\qquad \times \left|\frac{\tau}{\omega(\sigma+iy)}\right|^{i(\sigma_2+y)} \frac{1}{e^{-\pi y} - e^{\pi y}} dy.
\end{aligned}$$

For $|y| \ge 1$, $(1+|\sigma|)^{k+\frac{1}{2}} \le c(1+|\sigma+iy|)^{k+\frac{1}{2}}|y|^{k+\frac{1}{2}}$ and from Lemma 3.1.5, $\left|\frac{K_0(\sigma)}{K_0(\sigma+iy)} \frac{\tau^{iy}}{e^{-\pi y}-e^{\pi y}}\right| \le c\frac{e^{(\pi-\epsilon_1)|y|}}{|e^{-\pi y}-e^{\pi y}|}$. Hence, we have the inequality

$$|I(\sigma)| \le c\int_{|y|\ge 1} |d_1(\sigma+iy)|(1+|\sigma+iy|)^{k+\frac{1}{2}} \frac{e^{(\pi-\epsilon_1)|y|}(1+|y|)^{k+1/2}}{|e^{-\pi y} - e^{\pi y}|}.$$

From Lemma 3.1.6, II is bounded by

$$|II(\sigma)| \leq c\int_{-1}^{1}|d_1(\sigma+iy)(1+|\sigma+iy|)^{k+1/2}\frac{|y|}{|e^{-\pi y}-e^{\pi y}|}dy.$$

Observe that $\frac{e^{(\pi-\epsilon_1)|y|}(1+|y|)^{k+1/2}}{|e^{-\pi y}-e^{\pi y}|}, \frac{|y|}{|e^{-\pi y}-e^{\pi y}|} \in L^2(-\infty,\infty)$. By Young's inequality,

$$\begin{aligned}\|I\|_{L^2(\Lambda)} &\leq c\|d_1(\sigma_1+i\cdot)(1+|\sigma_1+i\cdot|)^{k+1/2}\|_{L^2(-\infty,\infty)}\\ \|II\|_{L^2(\Lambda)} &\leq c\|d_1(\sigma_1+i\cdot)(1+|\sigma_1+i\cdot|)^{k+1/2}\|_{L^2(-\infty,\infty)}.\end{aligned}$$

Finally, apply Calderón-Zygmund theorem to III, then we obtain

$$\|III\|_{L^2(\Lambda)} \leq c\|d_1(\sigma_1+i\cdot)(1+|\sigma_1+i\cdot|)^{k+1/2}\|_{L^2(-\infty,\infty)}.$$

Note that $\|d_1(\sigma_1+i\cdot)(1+|\sigma_1+i\cdot|)^{k+1/2}\|_{L^2(-\infty,\infty)} = \|d_1(\cdot)(1+|\cdot|)^{k+1/2}\|_{L^2(\Lambda)}$. Therefore we have the desired estimate for M

$$\begin{aligned}&\int_{\mathrm{Re}(\sigma)=\mu-k-1}|M(\sigma)|^2(1+|\sigma|)^{2k+1}\frac{d\sigma}{i}\\ &\leq c\int_{\mathrm{Re}(\sigma)=\mu-k-1}|d_1(\sigma)|^2(1+|\sigma|)^{2k+1}\frac{d\sigma}{i}.\end{aligned}$$

3.1.2. *Uniqueness.* From the asymptotic behavior of $K_0(\sigma)$, we observe that $|M_0| \geq c$ for some positive constant. In the previous section, our constructed solution $M = M_0M_1 \in L^2(\Lambda)$, where $\Lambda = \{\sigma = \sigma_1 + i\sigma_2 : 0 < \sigma_1 < 1+\epsilon_0\}$. Below, we show the uniqueness of solution M (3.12) in the class $L^2(\Lambda)$.

LEMMA 3.1.7. *Any solution $M \in L^2(\Lambda)$ of (3.12) is of form $M = M_0M_1$.*

Proof. Suppose that $M \in L^2(\Lambda)$ is a solution of (3.12). Then, we observe that $(\frac{M}{M_0} - M_1)(\sigma+1) = (\frac{M}{M_0} - M_1)(\sigma)$. This implies that $(\frac{M}{M_0} - M_1)(\sigma)$ is a periodic function of period 1. Hence, $M = M_0M_1 + M_0E$ for some analytic function E with period 1.

Observe $M - M_0M_1 = M_0E \in L^2(\Lambda)$, $|M_0| \geq c$. It is not difficult to show that a periodic function with period one is either constant or grows no slower than $e^{\pi|\mathrm{Im}(\sigma)|}$ as $\mathrm{Im}(\sigma) \to \pm\infty$. (Refer section 5.2, chapter VII of [4].) Therefore $E(\sigma)$ must be identically zero to be $M_0E \in L^2(\Lambda)$. This completes the proof of our lemma. □

Appendix

4. In case $\theta_0 = \pi$

If $\theta_0 = \pi$, $\omega(\sigma) = \frac{\alpha}{2}(\sigma+1)\tan\sigma\theta_0$. From the identity $\sin z = z\Pi_{n=1}^{\infty}(1 - z^2/(n\pi)^2)$, $\tan(\sigma\theta_0) = \frac{\sin\sigma\theta_0}{\sin(\sigma\theta_0 - \frac{\pi}{2})}$ is represented by

$$\tan\sigma\theta_0 = \theta_0\sigma\Pi_{n=1}^{\infty}\frac{(1-\frac{\sigma}{y_n})(1+\frac{\sigma}{y_n})}{(1-\frac{\sigma}{z_n})(1+\frac{\sigma}{z_n})},$$

where $y_n = \frac{n\pi}{\theta_0}$, $z_n = \frac{n\pi}{\theta_0} - \frac{\pi}{2\theta_0}$. Hence $\omega(\sigma)$ can be represented by

$$\omega(\sigma) = \frac{\alpha\theta_0}{2}\sigma(\sigma+1)\Pi_{n=1}^{\infty}\frac{(1-\frac{\sigma}{y_n})(1+\frac{\sigma}{y_n})}{(1-\frac{\sigma}{z_n})(1+\frac{\sigma}{z_n})}.$$

Now we construct K_0 by

$$\begin{aligned} & K_0(\sigma) \\ = & \left(\frac{\alpha\theta_0}{2}\right)^{\sigma-1/2}\Gamma(\sigma)\Gamma(\sigma+1)\Pi_{n=1}^{\infty}\frac{\Gamma(y_n+\sigma)\Gamma(1+z_n-\sigma)}{\Gamma(1+y_n-\sigma)\Gamma(z_n+\sigma)}\left(\frac{z_n^2}{y_n^2}\right)^{\sigma-1/2}. \end{aligned}$$

Formally, it is easy to see $K_0(\sigma+1) = \omega(\sigma)K_0(\sigma)$.

We refer Proposition 5.1 of [18] to say the infinite product of K_0 is convergent to an analytic function for all σ with the exception of the poles of $\Gamma(\sigma), \Gamma(\sigma+1), \Gamma(\sigma+y_n)$, $\Gamma(1+z_n-\sigma)$.

Since $\Gamma(\sigma)$ has simple poles for $\sigma = 0, -1, \ldots$, $K_0(\sigma)$ has simple poles for $\sigma = 0, -1, -2, \ldots, -y_n, -y_n-1, \ldots, z_n+1, z_n+2, \ldots$. Hence $K_0(\sigma)$ is analytic in the strip $0 < \mathrm{Re}(\sigma) < z_1 + 1$. By the same reasoning, $K_0(\sigma)$ has zeros of multiplicity one for $\sigma = y_n+1, y_n+2, \ldots, -z_n, -z_n-1, \ldots$. Hence $K_0(\sigma)$ does not have any zeros in the strip $0 < \mathrm{Re}(\sigma) < y_1 + 1$. We set $\epsilon_0 = \min\{z_1, y_1\}$. Then $K_0(\sigma)$ is analytic and does not vanish in the strip $0 < \mathrm{Re}(\sigma) < 1 + \epsilon_0$.

Moreover, applying Proposition 5.2 of [18], we can derive the following asymptotic formulas for $\ln K_0(\sigma)$;

$$\begin{aligned} \ln K_0(\sigma) &= (\sigma - 1/2)\ln\omega(\sigma) + c\sigma + r_1(\sigma), \\ (\ln K_0(\sigma))' &= \ln\omega(\sigma) + r_2(\sigma), \end{aligned}$$

for some $r_1(\sigma) = O(1)$ and $r_2(\sigma) = O(1)$ as $|\sigma| \to \infty$.

5. In case $\theta_0 \neq \pi \in (0, 2\pi)$

Throughout this section, we only consider the case $\theta_0 \neq \pi$. Let us denote the set of zero of f by $Z[f] = \{\sigma : f(\sigma) = 0\}$. We denote the complex conjugate of $\sigma = \sigma_1 + i\sigma_2$ by $\bar{\sigma} = \sigma_1 - i\sigma_2$. For real a, we denote by $[a]$ the largest integer less than or equal to a.

5.1. Classification of $Z[\cos^2 \sigma\theta_0 - \sigma^2 \sin^2 \theta_0]$

In the below, we will show $\cos^2 \sigma\theta_0 - \sigma^2 \sin^2 \theta_0$ has only finite real zeros and countably many complex zeros, by observing the followings.

i. $\sigma \in Z[\cos^2(\theta_0\sigma) - \sigma^2 \sin^2 \theta_0]$, if and only if $-\sigma \in Z[\cos^2(\theta_0\sigma) - \sigma^2 \sin^2 \theta_0]$, if and only if $\bar{\sigma} \in Z[\cos^2(\theta_0\sigma) - \sigma^2 \sin^2 \theta_0]$.

ii. $Z[\cos^2(\theta_0\sigma) - \sigma^2 \sin^2 \theta_0] \cap \{\frac{n\pi}{\theta_0} + \frac{\pi}{2\theta_0} + iy : y \in (-\infty, \infty)\}$ is empty for all integer n, since for all integer n and for all real y,

$$\begin{aligned} & \cos^2 \theta_0(\frac{n\pi}{\theta_0} + iy) - (\frac{n\pi}{\theta_0} + \frac{\pi}{2\theta_0} + iy)^2 \sin^2 \theta_0 \\ = & -\sinh^2 \theta_0 y - (\frac{n\pi}{\theta_0} + \frac{\pi}{2\theta_0} + iy)^2 \sin^2 \theta_0 \neq 0. \end{aligned}$$

iii. Observe that $Z[\cos^2 \theta_0\sigma - \sigma^2 \sin^2 \theta_0] = Z[\cos \theta_0\sigma + \sigma \sin \theta_0] \cup Z [\cos \theta_0\sigma - \sigma \sin \theta_0]$.

iv. Let $c_0 = |\frac{\sin \theta_0}{\theta_0}|$. Obviously $0 < c_0 < 1$. Let n be a positive integer. Then,

$$\begin{aligned} & \left|\{z \in Z(\cos z - zc_0) : \operatorname{Re}(z) \in (n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2})\}\right| \\ = & \begin{cases} 3 & \text{if } n \text{ is even }, c_0 n\pi \leq 1 \\ 2 & \text{if } n \text{ is even } c + 0n\pi > 1 \\ 0 & \text{if } n \text{ is odd} \end{cases} \\ & \left|\{z \in Z(\cos z + zc_0) : \operatorname{Re}(z) \in (n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2})\}\right| \\ = & \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd }, c_0 n\pi > 1 \\ 3 & \text{if } n \text{ is odd }, c_0 n\pi \leq 1. \end{cases} \end{aligned}$$

The proof can be shown by applying Rouché's theorem to the strip strip $n\pi - \frac{\pi}{2} < \operatorname{Re}(z) < n\pi + \frac{\pi}{2}$, taking two analytic function $f(z) = \cos z \pm c_0 z$ and $g(z) = \cos z \pm c_0 n\pi$.

v. Finally, applying Rouché's theorem to the strip $0 < \operatorname{Re}(z) < \frac{\pi}{2}$, taking two analytic function $f(z) = \cos z \pm c_0 z$ and $g(z) = \cos z \pm$

$\frac{\pi c_0}{4}$, we see

$$\Big|\{z \in Z(\cos z - zc_0) : \operatorname{Re}(z) \in (0, \frac{\pi}{2})\}\Big| = 1$$
$$\Big|\{z \in Z(\cos z + zc_0) : \operatorname{Re}(z) \in (0, \frac{\pi}{2})\}\Big| = 0.$$

Set $N_1 = \max\{0, [\frac{1}{c_0\pi}]\}$. From the above items i-v, we deduce that there are complex numbers $\beta_1, \beta_2, \ldots$ with $\operatorname{Re}(\beta_n) \in (\frac{n\pi}{\theta_0} - \frac{\pi}{2\theta_0}, \frac{n\pi}{\theta_0} + \frac{\pi}{2\theta_0})$, $\operatorname{Im}(\beta_n) > 0$, and real numbers $w_0 \in (0, \frac{\pi}{2\theta_0})$, $w_i \in (\frac{n\pi}{\theta_0} - \frac{\pi}{2\theta_0}, \frac{n\pi}{\theta_0} + \frac{\pi}{2\theta_0})$, $i = 1, \ldots, N_1$, such that

$$Z[\cos^2(\theta_0\sigma) - \sigma^2 \sin^2\theta_0] = \{\beta_j, -\beta_j, \overline{\beta_j}, -\overline{\beta_j}\}_{j=1}^{\infty} \cup \{w_i, -w_i\}_{i=0}^{N_1}.$$

Hence $\cos^2(\theta_0\sigma) - \sigma^2\sin^2\theta_0$ can be represented by the following infinite product

$$\begin{aligned} \text{(5.1)} \qquad & \cos^2(\theta_0\sigma) - \sigma^2\sin^2\theta_0 \\ = \;& e^{g_2(\sigma)}\Pi_{i=0}^{N_1}\Big(1 - \frac{\sigma^2}{w_i^2}\Big)\Pi_{j=1}^{\infty}\Big(1 - \frac{\sigma^2}{\beta_j^2}\Big)\Big(1 - \frac{\sigma^2}{\overline{\beta_j}^2}\Big) \end{aligned}$$

for some entire function g_2. Now, we claim $g_2(\sigma) \equiv 0$. Then, we have the identity

$$\cos^2\theta_0\sigma - \sigma^2\sin^2\theta_0 = \Big(1 - \frac{\sigma^2}{w_i^2}\Big)\Pi_{j=1}^{\infty}\Big(1 - \frac{\sigma^2}{\beta_j^2}\Big)\Big(1 - \frac{\sigma^2}{\overline{\beta_j}^2}\Big).$$

Proof of claim. Taking $\sigma = 0$ to the both sides of the above infinite product, we observe that $g_2(0) = 0$. By taking ln to the both sides of (5.1) and then differentiating, we have the identity

$$\begin{aligned} & g_2'(\sigma) \\ = & -\frac{\theta_0\sin 2\theta_0\sigma + 2\sigma\sin^2\theta_0}{\cos^2\theta_0\sigma - \sigma^2\sin^2\theta_0} - \sum_{i=0}^{N_1}\frac{2\sigma}{w_i^2 - \sigma^2} - \sum_{j=1}^{\infty}\frac{2\sigma}{\beta_j^2 - \sigma^2} + \frac{1}{\overline{\beta_j}^2 - \sigma^2}. \end{aligned}$$

Now, we observe the right hand side of the above identity is bounded and also is equal to zero for $\sigma = 0$. This implies $g_2'(\sigma) \equiv 0$ by Louville's theorem. Hence $g_2 = const = g_2(0) = 0$. □

5.2. Classification of $Z[\sin 2\sigma\theta_0 - \sigma\sin 2\theta_0]$

5.2.1. *In case $\theta_0 = \frac{\pi}{2}$ or $\theta_0 = \frac{3\pi}{2}$.* If $\theta_0 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, then $\sin 2\sigma\theta_0 - \sigma\sin 2\theta_0 = \sin 2\sigma\theta_0$. Hence, $\sin 2\sigma\theta_0 - \sigma\sin 2\theta_0$ has only real zeros $\frac{n\pi}{2\theta_0}$,

$n = 0, \pm 1, \pm 2, \dots$ with multiplicity one. Moreover, by the well known representation for $\sin 2\sigma\theta_0$

$$\sin 2\sigma\theta_0 - \sigma \sin 2\theta_0 = \sin 2\theta_0\sigma = 2\theta_0\sigma\Pi_{j=1}^{\infty}(1 - \frac{\sigma^2}{y_j^2}),$$

where $y_n = \frac{n\pi}{2\theta_0}$.

5.2.2. *In case* $\theta_0 \neq \frac{\pi}{2}$, $\theta_0 \neq \frac{3\pi}{2}$. In the below, we will show $\sin 2\sigma\theta_0 - \sigma \sin 2\theta_0$ has only finite real zeros and countably many complex zeros by observing the followings.

i. $\sigma \in Z[\sin(2\theta_0\sigma) - \sigma \sin 2\theta_0]$, if and only if $-\sigma \in Z[\sin(2\theta_0\sigma) - \sigma \sin 2\theta_0]$, if and only if $\bar{\sigma} \in Z[\sin(2\theta_0\sigma) - \sigma \sin 2\theta_0]$.

ii. $Z[\sin(2\theta_0\sigma) - \sigma \sin 2\theta_0] \cap \{\frac{m\pi}{2\theta_0} + iy : \ y \in (-\infty, \infty)\}$ is empty for $m \neq 0$ and $Z[\sin(2\theta_0\sigma) - \sigma \sin 2\theta_0] \cap \{iy : \ y \in (-\infty, \infty)\} = \{0\}$, since

$$\begin{aligned} & \sin(m\pi + 2\theta_0 iy) - (\frac{m\pi}{2\theta_0} + iy)\sin 2\theta_0 \\ = \ & (-1)^m i \sinh 2\theta_0 y - (\frac{m\pi}{2\theta_0} + iy)\sin 2\theta_0 \\ = \ & \begin{cases} \neq 0 & \text{for all integer } m \neq 0 \text{ or for all real } y \neq 0 \\ = 0 & \text{for } m = 0 \text{ and } y = 0 \end{cases} \end{aligned}$$

iii. Let $c_1 = |\frac{\sin 2\theta_0}{2\theta_0}|$. Obviously $0 < c_1 < 1$. Let n be a positive integer.

$$\begin{aligned} & |Z(\sin z + zc_1) \cap \{z : \ n\pi - \pi < \mathrm{Re}(z) < n\pi\}| \\ = & \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even and } c_1(n\pi - \pi/2) > 1, \\ 3 & \text{if } n \text{ is even and } c_1(n\pi - \pi/2) \leq 1. \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} & |Z(\sin z - zc_1) \cap \{z : \ n\pi - \pi < \mathrm{Re}(z) < n\pi\}| \\ = & \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd and } c_1(n\pi - \pi/2) > 1, \\ 3 & \text{if } n \text{ is odd and } c_1(n\pi - \pi/2) \leq 1. \end{cases} \end{aligned}$$

This follows from applying Rouché's theorem to the following functions

$$f(z) = \sin z \pm zc_1, \quad g(z) = \sin z \pm (n\pi - \frac{\pi}{2})c_1$$

in the strip $n\pi - \pi < \mathrm{Re}(z) < n\pi$.

Let us define

$$N_0 = \max\{0, [\frac{1}{c_1\pi} + \frac{1}{2}]\}.$$

From the above item i, ii, iii, we deduce that there are positive numbers $x_1, x_2, \ldots, x_{N_0}$ with $x_i \in (\frac{n\pi-\pi}{\theta_0}, \frac{n\pi}{\theta_0})$, and $\alpha_1, \alpha_2, \ldots$ with $\mathrm{Re}(\alpha_n) \in (\frac{n\pi-\pi}{\theta_0}, \frac{n\pi}{\theta_0})$, $\mathrm{Im}(\alpha_n) > 0$, such that

$$Z[\sin(2\theta_0\sigma) - \sigma\sin 2\theta_0] = \{0\} \cup \{x_i, -x_i\}_{i=1}^{N_0} \cup \{\alpha_j, -\alpha_j, \overline{\alpha_j}, -\overline{\alpha_j}\}_{j=1}^{\infty}.$$

Hence $\sin 2\theta_0\sigma - \sigma\sin 2\theta_0$ can be represented by the following infinite product

$$\begin{aligned} (5.2)\qquad & \sin 2\theta_0\sigma - \sigma\sin 2\theta_0 \\ = \;& e^{g_1(\sigma)}\sigma\Pi_{i=1}^{N_0}\Big(1-\frac{\sigma^2}{x_i^2}\Big)\Pi_{j=1}^{\infty}\Big(1-\frac{\sigma^2}{\alpha_j^2}\Big)\Big(1-\frac{\sigma^2}{\overline{\alpha_j}^2}\Big) \end{aligned}$$

for some entire function g_1. We claim $g_1(\sigma) \equiv \ln(2\theta_0 - \sin 2\theta_0)$. Hence

$$\begin{aligned} & \sin 2\theta_0\sigma - \sigma\sin 2\theta_0 \\ = \;& (2\theta_0 - \sin 2\theta_0)\sigma\Pi_{i=1}^{N_0}\Big(1-\frac{\sigma^2}{x_i^2}\Big)\Pi_{j=1}^{\infty}\Big(1-\frac{\sigma^2}{\alpha_j^2}\Big)\Big(1-\frac{\sigma^2}{\overline{\alpha_j}^2}\Big). \end{aligned}$$

Proof of the claim. Take ln to the both sides of the above identity and differentiating, then we have the identity

$$\begin{aligned} g_1'(\sigma) \;=\; & \frac{2\theta_0\sigma\cos 2\theta_0\sigma - \sin 2\theta_0\sigma}{\sin 2\theta_0\sigma - \sigma\sin 2\theta_0} - \frac{1}{\sigma} \\ & + \sum_{i=1}^{N_0}\frac{2\sigma}{\sigma^2 - x_i^2} + \sum_{j=1}^{\infty}\frac{2\sigma}{\alpha_j^2 - \sigma^2} + \frac{2\sigma}{\overline{\alpha_j}^2 - \sigma^2}. \end{aligned}$$

Note that g_1' is bounded entire function with removable singularity at $\sigma = 0$ with $g_1'(0) = 0$. By Louville's theorem, this implies that $g_1' \equiv 0$. Hence $g_2 \equiv$ *const*. Observing that $\lim_{\sigma\to 0}\frac{\sin 2\theta_0\sigma - \sigma\sin 2\theta_0}{\sigma} = 2\theta_0 - \sin 2\theta_0$, obtain $e^{g_1(0)} = 2\theta_0 - \sin 2\theta_0$. □

5.3. Construction of K_0

Therefore, ω can be represented by

$$\omega(\sigma) = \frac{\alpha\theta_0}{2}\sigma(\sigma+1)\frac{\Pi_{i=1}^{N_0}\Big(1-\frac{\sigma^2}{x_i^2}\Big)}{\Pi_{i=0}^{N_1}\Big(1-\frac{\sigma^2}{w_i^2}\Big)}\Pi_{j=1}^{\infty}\frac{\Big(1-\frac{\sigma^2}{y_j^2}\Big)}{\Big(1-\frac{\sigma^2}{\beta_j^2}\Big)\Big(1-\frac{\sigma^2}{\overline{\beta_j}^2}\Big)}$$

for the case $\theta_0 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, and

$$\omega(\sigma) = \frac{\alpha(2\theta_0 - \sin 2\theta_0)}{4}\sigma(\sigma+1)\frac{\Pi_{i=1}^{N_0}\left(1-\frac{\sigma^2}{x_i^2}\right)}{\Pi_{i=0}^{N_1}\left(1-\frac{\sigma^2}{w_i^2}\right)}\Pi_{j=1}^{\infty}\frac{\left(1-\frac{\sigma^2}{\alpha_j^2}\right)\left(1-\frac{\sigma^2}{\overline{\alpha_j}^2}\right)}{\left(1-\frac{\sigma^2}{\beta_j^2}\right)\left(1-\frac{\sigma^2}{\overline{\beta_j}^2}\right)}$$

for the case $\theta_0 \neq \frac{\pi}{2}, \frac{3\pi}{2}$.

From now on,we only consider the case $\theta_0 \neq \frac{\pi}{2}$, $\frac{3\pi}{2}$, since the other case can be treated by the same way.

Set $K_0(\sigma)$ as follows;

$$\begin{aligned} &K_0(\sigma)\\ =&\gamma^{\sigma-1/2}\Gamma(\sigma)\Gamma(\sigma+1)\\ &\times \Pi_{i=1}^{N_0}\frac{\Gamma(x_i+\sigma)}{\Gamma(x_i+1-\sigma)}(x_i^2)^{-(\sigma-1/2)}\Pi_{i=0}^{N_1}\frac{\Gamma(w_i+1-\sigma)}{\Gamma(w_i+\sigma)}(w_i^2)^{\sigma-1/2}\\ &\times \Pi_{j=1}^{\infty}\frac{\Gamma(\sigma+\alpha_j)\Gamma(\sigma+\overline{\alpha_j})\Gamma(\beta_j+1-\sigma)\Gamma(\overline{\beta_j}+1-\sigma)}{\Gamma(\alpha_j+1-\sigma)\Gamma(\overline{\alpha_j}+1+\sigma)\Gamma(\sigma+\beta_j)\Gamma(\sigma+\overline{\beta}_j)}\frac{|\alpha_j|^{-4(\sigma-1/2)}}{|\beta_j|^{-4(\sigma-1/2)}}. \end{aligned}$$

Here $\gamma = \alpha\frac{2\theta_0 - \sin 2\theta_0}{4}$.

Formally, $K_0(\sigma)$ satisfies $K_0(\sigma+1) = \omega(\sigma)K_0(\sigma)$. Now, we will show that the infinite product $K_0(\sigma)$ converges to an analytic function and does not vanish in the strip $0 < \mathrm{Re}(\sigma) < 1 + \epsilon_0$ for some positive ϵ_0.

PROPOSITION 5.3.1. *Suppose that the sequences of complex numbers* $\{a_n\}$, $\{b_n\}$, $\{\rho_n\}$, $\{\gamma_n\}$ *satisfy the conditions*

(1) $a_m = O(m)$, $b_m = O(m)$, $\rho_m = O(m)$, $\gamma_m = O(m)$ *as* $m \to \infty$;

(2) *the series* $\Sigma_{m=1}^{\infty}\left|1/\rho_m + 1/\overline{\rho_m} - 1/a_m - 1/\overline{a_m} + 1/b_m + 1/\overline{b_m} - 1/\gamma_m - 1/\overline{\gamma_m}\right|$ *converges.*

Then the following infinite product

$$\Pi_{m=1}^{\infty}\frac{\Gamma(\rho_m+p)\Gamma(\overline{\rho_m}+p)\Gamma(1+b_m-p)\Gamma(1+\overline{b_m}-p)}{\Gamma(a_m+p)\Gamma(\overline{a_m}+p)\Gamma(1+\gamma_m-p)\Gamma(1+\overline{\gamma_m}-p)}\left(\frac{|a_m|^2|b_m|^2}{|\rho_m|^2|\gamma_m|^2}\right)^{p-1/2}e^{R(m)}$$

converges for all p *with the exception of the poles of* $\Gamma(p+\rho_m)$, $\Gamma(p+\overline{\rho_m})$, $\Gamma(1+b_m-p)$ *and* $\Gamma(1+\overline{b_m}-p)$, *where* $R(m) = a_m(\ln a_m - 1) + \overline{a_m}(\ln \overline{a_m} - 1) + \gamma_m(\ln\gamma_m - 1) + \overline{\gamma_m}(\ln\overline{\gamma_m} - 1) + \rho_m(1 - \ln\rho_m) + \overline{\rho_m}(1 - \ln\overline{\rho_m}) + b_m(1 - \ln b_m) + \overline{b_m}(1 - \ln\overline{b_m})$.

Set $\rho_m = \gamma_m = \alpha_m$, $a_m = b_m = \beta_m$. Observe that α_n, $\overline{\alpha_n}$. $\beta_n, \overline{\beta_n} = O(\frac{1}{|n|})$. Then $\rho_m, \gamma_m, a_m, b_m$ satisfies the hypothesis (1) and (2) and $R(m) = 0$. Moreover the each term of the infinite series of (2) is equal to zero. Hence the infinite product of K_0 is well defined, analytic with exceptions of the poles of $\Gamma(\sigma+\alpha_j), \Gamma(\sigma+\overline{\alpha_j}), \Gamma(\sigma+\beta_j), \Gamma(\sigma+\overline{\beta_j})$.

Recalling $\Gamma(z)$ has only simple zeros at the points $z = 0, -1, -2, \ldots$, $K_0(\sigma)$ is analytic except $0, -1, -2, \ldots; -x_i, -x_i - 1, \ldots; w_i + 1, w_i + 2, \ldots; -\alpha_j, -\alpha_j - 1, \ldots; -\overline{\alpha_j}, -\overline{\alpha_j} - 1; \beta_j + 1, \beta_j + 2, \ldots; \overline{\beta_j} + 1, \overline{\beta_j} + 2, \ldots$, and is not zero with exceptions $x_i + 1, x_i + 2, \ldots; -w_i, -w_i - 1, \ldots; \alpha_j + 1, \alpha_j + 2, \ldots; \overline{\alpha_j} + 1, \overline{\alpha_j} + 2, \ldots; -\beta_j, -\beta_j - 1, -\overline{\beta_j}, -\overline{\beta_j} - 1, \ldots$. Hence, by setting $\epsilon_0 = \min\{x_1, w_0, \mathrm{Re}(\alpha_1), \mathrm{Re}(\beta_1)\}$, we observe that K_0 is nonzero analytic in the strip $0 < \mathrm{Re}(\sigma) < 1 + \epsilon_0$.

Hence our infinite product of K_0 is well defined and analytic and does not vanish in the strip $0 < \mathrm{Re}(\sigma) < 1 + \epsilon_0$.

PROPOSITION 5.3.2.

$$\begin{aligned} \ln K_0(\sigma) &= (\sigma - 1/2)\ln\omega(\sigma) + \ln(\sigma+1) + c\sigma + r_1(\sigma), \\ (\ln K_0(\sigma))' &= \ln\omega(\sigma) + r_2(\sigma), \end{aligned}$$

for some $r_1(\sigma) = O(1)$ *and* $r_2(\sigma) = O(1)$ *as* $|\sigma| \to \infty$.

For the proof of Proposition 5.3.1 and Proposition 5.3.2, the following observation is useful. (We refer [1] for details.)

LEMMA 5.3.3.

$$\begin{aligned} \ln\Gamma(z) &= (z - 1/2)\ln z - z + \frac{1}{2}\ln 2\pi + \frac{1}{12z} + \xi_1(z), \\ (\ln\Gamma(z))' &= \ln z - \frac{1}{2z} + \xi_2(z), \\ (\ln\Gamma(z))'' &= \frac{1}{z} + \xi_3(z), \end{aligned}$$

where $\xi_1(z) = O(\frac{1}{z^3})$, $\xi_2(z) = O(\frac{1}{z^2})$, $\xi_3(z) = O(\frac{1}{z^2})$ *as* $|z| \to \infty$, $|\arg z| < \pi$, *and* $|\xi_1(z)| \le c_3(\delta)/|z|^3$, $|\xi_2(z)| \le c_4(\delta)/|z|^2$, *if* $|\arg z| \le \pi - \delta$.

Proof of Proposition 5.3.1. The arguments in this proof is based on the proof of proposition 5.1 of [18].

The convergence of infinite product of $A(p)$ is equivalent to the convergence of the infinite series

$$\begin{aligned} \ln A(p) &= \sum_{m=1}^{\infty} \ln \frac{\Gamma(\rho_m + p)\Gamma(\overline{\rho_m} + p)\Gamma(1 + b_m - p)\Gamma(1 + \overline{b_m} - p)}{\Gamma(a_m + p)\Gamma(\overline{a_m} + p)\Gamma(1 + \gamma_m - p)\Gamma(1 + \overline{\gamma_m} - p)} \\ &\quad + (p - 1/2)\ln\frac{|a_m|^2|b_m|^2}{|\rho_m|^2|\gamma_m|^2} + R(m). \end{aligned}$$

For a given p, take n so large that for $m \geq n$

$$|p| \leq \min\{|a_m|, |b_m|, |\rho_m|, |\gamma_m|\}.$$

Now, it is enough to show the convergence of the infinite series

$$\begin{aligned} B(p) &= \sum_{m=n}^{\infty} \ln \frac{\Gamma(\rho_m + p)\Gamma(\overline{\rho_m} + p)\Gamma(1 + b_m - p)\Gamma(1 + \overline{b_m} - p)}{\Gamma(a_m + p)\Gamma(\overline{a_m} + p)\Gamma(1 + \gamma_m - p)\Gamma(1 + \overline{\gamma_m} - p)} \\ &\quad + (p - 1/2) \ln \frac{|a_m|^2 |b_m|^2}{|\rho_m|^2 |\gamma_m|^2} + R(m). \end{aligned}$$

From Taylor series expansion, we have the equality

$$\ln \Gamma(z \pm p) = \ln \Gamma(z) + \sum_{s=1}^{\infty} \Psi^{(s-1)}(z) \frac{(\pm p)^s}{s!},$$

where $\Psi(z) = \frac{d}{dz} \ln \Gamma(z)$. Apply the above series expansion to each term of the infinite series, then we have the equality

$$\begin{aligned} & B(p) \\ = & \sum_{m=n}^{\infty} \ln \frac{\Gamma(\rho_m)\Gamma(\overline{\rho_m})\Gamma(1 + b_m)\Gamma(1 + \overline{b_m})}{\Gamma(a_m)\Gamma(\overline{a_m})\Gamma(1 + \gamma_m)\Gamma(1 + \overline{\gamma_m})} \\ & + (p - 1/2) \ln \frac{|a_m|^2 |b_m|^2}{|\rho_m|^2 |\gamma_m|^2} + R(m) \\ & + p[\Psi(\rho_m) + \Psi(\overline{\rho_m}) - \Psi(1 + b_m) - \Psi(1 + \overline{b_m}) \\ & - \Psi(a_m) - \Psi(\bar{a}_m) + \Psi(\gamma_m + 1) + \Psi(\overline{\gamma_m} + 1)] \\ & + \frac{p^2}{2}[\Psi^{(1)}(\rho_m) + \Psi^{(1)}(\overline{\rho_m}) + \Psi^{(1)}(1 + b_m) + \Psi^{(1)}(1 + \overline{b_m}) \\ & - \Psi^{(1)}(a_m) - \Psi^{(1)}(\bar{a}_m) - \Psi^{(1)}(\gamma_m + 1) - \Psi^{(1)}(\overline{\gamma_m} + 1)] \\ & + \sum_{m=n}^{\infty} \sum_{s=3}^{\infty} \Psi^{(s-1)}(\rho_m) \frac{(p)^s}{s!} + \sum_{m=n}^{\infty} \sum_{s=3}^{\infty} \Psi^{(s-1)}(\overline{\rho_m}) \frac{(p)^s}{s!} \\ & + \sum_{m=n}^{\infty} \sum_{s=3}^{\infty} \Psi^{(s-1)}(b_m + 1) \frac{(-p)^s}{s!} + \sum_{s=3}^{\infty} \Psi^{(s-1)}(\bar{b}_m + 1) \frac{(-p)^s}{s!} \\ & - \sum_{m=n}^{\infty} \sum_{s=3}^{\infty} \Psi^{(s-1)}(a_m) \frac{(p)^s}{s!} - \sum_{s=3}^{\infty} \Psi^{(s-1)}(\bar{a}_m) \frac{(p)^s}{s!} \\ & - \sum_{m=n}^{\infty} \sum_{s=3}^{\infty} \Psi^{(s-1)}(\gamma_m + 1) \frac{(-p)^s}{s!} - \sum_{s=3}^{\infty} \Psi^{(s-1)}(\overline{\gamma_m} + 1) \frac{(-p)^s}{s!} \\ = & A_1(p) + A_2(p) + A_3(p) + \cdots + A_8(p) + A_9(p). \end{aligned}$$

Here $A_k(p)$ denote the k-th infinite series. The convergence of the form

$$\sum_{m=n}^{\infty}\sum_{s=3}^{\infty}\Psi^{(s-1)}(z)\frac{(\pm p)^s}{s!}$$

follows from the result of Barnes. Hence it remains only to show the convergence of the series $A_1(p)$. Using recurrence formulas $\Psi(z+1) = \Psi(z) + 1/z$, $\Psi^{(1)}(1+z) = \Psi^{(1)}(z) - 1/z^2$ and the asymptotic formula from Lemma 5.3.3, $\ln\Gamma(z) = (z-1/2)\ln z - z + \frac{1}{2}\ln 2\pi + \frac{1}{12z} + O(\frac{1}{z^3})$, $\Psi(z) = \log z - \frac{1}{2z} + \xi_2(z)$, $\Psi^{(1)}(z) = \frac{1}{z} + \xi_3(z)$, for $\xi_2(z) = O(\frac{1}{z^3})$, $\xi_3(z) = O(\frac{1}{z^2})$. Hence $A_1(p)$ has the representation

$$A_1(p) = \frac{p^2}{2}B_1 - \frac{p}{2}B_2 + B_3,$$

where

$$\begin{aligned}
B_1 &= \sum_{m=n}^{\infty}\left[\frac{1}{\rho_m} + \frac{1}{\overline{\rho_m}} - \frac{1}{a_m} - \frac{1}{\bar{a}_m} + \frac{1}{b_m} + \frac{1}{\bar{b}_m} - \frac{1}{\gamma_m} - \frac{1}{\overline{\gamma_m}}\right] + O(L),\\
B_2 &= B_1 + O(L),\\
B_3 &= \frac{1}{12}B_1 + O(L),\\
L &= \sum_{m=n}^{\infty}\left[\frac{1}{\rho_m^2} + \frac{1}{\overline{\rho_m}^2} + \frac{1}{a_m^2} + \frac{1}{\bar{a}_m^2} + \frac{1}{b_m^2} + \frac{1}{\bar{b}_m^2} + \frac{1}{\gamma_m{}^2} + \frac{1}{\overline{\gamma_m}^2}\right].
\end{aligned}$$

Now, the absolute convergence of the series B_1 and L follows from the hypotheses (1), (2) of the proposition. Hence we have the absolute convergence of B_2, B_3, too. This completes the proof of the convergence of the infinite product A. □

Proof of Proposition 5.3.2. 1. Taking ln to K_0, we have the identity

$$\begin{aligned}
&\ln K_0(\sigma)\\
=\ &(\sigma - 1/2)\ln\gamma + \ln\Gamma(\sigma) + \ln\Gamma(\sigma+1)\\
&+\sum_{i=1}^{N_0}\left[\ln\Gamma(x_i+\sigma) - \ln\Gamma(x_i+1-\sigma) - (\sigma-1/2)\ln x_i^2\right]\\
&-\sum_{i=0}^{N_1}\left[\ln\Gamma(w_i+\sigma) - \ln\Gamma(w_i+1-\sigma) - (\sigma-1/2)\ln w_i^2\right]
\end{aligned}$$

$$+\sum_{j=1}^{\infty}\Big[\ln\Gamma(\sigma+\alpha_j)+\ln\Gamma(\sigma+\overline{\alpha_j})+\ln\Gamma(\beta_j+1-\sigma)$$
$$+\ln\Gamma(\overline{\beta_j}+1-\sigma)-\ln\Gamma(\alpha_j+1-\sigma)-\ln\Gamma(\overline{\alpha_j}+1-\sigma)$$
$$-\ln\Gamma(\beta_j+\sigma)-\ln\Gamma(\overline{\beta_j}+\sigma)-(\sigma-1/2)\ln\frac{|\alpha_j|^4}{|\beta_j|^4}\Big].$$

From Lemma 5.3.3, we have $\ln\Gamma(z)=(z-1/2)\ln z-z+\frac{1}{2}\ln 2\pi+\frac{1}{12z}+\xi_1(z)$, for $\xi_1(z)=O(\frac{1}{z^3})$. Using the above asymptotic behavior and the property $\Gamma(z+1\pm\sigma)=(z\pm\sigma)\Gamma(z\pm\sigma)$, the above identity is rearranged by

$$\begin{aligned}&\ln K_0(\sigma)\\ =\;&(\sigma-1/2)\ln\omega(\sigma)-2(N_0+1)\sigma+\ln(\sigma+1)+\ln 2pi\\ &+Q_1(\sigma)+Q_2(\sigma)+Q_3(\sigma)+Q_4(\sigma),\end{aligned}$$

where

$$Q_1(\sigma)$$
$$=\sum_{j=1}^{\infty}\Big[\alpha_j\ln\frac{\sigma+\alpha_j}{\alpha_j-\sigma}+\overline{\alpha_j}\ln\frac{\sigma+\overline{\alpha_j}}{\overline{\alpha_j}-\sigma}+\beta_j\ln\frac{\beta_j-\sigma}{\beta_j+\sigma}+\overline{\beta_j}\ln\frac{\overline{\beta_j}-\sigma}{\overline{\beta_j}+\sigma}\Big],$$
$$Q_2(\sigma)$$
$$=\frac{1}{12}\sum_{j=1}^{\infty}\Big[\frac{1}{\sigma+\alpha_j}-\frac{1}{\alpha_j-\sigma}+\frac{1}{\overline{\alpha_j}+\sigma}-\frac{1}{\overline{\alpha_j}-\sigma}$$
$$-\frac{1}{\beta_j+\sigma}+\frac{1}{\beta_j-\sigma}-\frac{1}{\overline{\beta_j}+\sigma}+\frac{1}{\overline{\beta_j}-\sigma}\Big]$$
$$Q_3(\sigma)$$
$$=\sum_{j=1}^{\infty}\Big[\xi_1(\sigma+\alpha_j)+\xi_1(\sigma+\overline{\alpha_j})+\xi_1(\beta_j-\sigma)+\xi_1(\overline{\beta_j}-\sigma)$$
$$-\xi_1(\alpha_j-\sigma)-\xi_1(\overline{\alpha_j}-\sigma)-\xi_1(\beta_j+\sigma)+\xi_1(\overline{\beta_j}+\sigma)\Big]$$
$$Q_4(\sigma)$$
$$=\sum_{i=1}^{N_0}x_i\ln\frac{x_i+\sigma}{x_i-\sigma}-\sum_{i=0}^{N_1}w_i\ln\frac{w_i+\sigma}{w_i-\sigma}$$
$$+\frac{1}{12}(\frac{1}{\sigma}+\frac{1}{\sigma+1})+\xi_1(\sigma)+\xi_1(\sigma+1)$$

$$+\frac{1}{12}\sum_{i=1}^{N_0}\Big[\frac{1}{x_i+\sigma}-\frac{1}{x_i-\sigma}\Big]-\frac{1}{12}\sum_{i=0}^{N_1}\Big[\frac{1}{w_i+\sigma}-\frac{1}{w_i-\sigma}\Big]$$

$$+\sum_{i=1}^{N_0}\Big[\xi_1(x_i+\sigma)-\xi_1(x_i-\sigma)\Big]-\sum_{i=1}^{N_1}\Big[\xi_1(w_i+\sigma)-\xi_1(w_i-\sigma)\Big].$$

It is clear $Q_4(\sigma)=O(1)$. Hence it only remains to show the infinity behavior of Q_1, Q_2 and Q_3.

Recall $\frac{\sin 2\sigma\theta_0-\sigma\sin 2\theta_0}{\cos^2\theta_0\sigma-\sigma^2\sin^2\theta_0}=(2\theta_0-\sin 2\theta_0)\sigma\frac{\Pi_{i=1}^{N_0}(1-\frac{\sigma^2}{x_i^2})}{\Pi_{i=0}^{N_1}(1-\frac{\sigma^2}{w_i^2})}\Pi_{j=1}^{\infty}\frac{(1-\frac{\sigma^2}{\alpha_j^2})(1-\frac{\sigma^2}{\overline{\alpha_j}^2})}{(1-\frac{\sigma^2}{\beta_j^2})(1-\frac{\sigma^2}{\overline{\beta_j}^2})}$.

Take ln to the both sides, and differentiate. Then, we have

$$\begin{aligned}&\frac{2\theta_0\cos 2\theta_0\sigma-\sin 2\theta_0}{\sin 2\theta_0\sigma-\sigma\sin 2\theta_0}+\frac{\theta_0\sin 2\sigma\theta_0+2\sigma\sin^2\theta_0}{\cos^2\theta_0\sigma-\sigma^2\sin^2\theta_0}\\ &-\frac{1}{\sigma}-\sum_{i=1}^{N_0}\frac{2\sigma}{\sigma^2-x_i^2}+\sum_{i=0}^{N_1}\frac{2\sigma}{\sigma^2-w_i^2}\\ =&-\sum_{j=1}^{\infty}\Big(\frac{2\sigma}{\alpha_j^2-\sigma^2}+\frac{2\sigma}{\overline{\alpha_j}^2-\sigma^2}-\frac{2\sigma}{\beta_j^2-\sigma^2}-\frac{2\sigma}{\overline{\beta_j}^2-\sigma^2}\Big)\\ =&\ 12Q_2(\sigma).\end{aligned}$$

Since the left hand side of the above identity is $O(1)$ for $\sigma_2\to\infty$, we conclude $Q_2(\sigma)=O(1)$ for $\sigma_2\to\infty$.

Since $\xi_1(z)=O(1/z^3)$ and $\frac{1}{|z\pm\sigma|}\le\frac{1}{|\text{Re}(z\pm\sigma)|}$, we note that

$$\sum_{j=1}^{\infty}|\xi_1(\alpha_j\pm\sigma)|\le\sum_{j=1}^{\infty}\frac{1}{|\text{Re}(\alpha_j)+\sigma_1|^3}\le c\sum_{j=1}^{\infty}\frac{1}{|n|^3}<\infty$$

for $|\sigma_2|\to\infty$. The same holds for $\xi_1(\sigma+\overline{\alpha_j}),\ \xi_1(\sigma+\beta_j),\ \xi_1(\sigma+\overline{\beta_j})$. Hence, we conclude $Q_3(\sigma)=O(1)$.

Now, we claim that $Q_1'(\sigma)=2(N_0-N_1)+O(\frac{1}{|\sigma|^2})$ as $|\sigma_2|\to\infty$. Then it follows that $Q_1(\sigma)=2(N_0-N_1)\sigma+O(1)$. □

Proof of the claim. By differentiating the both sides of $Q_1(\sigma)$, we obtain the following identity

$$\begin{aligned}Q_1'(\sigma)=\sum_{j=1}^{\infty}&\alpha_j\big(\frac{1}{\sigma+\alpha_j}+\frac{1}{\alpha_j-\sigma}\big)+\overline{\alpha_j}\big(\frac{1}{\sigma+\overline{\alpha_j}}+\frac{1}{\overline{\alpha_j}-\sigma}\big)\\ &-\beta_j\big(\frac{1}{\sigma+\beta_j}+\frac{1}{\beta_j-\sigma}\big)-\overline{\beta_j}\big(\frac{1}{\sigma+\overline{\beta_j}}+\frac{1}{\overline{\beta_j}-\sigma}\big).\end{aligned}$$

On the other hand, observe that

$$\begin{aligned}
&\alpha_j(\frac{1}{\sigma+\alpha_j}+\frac{1}{\alpha_j-\sigma})+\overline{\alpha_j}(\frac{1}{\sigma+\overline{\alpha_j}}+\frac{1}{\overline{\alpha_j}-\sigma})\\
&-\beta_j(\frac{1}{\sigma+\beta_j}+\frac{1}{\beta_j-\sigma})-\overline{\beta_j}(\frac{1}{\sigma+\overline{\beta_j}}+\frac{1}{\overline{\beta_j}-\sigma})\\
= &-\sigma\Big(\frac{1}{\sigma+\alpha_j}-\frac{1}{\alpha_j-\sigma}+\frac{1}{\sigma+\overline{\alpha_j}}-\frac{1}{\overline{\alpha_j}-\sigma}\\
&-\frac{1}{\sigma+\beta_j}+\frac{1}{\beta_j-\sigma}-\frac{1}{\sigma+\overline{\beta_j}}+\frac{1}{\overline{\beta_j}-\sigma}\Big)\\
= &-12\sigma Q_2(\sigma)\\
= &-\sigma\Big(\frac{\sin 2\theta_0-2\theta_0\cos 2\sigma\theta_0}{\sigma\sin 2\theta_0-\sin 2\sigma\theta_0}+\frac{\theta_0\sin 2\sigma\theta_0+2\sigma\sin^2\theta_0}{\cos^2\sigma\theta_0-\sigma^2\sin^2\theta_0}\Big)\\
&+1+\sum_{i=1}^{N_0}\frac{2\sigma^2}{\sigma^2-x_i^2}-\sum_{i=0}^{N_1}\frac{2\sigma^2}{\sigma^2-w_i^2}\\
= &\ 2(N_0-N_1)+I+II.
\end{aligned}$$

Here $I=\sum_{i=1}^{N_0}\frac{2x_i^2}{\sigma^2-x_i^2}-\sum_{i=0}^{N_1}\frac{2w_i^2}{\sigma^2-w_i^2}$,

$$II=-\sigma\Big(\frac{\sin 2\theta_0-2\theta_0\cos 2\sigma\theta_0}{\sigma\sin 2\theta_0-\sin 2\sigma\theta_0}+\frac{\theta_0\sin 2\sigma\theta_0+2\sigma\sin^2\theta_0}{\cos^2\sigma\theta_0-\sigma^2\sin^2\theta_0}\Big).$$

Clearly, $I=O(\frac{1}{|\sigma|^2})$. Direct computation shows that

$$\begin{aligned}
II &= -\sigma\Big[\frac{\theta_0-\frac{\sin 2\theta_0}{2}+(\theta_0-\frac{\sin 2\theta_0}{2}-2\theta_0\sigma^2\sin^2\theta_0)\cos 2\theta_0\sigma}{(\sigma\sin 2\theta_0-\sin 2\sigma\theta_0)(\cos^2\sigma\theta_0+\sigma^2\sin^2\theta_0)}\\
&\quad+\frac{2\sin^2\theta_0-\theta_0\sin 2\theta_0)\sigma\sin 2\theta_0\sigma-\sigma^2\sin^2\theta_0\sin 2\theta_0}{(\sin 2\sigma\theta_0-\sigma\sin 2\theta_0)(\cos^2\sigma\theta_0-\sigma^2\sin^2\theta_0)}\Big]\\
&= O(\frac{1}{|\sigma|^2}).
\end{aligned}$$

2. Taking differentiation of $\ln K_0(\sigma)$, we have the identity

$$\begin{aligned}
&(\ln K_0(\sigma))'=\frac{K_0'(\sigma)}{K_0(\sigma)}\\
= &\ \ln\gamma+(\ln\Gamma(\sigma))'+(\ln\Gamma(\sigma+1))'\\
&+\sum_{i=1}^{N_0}(\ln\Gamma(x_i+\sigma))'-(\ln\Gamma(x_i+1-\sigma))'-\ln x_i^2
\end{aligned}$$

$$-\sum_{i=0}^{N_1}(\ln\Gamma(w_i+\sigma))' - (\ln\Gamma(w_i+1-\sigma))' - \ln w_i^2$$
$$+\sum_{j=1}^{\infty}\Big[(\ln\Gamma(\sigma+\alpha_j))' + (\ln\Gamma(\sigma+\overline{\alpha_j}))' - (\ln\Gamma(\alpha_j+1-\sigma))'$$
$$-(\ln\Gamma(\overline{\alpha_j}+1-\sigma))' - (\ln\Gamma(\sigma+\beta_j))' - (\ln\Gamma(\sigma+\overline{\beta_j}))'$$
$$+(\ln\Gamma(\beta_j+1-\sigma))' + (\ln\Gamma(\overline{\beta_j}+1-\sigma))' - \ln\frac{|\alpha_j|^4}{|\beta_j|^4}\Big]. \tag{5.3}$$

Since $\Gamma(z+1\pm\sigma) = (z\pm\sigma)\Gamma(z\pm\sigma)$, we observe that $(\ln\Gamma(z+1\pm\sigma))' = \frac{\pm1}{z\pm\sigma} \pm (\ln\Gamma)'(z\pm\sigma)$. Recall $(\ln\Gamma(z))' = \ln z - \frac{1}{2z} + \xi_2(z)$ from Lemma 5.3.3. Hence, the right hand side of (5.3) can be rewritten by

$$(\ln K_0(\sigma))' \;=\; \ln\omega(\sigma) - \frac{1}{2}Q_5 - \frac{1}{2}Q_6 + Q_7,$$

where

$$Q_5 = \frac{1}{\sigma} + \frac{1}{\sigma+1} + \sum_{i=1}^{N_0}\frac{1}{x_i+\sigma} + \frac{1}{x_i-\sigma} - \sum_{i=0}^{N_1}\frac{1}{w_i+\sigma} + \frac{1}{w_i-\sigma}$$
$$+\xi_2(\sigma) + \xi_2(\sigma+1) + \sum_{i=1}^{N_0}\xi_2(x_i+\sigma) + \xi_2(x_i-\sigma),$$

$$Q_6 = \sum_{j=1}^{\infty}\frac{1}{\sigma+\alpha_j} - \frac{1}{\alpha_j-\sigma} + \frac{1}{\sigma+\overline{\alpha_j}} - \frac{1}{\overline{\alpha_j}-\sigma}$$
$$-\frac{1}{\sigma+\beta_j} + \frac{1}{\beta_j-\sigma} - \frac{1}{\sigma+\overline{\beta_j}} + \frac{1}{\overline{\beta_j}-\sigma}.$$

$$Q_7 = \sum_{j=1}^{\infty}\xi_2(\sigma+\alpha_j) + \xi_2(\sigma+\overline{\alpha_j}) + \xi_2(\alpha_j-\sigma) + \xi_2(\overline{\alpha_j}-\sigma)$$
$$+\xi_2(\sigma+\beta_j) + \xi_2(\sigma+\overline{\beta_j}) + \xi_2(\beta_j-\sigma) + \xi_2(\overline{\beta_j}-\sigma).$$

Clearly, $Q_5 = O(1)$. Since $\frac{1}{|z\pm\sigma|^2} \le \frac{1}{|\mathrm{Re}(z\pm\sigma)|^2}$, $\sum_{j=1}^{\infty}|\xi_2(\alpha_j\pm\sigma)| \le c\sum_{j=1}^{\infty}\frac{1}{|\mathrm{Re}(\alpha_j+\sigma)|^2} \le c\sum_{j=1}^{\infty}\frac{1}{|n|^2} < \infty$. Hence $Q_7 = O(1)$. On the other hand, we observe that

$$Q_6 = 12Q_2(\sigma) = O(1).$$

□

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Division of Mathematics
College of Science
Mokpo National University
Muan 534-729, Korea
E-mail: bumjajin@hanmail.net