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SLANT SUBMANIFOLDS OF AN ALMOST PRODUCT RIEMANNIAN MANIFOLD

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Abstract. In this paper, we study both slant and semi-slant submanifolds of an almost product Riemannian manifold. We give characterization theorems for slant and semi-slant submanifolds and investigate special class of slant submanifolds which are product version of Kaehlerian slant submanifold. We also obtain integrability conditions for the distributions which are involved in the definition of a semi-slant submanifold. Finally, we prove a theorem on the geometry of leaves of distributions under a condition.

1. Introduction

The geometry of slant submanifolds was initiated by B. Y. Chen, as a generalization of both holomorphic and totally real submanifolds in complex geometry [3], [4]. Since then, many mathematicians have studied these submanifolds. In particular, N. Papaghiuc [7] introduced semi-slant submanifolds. A. Lotta [5], [6], defined and studied slant submanifolds in contact geometry. J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez studied slant, semi-slant and bi-slant submanifolds in contact geometry [1], [2].

In this paper, we introduce slant and semi-slant submanifolds of an almost product Riemannian manifold. In section 2, we review some formulas and definitions for an almost product Riemannian manifold and their submanifolds. In section 3, we define slant submanifolds for an almost product Riemannian manifold and give characterization theorems for a slant submanifold. We observe that slant surfaces of an almost product manifold quite different from slant surfaces of complex and contact manifolds. In section 4, we define and study semi-slant submanifolds in

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an almost product Riemannian manifold. We give characterization theorems for a semi-slant submanifold. After we find integrability conditions of the distributions, we investigate a special class of a slant submanifolds satisfying $\nabla T = 0$ which are product version of Kaehlerian slant submanifolds for an almost Hermitian manifold and obtain a necessary and sufficient condition for $\nabla T = 0$ in terms of the shape operator on a semi-slant submanifold. In the last part of section 4, we obtain that the distributions are integrable and their leaves are totally geodesic in semi-slant submanifold under the condition $\nabla T = 0$. Finally, the paper contains several examples.

2. Preliminaries

Let $\bar{M}$ be an m-dimensional manifold with a tensor of type $(1,1)$ such that

$$F^2 = I, \tag{2.1}$$

where I denotes the identity transformation. Then we say that $\bar{M}$ is an almost product manifold with almost product structure F. If we put

$$P = \frac{1}{2}(I + F),\ Q = \frac{1}{2}(I - F) \tag{2.2}$$

then we have

$$P + Q = I, P^2 = P,\ Q^2 = Q,\ PQ = QP = 0 \tag{2.3}$$

and

$$F = P - Q. \tag{2.4}$$

If an almost product manifold $\bar{M}$ admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y) \tag{2.5}$$

for any vector fields X and Y on $\bar{M}$, then $\bar{M}$ is called an almost product Riemannian manifold [9]. Let $\bar{\nabla}$ denotes the Levi Civita connection on $\bar{M}$ with respect to g. In particular, if $\bar{\nabla}_X F = 0$, $X \in T\bar{M}$, where $T\bar{M}$ denotes the set of all vector fields of $\bar{M}$, then $\bar{M}$ is called a locally product Riemannian manifold.

Let M be a Riemannian manifold isometrically immersed in $\bar{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let TM be the Lie algebra of vector fields in M and $TM^{\perp}$ the set of all vector fields normal to M. Denote by ∇ the Levi-Civita connection of M. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.6}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{2.7}$$

for any $X, Y \in TM$ and any $N \in TM^{\perp}$, where $\nabla^{\perp}$ is the connection in the normal bundle $TM^{\perp}$, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form and the shape operator A are related by $g(A_N X, Y) = g(h(X,Y), N)$.

For any $X \in TM$ we write

$$FX = TX + \omega X, \tag{2.8}$$

where TX is the tangential component of FX and ωX is the normal component of FX. Similarly, for any vector field normal to M, We put

$$FN = BN + CN, \tag{2.9}$$

where BN and CN are the tangential and the normal components of FN, respectively. From (2.5) and (2.8) we have

$$g(TX,Y) = g(X,TY) \tag{2.10}$$

for any $X, Y \in TM$.

The submanifold M is said to be $F-$invariant if ω is identically zero, i.e., $FX \in TM$, for any $X \in TM$. On the other hand, M is said to be $F-$ anti -invariant submanifold if T is identically zero, i.e., $FX \in TM^{\perp}$, for any $X \in TM$ [9].

3. Slant submanifolds

In this section, we study slant immersions of an almost product Riemannian manifold $\bar{M}$. First, we present definition of a slant submanifold of an almost product Riemannian manifold following Chen's ([3]) definition for a Hermitian manifold. Let M be a Riemannian manifold isometrically immersed in an almost product Riemannian manifold $\bar{M}$. For each nonzero vector X tangent to M at x the angle $\theta(X)$, $0 \leq \theta(X) \leq \frac{\pi}{2}$ between FX and T_xM is called the wirtinger angle of X. Then M is said to be slant if the angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in TM$. The angle θ of a slant immersion is called the slant angle of the immersion. Thus, the $F-$invariant and $F-$

anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither $F-$invariant nor $F-$ anti-invariant is called a proper slant immersion. Finally, we say that a slant submanifold of $\bar{M}$ is a product slant if the endomorphism T is parallel.

Next we give an useful characterization of slant submanifolds in an almost product Riemannian manifold :

THEOREM 3.1. *Let M be a submanifold of an almost product Riemannian manifold $\bar{M}$. Then, M is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda I. \tag{3.1}$$

Note that, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Proof. Let us suppose that M is a slant submanifold of $\bar{M}$. Then $\cos\theta(X)$ is independent $x \in M$ and $X \in T_xM$. Thus, using (2.5) and (2.8), we obtain

$$\cos\theta(X) = \frac{g(TX,FX)}{|X||TX|} = \frac{g(FTX,X)}{|X||TX|} = \frac{g(T^2X,X)}{|X||TX|}. \tag{3.2}$$

On the other hand, we have $\cos\theta(X) = \frac{|TX|}{|FX|}$, thus using (3.2) we derive $\cos^2\theta(X) = \frac{g(T^2X,X)}{|X|^2}$. Hence we obtain $T^2X = \lambda X$, $\lambda \in [0,1]$.

Conversely, we suppose that $T^2X = \lambda X$ for any $X \in TM$ and $\lambda \in [0,1]$. Then, in a similar way, we have $\cos^2\theta(X) = \lambda$, hence $\theta(X)$ is constant on M. □

LEMMA 3.1. *Let M be a slant submanifold of an almost product Riemannian manifold $\bar{M}$. Then, for any $X,Y \in TM$, we have*

$$g(TX,TY) = \cos^2\theta g(X,Y) \tag{3.3}$$

and

$$g(\omega X,\omega Y) = \sin^2\theta g(X,Y). \tag{3.4}$$

Proof. Substituting Y by TY in (2.10) we have $g(TX,TY) = g(X, T^2Y)$ for any $X,Y \in TM$. Then (3.1) give us (3.3). Then the proof of (3.4) follows from (2.5) and (2.8). □

Let $R^m = R^{n_1} \times R^{n_2}$ be the Euclidean space of dimension $m = n_1 + n_2$ and endowed with the Euclidean metric. Let F be a product structure defined on $R^m = R^{n_1} \times R^{n_2}$ by

$$F(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = (\frac{\partial}{\partial x_i}, -\frac{\partial}{\partial x_j}),\ i \in \{1, \dots, n_1\},\ j \in \{n_1 + 1, \dots, m\}.$$

Now, we give some examples of slant submanifolds in almost product Riemannian manifolds.

EXAMPLE 3.1. Consider in $R^4 = R^2 \times R^2$ the submanifold given by

$$x(u, v) = (u \cos\theta,\ v \cos\theta,\ u \sin\theta,\ v \sin\theta)$$

for any $\theta > 0$. Then M is a slant plane with slant angle 2θ.

EXAMPLE 3.2. For any $u, v \in (0, \frac{\pi}{2})$ and positive constant $k \neq 1$,

$$x(u, v) = (u, v, -k \sin u, -k \sin v, k \cos u, k \cos v)$$

defines a proper slant surface in $R^6 = R^2 \times R^4$ with slant angle $\theta = \cos^{-1}(\frac{1-k^2}{1+k^2})$.

EXAMPLE 3.3. Consider a submanifold M in $R^4 = R^2 \times R^2$ given by

$$x(u_1, u_2) = (u_1 + u_2, u_1 + u_2, \sqrt{2}u_2, \sqrt{2}u_1).$$

Then M is a slant submanifold with the slant angle $\frac{\pi}{3}$.

EXAMPLE 3.4. Let us consider the almost product manifold R^4 with coordinates (x_1, x_2, y_1, y_2) and product structure

$$F(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i},\ F(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial x_i}.$$

Then consider a submanifold M given by

$$x(u, v) = (u \cos\theta, u \sin\theta, v, 0).$$

It is easy to see that M is a slant submanifold with the slant angle θ.

EXAMPLE 3.5. Consider the almost product Riemann manifold $R^7 = R^4 \times R^3$ with coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3)$ and product structure F given by

$$F(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial x_i},\ F(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial y_i}.$$

Then, consider a submanifold M given by

$$x(u_1, u_2, u_3) = (\sqrt{2}u_1, u_2, u_3, u_2 + u_3, u_1 + u_2 + u_3, u_1, u_2 + u_3).$$

Then it is easy to see that M is a slant submanifold with slant angle $\cos^{-1}(\frac{1}{4})$.

REMARK 3.1. It is known that in complex geometry, proper slant submanifolds are always even dimensional, while in contact geometry, proper slant submanifolds are always odd dimensional. However, in the product Riemann manifolds,the situation is quiet different from both geometries. For instance, in Examples 3.1–3.4, the slant submanifolds are even dimensional, while in Example 3.5, the proper slant submanifold is odd dimensional. Thus, one can conclude that there are even and odd dimensional proper slant submanifolds of almost product Riemann manifolds.

Let M be a Riemannian manifold isometrically immersed in an almost product Riemannian manifold $\bar{M}$. Then, from (2.10), we conclude that the T and T^2 are symmetric operators on the tangent space T_xM at $x \in M$. If λ is the eigenvalue of T^2 at $x \in M$, since T^2 is a composition of an isometry and projection, we have $\lambda \in [0,1]$. On the other hand, since T^2 is a self-adjoint endomorphism of TM, each tangent space T_xM of M at $p \in M$, admits an orthogonal direct decomposition of eigenspace of T^2 :

$$T_xM = D_x^1 \oplus \cdots \oplus D_x^{k(x)}.$$

If $\lambda_i \neq 0$ then the corresponding eigenspaces of D_x^i is invariant under the endomorphism T.

Now, we set $Q = T^2$ and define by

$$(\nabla_X Q)Y = \nabla_X QY - Q\nabla_X Y$$

for any $X, Y \in TM$.

THEOREM 3.2. *Let M be a submanifold of an almost product manifold $\bar{M}$. Then Q is parallel if and only if*

1. *Each eigenvalue of Q is constant on M.*
2. *Each distribution D^i (associated with the eigen valueλ_i) is completely integrable.*
3. *M is locally the Riemannian product $M_1 \times M_2 \times \cdots \times M_k$ of the leaves of the distributions.*

The proof is same as that of Lemma 3.1 in [4], p.21, so we omit it here.

THEOREM 3.3. *Let M be a submanifold of an almost product manifold $\bar{M}$. Then $\nabla T = 0$ if and only if M is locally product $M_1 \times \cdots \times M_k$, where each M_i is either F-invariant submanifold with $\nabla^i T_i = 0$, $F-$ anti-invariant submanifold or a product slant submanifold of $\bar{M}$, where $T_i = T\mid_{TM_i}$ and ∇^i is the Riemannian metric of M_i.*

Proof. Since T is parallel, Q is parallel. Thus, from Theorem 3.2, M is locally the Riemannian product $M_1 \times \cdots \times M_k$ of leaves of distributions defined by eigen vectors of Q and each eigenvalue λ_i is constant on M. If $\lambda_i = 0$, the corresponding leaf of M_i is an $F-$ anti-invariant. If $\lambda_i = 1$ then M_i is an $F-$ invariant submanifold. If $\lambda \neq 0, 1$, since D_i is invariant under the endomorphism T, following the proof of the Theorem 3.1, we obtain

$$\begin{aligned} \cos\theta(X) &= \frac{g(FX,TX)}{\mid X \mid\mid TX \mid} \\ &= \frac{g(X,FTX)}{\mid X \mid\mid TX \mid} \\ &= \frac{g(X,T^2X)}{\mid X \mid\mid TX \mid} \\ &= \lambda_i \frac{\mid X \mid}{\mid TX \mid}. \end{aligned}$$

On the other hand, we have $\cos\theta(X) = \frac{|TX|}{|FX|} = \frac{|TX|}{|X|}$. Thus we get $\cos\theta(X) = \sqrt{\lambda_i}$. So, Wirtinger angle $\theta(X)$ is a constant and $\lambda_i \neq 0, 1$. Thus M_i is a proper slant submanifold. Let us assume that $\lambda_i \neq 0$. If we put $T_i = T\mid_{TM_i}$, then since M_i is invariant under endomorphism T, T_i is an almost product structure (Theorem 3.1, p.425, [9]). Let ∇^i denote the Riemannian connection of M_i, since M_i is totally geodesic, we have $(\nabla^i_X T_i)Y = (\nabla_X T)Y = 0$ for any $X, Y \in TM$. Moreover if M_i is proper slant, then M_i is a product slant submanifold. The converse is clear. □

We note that $g(FX,X) \neq 0$ for any unit vector field $X \in \Gamma(T\bar{M})$ in a product Riemann manifold, in general, contrary to complex ($g(JX,X) = 0$) and contact ($g(\phi X, X) = 0$) manifolds. If M is a slant submanifold of $\bar{M}$, then it might be $g(FX,X) = \cos\theta$ for any unit vector field $X \in \Gamma(TM)$. On the other hand, from the definition of a slant submanifold it obvious that the angle between FY and T_pM, $p \in M$ is constant for a unit vector field $Y \in \Gamma(TM)$, thus we can also consider $g(FX,Y) = \cos\theta$ for any unit vector fields $X, Y \in \Gamma(TM)$. Next theorem shows us that there are restrictions for a proper slant submanifold of an almost product Riemann manifold.

THEOREM 3.4. *Let M be a proper slant surface of an almost product Riemann manifold $\bar{M}$ such that $\{e_1, e_2\}$ is an orthonormal basis of M. Then the following situations can not be occurred at the same time:*

$$g(Fe_i, e_i) = \cos\theta,\ g(Fe_i, e_j) = \cos\theta,\ i \neq j \in \{1, 2\},$$

where θ denotes the slant angle of M.

Proof. Let M be a proper slant surface of $\bar{M}$ such that $TM = \text{span}\{e_1, e_2\}$. Moreover suppose that $g(Fe_i, e_i) = \cos\,\theta$ and $g(Fe_i, e_j) = \cos\,\theta$. Then we can write

$$Te_1 = g(Fe_1, e_1)e_1 + g(Fe_1, e_2)e_2.$$

Then by assumption we obtain

$$Te_1 = \cos\,\theta e_1 + \cos\,\theta e_2.$$

In similar computations, we get

$$Te_2 = \cos\,\theta e_1 + \cos\,\theta e_2.$$

Hence we have

$$g(Te_1, Te_2) = 2\cos^2\,\theta.$$

Then from (3.3) we obtain

$$\cos^2\,\theta g(e_1, e_2) = 2\cos^2\,\theta.$$

Since e_1 and e_2 are orthogonal, we derive $\cos\,\theta = 0$. This shows that M is an anti-invariant surface. But this is a contradiction, which proves the assertion. □

LEMMA 3.2. *Let M be a proper slant surface of an almost product Riemannian manifold $\bar{M}$ with the slant angle θ and $\{e_1, e_2\}$ be an orthonormal frame of M. Then we have:*

(i) *If $g(Fe_i, e_i) = \cos\,\theta$, then*

$$Te_1 = \cos\theta e_1\,,\ Te_2 = \cos\theta e_2 \quad \text{and} \quad g(Fe_i, e_j) = 0.$$

(ii) *If $g(Fe_i, e_j) = \cos\,\theta$, then*

$$Te_1 = \cos\,\theta e_2,\ Te_2 = \cos\,\theta e_1 \quad \text{and} \quad g(Fe_i, e_i) = 0.$$

Proof. (i) By the definition of T, for any $X \in TM$ we have

$$TX = g(FX, e_1)e_1 + g(FX, e_2)e_2.$$

Thus, for $X = e_1, X = e_2$, we obtain

$$Te_1 = \cos\theta e_1 + g(Fe_1, e_2)e_2 \tag{3.5}$$

and

$$Te_2 = g(Fe_2, e_1)e_1 + \cos\theta e_2. \tag{3.6}$$

Then, from (3.5) and (3.6) we obtain

$$g(Te_1, Te_2) = 2\cos\theta g(Fe_1, e_2).$$

Using (3.3) we have

$$\cos^2\theta g(e_1, e_2) = 2\cos\theta g(Fe_1, e_2).$$

Since e_1 and e_2 are orthogonal and M is proper slant, we obtain $g(Fe_1, e_2) = 0$. Then from (3.5) and(3.6) we have (i). The second assertion can be obtained in a similar way. □

REMARK 3.2. Note that we can easily find examples satisfying Lemma 3.2 (i) and (ii). Indeed, Example 3.1 and Example 3.2 satisfy Lemma 3.2 (i). On the other hand, Example 3.3 and Example 3.4 have the conditions of Lemma 3.2 (ii). We also note that it is easy to see, Theorem 3.1 and Lemma 3.1 are valid for both cases.

THEOREM 3.5. *Let M be a proper slant surface of an almost product Riemann manifold $\bar{M}$ with the slant angle θ such that $\{e_1, e_2\}$ is an orthonormal basis of M.*

(1) *If $g(Fe_i, e_i) = \cos\theta$, then M is a product slant surface of $\bar{M}$.*
(2) *If $g(Fe_i, e_j) = \cos\theta$, then M is a product slant surface of $\bar{M}$ if and only if each of the e_i is parallel.*

Proof. (1) Let M be a proper slant surface of $\bar{M}$ with slant angle θ such that $g(Fe_i, e_i) = \cos\theta$. Then we can choose an orthonormal frame $\{e_1, e_2\}$ such that it satisfies Lemma 3.2 (i). By straightforward computations, we have $\nabla_X e_1 = g(\nabla_X e_1, e_2)e_2$ for any $X \in TM$. Thus by using Lemma 3.2 (i) we derive

$$\begin{aligned}(\nabla_X T)e_1 &= \nabla_X Te_1 - T\nabla_X e_1\\ &= \cos\theta\nabla_X e_1 - g(\nabla_X e_1, e_2)Te_2\\ &= \cos\theta\nabla_X e_1 - g(\nabla_X e_1, e_2)\cos\theta e_2\\ &= \cos\theta(\nabla_X e_1 - g(\nabla_X e_1, e_2)e_2)\\ &= \cos\theta(\nabla_X e_1 - \nabla_X e_1)\end{aligned}$$

Hence we obtain $(\nabla_X T)e_1 = 0$. In a similar way we have $(\nabla_X T)e_2 = 0$. Thus if a surface of $\bar{M}$ is a proper slant, then it is a product slant surface.

(2) From Lemma 3.2 (ii), we have

$$(\nabla_X T)e_1 = \cos\theta\nabla_X e_2 - T\nabla_X e_1.$$

Hence, we get

$$\begin{aligned}(\nabla_X T)e_1 &= \cos\theta\nabla_X e_2 - g(\nabla_X e_1, e_2)Te_2\\ &= \cos\theta\nabla_X e_2 + g(\nabla_X e_2, e_1)\cos\theta\, e_1.\end{aligned}$$

Thus we obtain

$$(\nabla_X T)e_1 = 2\cos\theta\,\nabla_X e_2. \tag{3.7}$$

In a similar way, we have

$$(3.8) \qquad (\nabla_X T)e_2 = 2\cos\theta \nabla_X e_1.$$

Thus (3.7) and (3.8) proves (2).

Finally in this section, we give an another useful characterization for slant submanifolds of an almost product manifold. □

THEOREM 3.6. *Let M be a submanifold of an almost Riemann manifold $\bar{M}$. Then M is a proper slant submanifold of $\bar{M}$ if and only if there exists a constant $\kappa \in [0,1]$ such that*

$$(3.9) \qquad B\omega X = \kappa X$$

for any $X \in \Gamma(TM)$. Moreover, if θ is the slant angle of M, then $\kappa = \sin^2\theta$.

Proof. Applying F to (2.8) and using (2.8) and (2.9) we obtain

$$X = T^2X + \omega TX + B\omega X + C\omega X$$

for any $X \in \Gamma(TM)$. Thus, taking the tangential parts of this equation, we get

$$(3.10) \qquad X = T^2X + B\omega X.$$

Now, if M is a slant submanifold, then from Theorem 3.1, we have $T^2X = \cos^2\theta$. Hence we obtain

$$B\omega X = \sin^2\theta X = \kappa X,\ \kappa \in [0,1]$$

Conversely, suppose that $B\omega X = \kappa X$, $\kappa \in [0.1]$. Then from (3.10) we have

$$X = T^2X + \kappa X.$$

Hence, we obtain

$$T^2X = (1-\kappa)X.$$

Put $(1-\kappa) = \lambda$ so that $\lambda \in [0,1]$. The proof follows from Theorem 3.1. □

4. Semi-slant submanifolds

In this section, we define and study semi-slant submanifolds of an almost product Riemannian manifold $\bar{M}$. In particular, we investigate integrability of distributions which are arisen in the definition of semi-slant submanifolds and give a necessary and sufficient condition on the endomorphism T to be parallel. We also show that distributions are integrable and their leaves are totally geodesic in a semi-slant submanifold if the endomorphism T is parallel. We notice that semi-slant submanifolds were studied in complex and contact geometry by N. Papaghiuc [7] and J. L. Cabrerizo et al [2], respectively.

DEFINITION 4.1. Let M be a submanifold of an almost product Riemannian manifold $\bar{M}$. We say that M is a semi-slant submanifold of $\bar{M}$, if there exist two orthogonal distributions D_1 and D_2 such that

1. $TM = D_1 \oplus D_2$
2. *D_1 is an invariant distribution, $F(D_1) = D_1$.*
3. The distribution D_2 is slant with angle $\theta \neq 0$.

We will call D_2 as a slant distribution. In particular, if $\theta = \frac{\pi}{2}$ then the semi-slant submanifold is a semi-invariant submanifold [8]. On the other hand, if $\dim(D_2) = 0$ then M is an $F-$ invariant submanifold. If $\dim(D_1) = 0$ and $\theta = \frac{\pi}{2}$, then M is an $F-$ anti-invariant submanifold. Finally, if $\dim(D_1) = 0$ and $\theta \neq \frac{\pi}{2}$ then M is a proper slant submanifold with angle θ.

We now denote the projection morphisms on the distributions D_1 and D_2 by P_1 and P_2, respectively. Then we can write

$$X = P_1X + P_2X \tag{4.1}$$

for any $X \in TM$. Then applying F to (4.1) we have

$$FX = TP_1X + TP_2X + \omega P_2X. \tag{4.2}$$

Now, we put $T_1 = TP_1$ and $T_2 = TP_2$. Then we obtain

$$FX = T_1X + T_2X + \omega P_2X. \tag{4.3}$$

Since $F(D_1) = D_1$, we derive

$$FP_1X = TP_1X \,,\ \omega P_1X = 0 \,,\ TP_2X \in D_2. \tag{4.4}$$

Moreover, using (4.4) we get

$$TX = FP_1X + TP_2X \tag{4.5}$$

for any $X \in TM$.

Using a similar method of the proof of the Theorem 3.1, we can obtain the following theorem

THEOREM 4.1. *Let M be a submanifold of an almost product manifold $\bar{M}$. Then M is a semi-slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

(i) $D' = \{X \in D' \mid T^2X = \lambda X\}$.
(ii) *For any $X \in \Gamma(TM)$, orthogonal to D', $\omega X = 0$.*

Moreover in this case $\lambda = \cos^2\theta$, where θ denotes the slant angle of M.

Proof. Let M be a semi-slant submanifold of $\bar{M}$. Then $\lambda = \cos^2\theta$ and $D' = D_2$. By the definition of semi-slant submanifold, (ii) is clear. Conversely (i) and (ii) imply $TM = D \oplus D'$. Since $T(D') \subseteq D'$, from (ii), D is an invariant distribution. Thus proof is complete. □

From Theorem 4.1 we obtain

$$g(TX, TP_2Y) = \cos^2\theta g(X, P_2Y) \tag{4.6}$$

and

$$g(\omega X, \omega P_2Y) = \sin^2\theta g(X, P_2Y). \tag{4.7}$$

for any $X, Y \in TM$.

We now present two examples of semi-slant submanifolds.

EXAMPLE 4.1. For any $\theta \in (0, \frac{\pi}{2})$,

$$x(u, v, t) = (u, v, t\cos\theta, t\sin\theta, o)$$

defines a three dimensional proper semi-slant submanifold M, with slant angle 2θ, in $R^5 = R^3 \times R^2$. It easy to see $D_1 = \mathrm{Span}\{X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2}\}$ and $D_2 = \mathrm{Span}\{X_3 = \cos\theta\frac{\partial}{\partial x_3} + \sin\theta\frac{\partial}{\partial x_4}\}$.

EXAMPLE 4.2. For any non-zero constant $k \neq 1$,

$$x(u, v, s, t) = (k\sin t, -k\cos t, k\sin s, -k\cos s, t, s, u, v),\ t, s \neq \frac{\pi}{2}$$

defines a four dimensional proper semi-slant submanifold M, with slant angle $\cos^{-1}(\frac{k^2-1}{k^2+1})$, in $R^8 = R^4 \times R^4$ with its usual product structure. Moreover, it easy to see that

$$\begin{array}{ll} X_1 = k\cos t\frac{\partial}{\partial x_1} + k\sin t\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5} & X_3 = \frac{\partial}{\partial x_7} \\ X_2 = k\cos s\frac{\partial}{\partial x_3} + k\sin s\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_6} & X_4 = \frac{\partial}{\partial x_8} \end{array}$$

form a local orthogonal frame of TM. Then we can define $D_1 = \mathrm{Span}\{X_3, X_4\}$ and $D_2 = \mathrm{Span}\{X_1, X_2\}$.

Next, we are going to investigate integrability conditions of the distributions D_1 and D_2.

THEOREM 4.2. *Let M be a semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$. Then we have:*

(i) *The distribution D_1 is integrable if and only if*

$$h(X, TY) = h(TX, Y), \forall X, Y \in D_1.$$

(ii) *The distribution* D_2 *is integrable if and only if*

$$P_1(\nabla_X T_2 Y - \nabla_Y T_2 X) = P_1(A_{\omega Y}X - A_{\omega X}Y), \forall X, Y \in D_2.$$

Proof. From (2.6) and (2.8), taking the normal components we obtain

$$\omega\nabla_X Y = h(X, FY) - Ch(X, Y) \tag{4.8}$$

for any $X, Y \in D_1$. Thus, interchanging role X and Y in (4.8) and subtracting we derive

$$\omega[Y, X] = h(X, FY) - h(FX, Y).$$

Thus if D_1 is integrable, then we obtain (i). Conversely, suppose that the condition (i) is satisfied. Then we derive $\omega[X, Y] = 0$. Thus from (4.7) we obtain

$$g(\omega[X, Y], \omega Z) = \sin^2\theta g([X, Y], Z) = 0$$

for $Z \in \Gamma(D_2)$. Then proper M implies $\sin\theta \neq 0$. Thus we conclude that $[X, Y] \in \Gamma(D_1)$. In a similar way, taking the tangential parts we have

$$(\nabla_X T)Y = A_{\omega Y}X + Bh(X, Y) \tag{4.9}$$

for any $X, Y \in TM$. Thus for $X, Y \in D_2$ in (4.9) and using (4.3), we derive

$$\nabla_X T_2 Y = A_{\omega Y}X + T_1\nabla_X Y + T_2\nabla_X Y + Bh(X, Y).$$

Hence we get

$$\nabla_X T_2 Y - \nabla_Y T_2 X = A_{\omega Y}X - A_{\omega X}Y + T_1[X, Y] + T_2[X, Y].$$

Thus applying P_1 to this equation we obtain assertion (ii). □

In the rest of this section we are going to find a condition for semi-slant submanifolds such that $\nabla T = 0$ and we investigate the geometry of leaves of the distributions D_1 and D_2 under the condition $\nabla T = 0$.

THEOREM 4.3. *Let* M *be a slant submanifold of a locally product Riemannian manifold* $\bar{M}$. *Then* T *is parallel if and only if*

$$A_{\omega P_2 Y}X = -A_{\omega P_2 X}Y$$

for any $X, Y \in TM$.

Proof. From (4.9) we have

$$(\nabla_X T)Y = A_{\omega P_2 Y}X + Bh(X, Y)$$

for any $X, Y \in TM$. Using (2.5), (2.9) and (4.2) we obtain

$$g((\nabla_X T)Y, Z) = g(A_{\omega P_2 Y}X, Z) + g(h(X, Y), \omega P_2 Z)$$

for any $Z \in TM$. Hence we get

$$g((\nabla_X T)Y, Z) = g(A_{\omega P_2 Y} X, Z) + g(A_{\omega P_2 Z} Y, X).$$

Since A is self-adjoint, we obtain the assertion of theorem. □

We note that semi-slant submanifolds satisfying $\nabla T = 0$ is the semi-slant version of product slant submanifolds

THEOREM 4.4. *Let M be a semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$. If $\nabla T = 0$, then the distributions D_1 and D_2 are integrable and their leaves are totally geodesic in M.*

Proof. If $\nabla T = 0$, then from (4.9) we obtain $Bh(X, Y) = 0$ for any $X \in TM$ and $Y \in D_1$. Thus, using (2.5) and (4.2) we derive

$$g(h(X, Y), \omega P_2 Z) = 0 \tag{4.10}$$

and

$$g(Fh(X, Y), \omega P_2 Z) = 0 \tag{4.11}$$

for any $X, Z \in TM$ and $Y \in D_1$. Since $\bar{M}$ is a locally product Riemannian manifold, from Gauss formula and (4.2), we have

$$\begin{aligned} & g(\omega P_2 \nabla_X Y, Fh(X, Y)) \\ = \; & g(\omega P_2 \nabla_X Y, F\bar{\nabla}_X Y - F\nabla_X Y) \\ = \; & g(\omega P_2 \nabla_X Y, \bar{\nabla}_X FY) - g(\omega P_2 \nabla_X Y, \omega P_2 \nabla_X Y) \end{aligned}$$

for any $X \in TM$ and $Y \in D_1$. Hence, using Gauss formula, we derive

$$\begin{aligned} & g(\omega P_2 \nabla_X Y, Fh(X, Y)) \\ & = g(\omega P_2 \nabla_X Y, h(X, FY)) - g(\omega P_2 \nabla_X Y, \omega P_2 \nabla_X Y). \end{aligned}$$

Thus from (4.10) we get

$$g(\omega P_2 \nabla_X Y, Fh(X, Y)) = -g(\omega P_2 \nabla_X Y, \omega P_2 \nabla_X Y).$$

Then by using (4.7) we obtain

$$g(\omega P_2 \nabla_X Y, Fh(X, Y)) = -\sin^2 \theta g(P_2 \nabla_X Y, P_2 \nabla_X Y). \tag{4.12}$$

Then, from (4.11) and (4.12) we have

$$-\sin^2 \theta g(P_2 \nabla_X Y, P_2 \nabla_X Y) = 0.$$

Since M is a proper semi-slant submanifold and g is the Riemannian metric we get, $P_2 \nabla_X Y = 0$. Hence $\nabla_X Y \in D_1$ for $X \in TM$ and $Y \in D_1$. Moreover, since D_1 and D_2 are orthogonal, we conclude that $\nabla_X Z \in D_2$ for $X \in TM$ and $Z \in D_2$. Hence proof is complete. □

COROLLARY 4.1. *Let M be a semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$ and suppose that the distribution D_1 is integrable and its each leaf is totally geodesic in M. Then $\nabla T = 0$ if and only if $(\nabla_X T_2)Y = 0, \forall Y \in D_2$.*

Proof. Suppose that D_1 is integrable and its each leaf is totally geodesic in M. Then, since $\bar{M}$ is a locally product Riemannian manifold, from Gauss formula, we have

$$g(h(X,Y), FZ) = g(F\bar{\nabla}_X Y - F\nabla_X Y, Z)$$

for $X \in TM$, $Y \in D_1$ and $Z \in D_2$. Since $\nabla_X Y \in D_1$, we obtain $g(F\nabla_X Y, Z) = 0$. Hence we derive

$$g(h(X,Y), FZ) = g(\bar{\nabla}_X FY, Z) = g(\nabla_X FY, Z) = 0$$

for $X \in TM$, $Y \in D_1$ and $Z \in D_2$. Thus we conclude that $Bh(X,Y) = 0$ for any $X \in TM$ and $Y \in D_1$. Then from (4.9) we have

$$(\nabla_X T)Y = 0, \forall X \in TM, Y \in D_1.$$

On the other hand, for $Y \in D_2$, from (4.5) we get

$$(\nabla_X T)Y = (\nabla_X T_2)Y$$

for any $X \in TM$ and $Y \in D_2$, hence we obtain the assertion of Corollary 4.1. □

REMARK 4.1. Note that, Theorem 4.4 implies that semi-slant submanifold satisfying $\nabla T = 0$ is a locally product Riemannian manifold. This shows us, the condition $\nabla T = 0$ is an effective tool to describe the geometry of slant submanifolds as well as semi-slant Submanifolds.

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