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EXOTIC SMOOTH STRUCTURE ON $\mathbb{CP}^2\sharp 13\overline{\mathbb{CP}}^2$

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ABSTRACT. In this paper, we construct a new exotic smooth 4-manifold X which is homeomorphic, but not diffeomorphic, to $\mathbb{CP}^2\sharp 13\overline{\mathbb{CP}}^2$. Moreover the manifold X has vanishing Seiberg-Witten invariants for all Spin^c-structures of X and has no symplectic structure.

1. Introduction

We say that a simply connected, oriented, smooth 4-manifold X has type $(1, k)$ for some integer $k \geq 1$ if the self-intersection form $q_M : H^2(X;\mathbb{Z}) \to \mathbb{Z}$ defined by $q_M(x) = \int_M x \cup x$ is isomorphic to the form $x_1^2 - y_1^2 - \cdots - y_k^2$ on $\mathbb{Z}^{k+1}$ with a basis $\{x_1, y_1, \ldots, y_k\}$.

By M. Freedman's result [13], any two simply-connected, oriented 4-manifolds of type $(1, k)$ are homeomorphic. However S. K. Donaldson showed in [11] that not all such manifolds are diffeomorphic. This provides the first example of simply-connected h-cobordant manifolds which are not diffeomorphic. In fact, he showed that there are two simply-connected algebraic surfaces of type (1,9) which are not diffeomorphic.

Many people consider the problem of classifying up to diffeomorphism the complex surfaces homeomorphic to some $\mathbb{CP}^2\sharp k\overline{\mathbb{CP}}^2$, that is, the problem to find exotic smooth structures on $\mathbb{CP}^2\sharp k\overline{\mathbb{CP}}^2$.

When $k = 0$, by Yau's result, any complex surface which is homeomorphic to $\mathbb{CP}^2$ is diffeomorphic to $\mathbb{CP}^2$.

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For $k = 1$, there are the Hirzebruch surfaces Σ_n (n : odd) which are known to be diffeomorphic to $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$.

When $0 < k < 9$, up to now it has not been known whether $\mathbb{CP}^2 \sharp k\overline{\mathbb{CP}}^2$ can have infinite family of smooth structures. For a long time the smallest known example was the Barlow surface [2]. D. Kotschick proved in [19] that the Barlow surface, which was known to be homeomorphic to $\mathbb{CP}^2 \sharp 8\overline{\mathbb{CP}}^2$, is not diffeomorphic to it.

Recently, J. Park [20] found an example with exotic structure on $\mathbb{CP}^2 \sharp 7\overline{\mathbb{CP}}^2$. A. Stipsicz and Z. Szabó [23] used a technique similar to Park's construction and constructed an exotic manifold of type $(1, 6)$. Furthermore, R. Fintushel and R. J. Stern [12] introduce a new technique to show that $\mathbb{CP}^2 \sharp k\overline{\mathbb{CP}}^2$ does have an infinite family of smooth structures when $k = 6, 7, 8$.

When $k = 5$, J. Park, A. Stipsicz, and Z. Szabó showed in [21] that there exist infinitely many pairwise non-diffeomorphic 4-manifolds which are all homeomorphic to $\mathbb{CP}^2 \sharp 5\overline{\mathbb{CP}}^2$ using Fintushel and Stern's technique of knot surgery in a double node neighborhood with a particular form of generalized rational blow-down.

When $k = 9$, S. K. Donaldson showed in [11] that two well-known simply-connected algebraic surfaces E(1) and S(2,3) of type (1,9) are not diffeomorphic. Here E(1) is $\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2$ as being equipped with an elliptic fibration and $S(p, q)$ is an algebraic surface which is obtained from $\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2$ by performing log transformations at two generic elliptic smooth fibers of $\pi : \mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2 \to \mathbb{CP}^1$ with multiplicities p and q, respectively.

I. Dolgachev showed in [10] that if the greatest common divisor of p and q, g.c.d $(p, q) = 1$, then $S(p, q)$ is simply-connected and of type $(1, 9)$.

When $k > 9$, R. Friedman and J. W. Morgan showed in [14] that $\mathbb{CP}^2 \sharp k\overline{\mathbb{CP}}^2$ has infinitely many smooth structures underlying algebraic surface. They found algebraic surfaces $\tilde{S}(p, q)$ with type $(1, 9+r), r > 0$, by blowing up at r points of $S(p, q)$ where p and q are relatively prime numbers greater than 1. They showed that $\tilde{S}(p, q)$ is not diffeomorphic to a rational surface.

It still will be interesting to find a new exotic 4-manifold with type $(1, k)$, $k \in \mathbb{N}$. In this paper, we construct a new exotic 4-manifold X which is not diffeomorphic to $\tilde{S}(p, q)$ with type (1, 13). The manifold X is homeomorphic to $\mathbb{CP}^2 \sharp 13\overline{\mathbb{CP}}^2$, but not diffeomorphic to it. Moreover

X has trivial Seiberg-Witten invariants for all Spin^c-structures of X and has no symplectic structure.

2. Construction of new four-manifold

Let (X, ω) be a closed, symplectic, 4-manifold with a symplectic structure ω. A smooth map $\sigma : X \to X$ is an anti-symplectic involution if and only if $\sigma^*\omega = -\omega$ and $\sigma^2 = \mathrm{Id}$. If X is a Kähler surface, then σ is anti-symplectic if and only if σ is anti-holomorphic, that is, $\sigma_* \circ J = -J \circ \sigma_*$ for the complex structure J on X.

When X is a Kähler surface, we will say that (X, σ) is a real manifold and X^σ is the fixed point sets of σ on X.

We start with a Silhol's real manifold (Y, ρ) which is constructed as follows: in $\mathbb{CP}^2$, take four real points x_i, $i = 1, \dots, 4$, in general position and choose a conic C_0 passing through all the x_i.

Choose another point b different from the x_i on C_0, $i = 1, \dots, 4$. If D_i denotes the line through b and x_i, then we can define a holomorphic involution

$$T : \mathbb{CP}^2 - \{C_0 \cup_{i=1}^4 D_i\} \longrightarrow \mathbb{CP}^2 - \{C_0 \cup_{i=1}^4 D_i\}$$

in the following way: for any point u in the domain above, the five points u, x_i, $i = 1, \dots, 4$, determine a unique conic C_u which intersects the line $D_u = \overline{ub}$ at u and another point, which is defined to be $T(u)$. See the following figure.

Since the complex conjugation c is an anti-holomorphic involution, we have $(T \circ c)_* \circ J = T_* \circ c_* \circ J = T_* \circ (-J \circ c_*)$.

Since T is a holomorphic involution, we have $T_* \circ J = J \circ T_*$ and then

$$\begin{aligned}(T \circ c)_* \circ J &= T_* \circ c_* \circ J = T_* \circ (-J \circ c_*) \\ &= -\, T_* \circ J \circ c_* = -J \circ (T_* \circ c_*) = -J \circ (T \circ c)_*.\end{aligned}$$

Thus composing T with the conjugation c, there is an anti-holomorphic involution $\rho_0 = T \circ c$ on $\mathbb{CP}^2 - \{C_0 \cup_{i=1}^4 D_i\}$ which extends to an anti-holomorphic involution ρ on the manifold obtained by blowing up $\mathbb{CP}^2$ at the five points b, x_1, x_2, x_3, x_4.

Let Y be the resulting manifold of type $\mathbb{CP}^2\sharp 5\overline{\mathbb{CP}}^2$. Then by R. Shilhol, the fixed point set Y^ρ of ρ is $S^2 \amalg S^2$ and the quotient $Y/\rho \cong \sharp 4\overline{\mathbb{CP}}^2$. For details, see [22].

In Y, take distinct four points x_i (not on the exceptional curves in $\mathbb{CP}^2\sharp 5\overline{\mathbb{CP}}^2$) such that $\rho(x_i) = x_{i+1}$, $i = 5, 7$ and assume that all points x_i , $i = 1, \dots, 8$, and b are distinct.

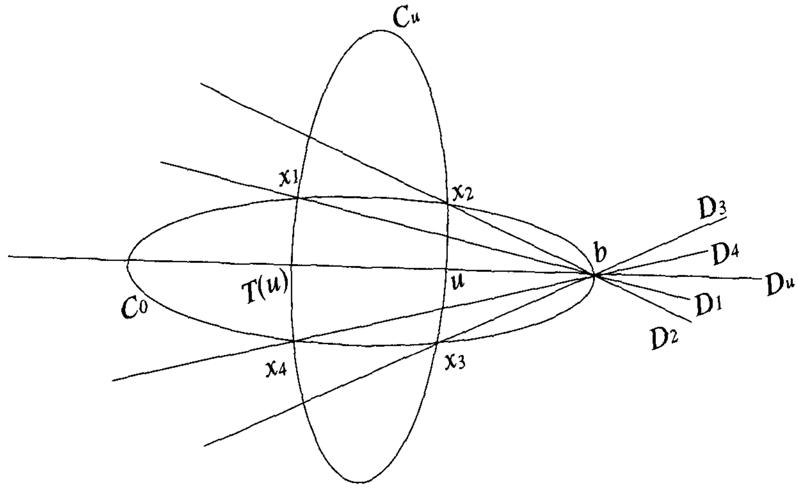

FIGURE 1

Let X_0 be the blow-up of (Y, ρ) at the four points x_i, $i = 5, 6, 7, 8$. Then the anti-holomorphic involution ρ on Y extends canonically to an anti-holomorphic involution σ_0 on the manifold $X_0 = \mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2$ such that the diffeomorphism type of its fixed point set and the quotient are respectively $X_0^\sigma = S^2 \amalg S^2$ and $X_0/\sigma_0 = \sharp 6\overline{\mathbb{CP}}^2$.

From now, let X_i be the manifold X_0 and $\sigma_i : X_i \to X_i$ be the anti-holomorphic involution σ_0, $i = 1, 2$.

Let F_i and $F_i' = \sigma_i(F_i)$ be generic fibers (Kähler torus) of X_i such that $F_i \cap F_i' = \emptyset$, $i = 1, 2$. Let $N(F_i)$ and $N(F_i')$ be small tubular neighborhoods of F_i and F_i' with radius $\epsilon > 0$ respectively, $i = 1, 2$.

The fibration on X_i determines a canonical normal framing of F_i, $i = 1, 2$. Thus there is a fiber-orientation reversing bundle isomorphism $\psi_1 : N(F_1) \to N(F_2)$, respecting the given framings and an orientation preserving diffeomorphism $\phi_1 : N(F_1) - F_1 \to N(F_2) - F_2$ by composing ψ_1 with the diffeomorphism

$$f : r \mapsto \sqrt{\epsilon^2 - r^2}, \qquad 0 < r < \epsilon,$$

that turns each punctured normal fiber inside out.

Similarly, the fibration on X_i determines a canonical normal framing of $\sigma_i(F_i) = F_i'$, $i = 1, 2$, so there is a fiber-orientation reversing bundle isomorphism $\psi_2 : N(F_1') \to N(F_2')$, respecting the given framings, and orientation preserving diffeomorphism $\phi_2 : N(F_1') - F_1' \to N(F_2') - F_2'$.

Let $X_1\sharp_{\phi_1,\phi_2}X_2$ be a smooth, closed, oriented 4-manifold obtained from $(X_1 - (F_1 \amalg F_1')) \amalg (X_2 - (F_2 \amalg F_2'))$ by using ϕ_1 and ϕ_2 to identify $N(F_1)-F_1$ and $N(F_2)-F_2$ and $N(F_1')-F_1'$ and $N(F_2')-F_2'$, respectively. Denote the resulting manifold $X_1\sharp_{\phi_1,\phi_2}X_2$ by $\bar{X}$.

LEMMA 2.1. *The manifold $\bar{X}$ is a symplectic 4-manifold.*

Proof. The 4-manifold $\bar{X}$ is obtained from the rational elliptic surfaces $X_i(= X_0)$ by two fiber sums, $i = 1, 2$. It is well known that the space $\bar{X}$ admits a Kähler structure.

Indeed, we can find a symplectic structure ω over $\bar{X}$ for any choice of the gluing maps ϕ_i, $i = 1, 2$. Let ω_0 be a Kähler form on X_i, $i = 1, 2$.

Let $K \supset F_1$ and $\tilde{K} \supset F_1'$ be compact subsets of $N(F_1)$ and $N(F_1')$, respectively. Furthermore, let η and η' be closed 2-forms compactly supported in $N(F_2)$ and $N(F_2')$, respectively.

Then, for some $O(2)$-bundle isomorphisms $\psi_1' : N(F_1) \to N(F_2)$ and $\psi_2' : N(F_1') \to N(F_2')$ which are fiber isotopic to ψ_1 and ψ_2 respectively, the manifold $(\bar{X}, \omega)$ is obtained from $(X_1 \amalg X_2 - (K \cup \tilde{K} \cup F_2 \cup F_2'), \omega_0 + t_0\eta + t_1\eta')$ by gluing via

$$\xi_1 = f \circ \psi_1' : N(F_1) - F_1 \to N(F_2) - F_2,$$
$$\xi_2 = f \circ \psi_2' : N(F_1') - F_1' \to N(F_2') - F_2',$$

for some sufficiently small, real values $0 < t_0, t_1 < 1$.

The gluing maps ξ_1 and ξ_2 are symplectic with respect to the symplectic form $(\omega_0 + t_0\eta + t_1\eta')$. For details, see [15]. □

LEMMA 2.2. *There is an involution σ on $\bar{X}$ with $\amalg_{i=1}^4 S_i^2$ as fixed point sets where S_i^2 is diffeomorphic to the standard 2-sphere, $i = 1, 2, 3, 4$.*

Proof. Since the manifolds $X_i = X_0$ and the anti-holomorphic involutions $\sigma_i = \sigma_0$, $i = 1, 2$, we have

$$\phi_2(\sigma_1(x)) = \sigma_2(\phi_1(x)), \quad \phi_1(\sigma_1(x')) = \sigma_2(\phi_2(x'))$$

for all $x \in N(F_1) - F_1$ and $x' \in N(F_1') - F_1'$, and so there is an involution σ on $\bar{X}$ induced from anti-holomorphic involutions $\sigma_i = \sigma_0$, $i = 1, 2$. In detail, there is a well-defined involution σ on $\bar{X}$ such that

$$\sigma = \begin{cases} \sigma_i \qquad X_i - (N(F_i) \amalg N(F_i')) \subset \bar{X}, \quad i = 1, 2, \\ \sigma_1(x') = \sigma_2(\phi_2(x')) \qquad (N(F_1) - F_1)\sharp_{\phi_1}(N(F_2) - F_2), \\ \sigma_1(x) = \sigma_2(\phi_1(x)) \qquad (N(F_1') - F_1')\sharp_{\phi_2}(N(F_2') - F_2') \end{cases}$$

for all $x \in N(F_1) - F_1$ and $x' \in N(F_1') - F_1'$.

Since $X_i^{\sigma_i} = S^2 \amalg S^2 \subset (X_i - (N(F_i) \amalg N(F_i')))$, $i = 1, 2$, the fixed point sets of σ in $\bar{X}$ is the disjoint union of 4 copies of 2-sphere, i.e.,

$$\bar{X}^\sigma = \amalg_{i=1}^4 S_i^2.$$

□

3. Exotic four-manifold

Let $\bar{X}$ be the symplectic 4-manifold $X_1 \sharp_{\phi_1,\phi_2} X_2$ in Lemma 2.1 which is obtained from $(X_1 - (F_1 \amalg F_1')) \amalg (X_2 - (F_2 \amalg F_2'))$ by using ϕ_1 and ϕ_2 to identify $N(F_1) - F_1$ and $N(F_2) - F_2$, and $N(F_1') - F_1'$ and $N(F_2') - F_2'$, respectively.

Then the involution σ on $\bar{X}$ in Lemma 2.2 has fixed point sets $\bar{X}^\sigma = \amalg_{i=1}^4 S_i^2$ where S_i^2 is diffeomorphic to the standard 2-sphere, $i = 1, 2, 3, 4$. Denote the quotient $\bar{X}/\sigma$ by X. Let X_i, σ_i, $i = 1, 2$, be the same as in Lemma 2.1.

For the proof of the Theorem 3.2, we briefly review the Seiberg-Witten invariant of X.

Let $L \to X$ be a complex line bundle satisfying $c_1(L) = w_2(TX)$ mod 2. This determines a principal Spin^c-structure on X which induces a unique complex spinor bundle $W \cong W^+ \oplus W^-$, where $W^\pm$ is the $(\pm\frac{1}{2})$-twisted spinor bundles on X with $\det(W^\pm) \cong L$.

For a unitary connection A in the set of all Riemannian connections on L, a positive spinor field $\Psi \in \Gamma(W^+)$, and a real valued, self-dual 2-form δ on X, the perturbed Seiberg-Witten equations are defined by

$$\begin{cases} F_A^+ + i\delta = q(\Psi) \\ D_A\Psi = 0, \end{cases}$$

where $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is the Dirac operator associated with the connection A. $q : C^\infty(W^+) \to \Omega_X^+(i\mathbb{R})$ is a quadratic map defined by $q(\Psi) = \Psi \otimes \Psi^* - \frac{||\Psi||^2}{2}\mathrm{Id}$.

Let M be the moduli space of the gauge equivalence classes of all solutions of the perturbed Seiberg-Witten equations. Then M is a smooth manifold with its dimension $\dim M = \frac{1}{4}(c_1(L)^2[X] - 2\chi(X) - 3\mathrm{sign}(X))$, where $\chi(X)$ is the Euler characteristic of X and $\mathrm{sign}(X)$ is the signature of X.

Note that if the metric on X is chosen so that the perturbed Seiberg-Witten equations admit no reducible solutions, then M is compact. Under these conditions, if $\dim M = 2d \geq 0$, then the Seiberg-Witten invariant is defined by

$$\int_M c_1(M_0)^d,$$

the integral of the maximal power of the Chern class of the circle bundle $M_0 \longrightarrow M$, where M_0 is the framed moduli space.

If $\dim M$ is odd or negative then the Seiberg-Witten invariant is defined to be zero. For details, see [6] and [7].

THEOREM 3.1. *The quotient X is simply-connected, smooth 4-manifold which is homeomorphic to* $\mathbb{CP}^2\sharp 13\overline{\mathbb{CP}}^2$.

Proof. For $\bar{X}$, since we have a map $\tilde{p} : \bar{X} \to T^2$ which induces an isomorphism between $\pi_1(\bar{X})$ and $\pi_1(T^2)$, if we consider the quotient map $\tilde{p}' : \bar{X}/\sigma \to T^2/c$ then $\tilde{p}'$ is an isomorphism between $\pi_1(\bar{X}/\sigma)$ and $\pi_1(T^2/c)$.

Thus we have $\pi_1(X) = \pi_1(T^2/c) = \pi_1(S^4) = 0$ and so X is simply-connected.

The Euler characteristic and the signature of $\bar{X}$ are

$$\chi(\bar{X}) = \chi(X_1) + \chi(X_2) = 24,$$
$$\mathrm{sign}(\bar{X}) = \mathrm{sign}(X_1) + \mathrm{sign}(X_2) = -16.$$

Let $\pi_i : X_i \to X_i/\sigma_i = X_i'$ be the projection map, $i = 1, 2$. Since X_i is a smooth, simply-connected double cover of X_i' branched along $S^2 \amalg S^2$, by [5] and [25] the quotient X_i' is smooth and simply-connected, $i = 1, 2$.

The Euler characteristic and signature of X_i' are

$$\chi(X_i') = \frac{1}{2}(\chi(X_i) + 2\chi(S^2)) = 6 + \chi(S^2),$$
$$\mathrm{sign}(X_i') = \frac{1}{2}(\mathrm{sign}(X_i) + 2S^2 \cdot S^2) = -4 + S^2 \cdot S^2, i = 1, 2.$$

Since each $S^2 \subset X_i^{\sigma_i} = S^2 \amalg S^2$ is a Lagrangian surface, it satisfies $\chi(S^2) + S^2 \cdot S^2 = 0$ and its self-intersection number $S^2 \cdot S^2 = -2$, $i = 1, 2$. Then $b_2^+(X_i') = 0$ and $b_2^-(X_i') = 6$ and we conclude that the quotient $X_i/\sigma_i = X_i' = \sharp 6\overline{\mathbb{CP}}^2$ is not a symplectic 4-manifold, $i = 1, 2$.

Since $\sigma_i(F_i) = F_i'$, we have $\pi_i(F_i) = \pi_i(F_i') \subset X_i'$, $i = 1, 2$.

Denote $\pi_i(F_i) = \pi_i(F_i')$ by $\tilde{F}_i$ and let $N(F_i')$ be a small tubular neighborhood of F_i' with radius $\epsilon > 0$, $i = 1, 2$.

By [4] and [9], the anti-holomorphic involution σ_i sends $N(F_i)$ to $N(F_i')$ respectively, $i = 1, 2$. Then $\pi_i(N(F_i)) = \pi_i(N(F_i'))$ is a small tubular neighborhood of $\tilde{F}_i$ with radius $\epsilon > 0$, $i = 1, 2$. Let $N(\tilde{F}_i)$ be the tubular neighborhood of $\tilde{F}_i$, $i = 1, 2$.

By [4] and [5], $\tilde{F}_i \cdot \tilde{F}_i = 2F_i \cdot F_i = 2F_i' \cdot F_i' = 0$, $i = 1, 2$. Thus the $\tilde{F}_i$ are tori with trivial self-intersection numbers and so we can identify tubular neighborhoods $N(\tilde{F}_i)$ with trivial normal bundles, $i = 1, 2$. Then there are fiber-orientation reversing bundle isomorphisms $\tilde{\psi} : N(\tilde{F}_1) \to N(\tilde{F}_2)$.

Let $X_1' \sharp_{\tilde{\phi}} X_2'$ be the smooth, closed, oriented 4-manifold obtained from $(X_1' - \tilde{F}_1) \amalg (X_2' - \tilde{F}_2)$ identifying $N(\tilde{F}_1) - \tilde{F}_1$ with $N(\tilde{F}_2) - \tilde{F}_2$ by using $\tilde{\phi} = f \circ \tilde{\psi}$.

Since $X_i/\sigma_i = X_i'$, $i = 1, 2$, and $\phi_2 \circ \sigma_1 = \sigma_2 \circ \phi_1$, we conclude that the quotient X is diffeomorphic to $X_1' \sharp_{\tilde{\phi}} X_2'$.

Since $\chi(S^2) = 2$ and $S^2 \cdot S^2 = -2$, we have Euler characteristic and signature of X as follows:

$$\chi(X) = \chi(X_1') + \chi(X_2') = 12 + 2\chi(S^2) = 16,$$
$$\operatorname{sign}(X) = \operatorname{sign}(X_1') + \operatorname{sign}(X_2') = -8 + 2S^2 \cdot S^2 = -12.$$

Thus we have $b_2^+(X) = 1$ and $b_2^-(X) = 13$ and so the space X is of type $(1, 13)$. By M. Freedman [13] X is homeomorphic to $\mathbb{CP}^2 \sharp 13 \overline{\mathbb{CP}}^2$. □

THEOREM 3.2. *The quotient X is not symplectic and has vanishing Seiberg-Witten invariants for all Spin^c-structures of X.*

Proof. By Theorem 3.1, the quotient $X = \bar{X}/\sigma$ is diffeomorphic to $X_1' \sharp_{\tilde{\phi}} X_2'$. If $X_1' \sharp_{\tilde{\phi}} X_2'$ is a symplectic 4-manifold then there is a non-trivial solution (A, ψ) of the Seiberg-Witten equations for the canonical class of $X_1' \sharp_{\tilde{\phi}} X_2'$.

Let $T^3 \subset X_1' \sharp_{\tilde{\phi}} X_2'$ be a 3-dimensional torus dividing $X_1' \sharp_{\tilde{\phi}} X_2'$ into two pieces $X_1' - N(\tilde{F}_1)$ and $X_2' - N(\tilde{F}_2)$.

Cutting $X_1' \sharp_{\tilde{\phi}} X_2'$ along the T^3, (A, ψ) sends to $(A_1 \vee A_2, \psi_1 \vee \psi_2)$ where (A_i, ψ_i) are solutions of the Seiberg-Witten equations on the spaces $X_i' - N(\tilde{F}_i)$ with cylindrical ends, $i = 1, 2$.

This means that if (A, ψ) is a non-trivial solution of the Seiberg-Witten equations on $X_1' \sharp_{\tilde{\phi}} X_2'$, then at least one of (A_i, ψ_i) is a non-trivial solution of the Seiberg-Witten equations, $i = 1, 2$.

However, it is impossible. Indeed, by the additivity of Euler characteristic

$$\chi(X_i') = \chi(X_i' - N(\tilde{F}_i)) + \chi(N(\tilde{F}_i)) - \chi((X_i' - N(\tilde{F}_i)) \cap N(\tilde{F}_i))$$
$$= \chi(X_i' - N(\tilde{F}_i)), \quad i = 1, 2.$$

By the Novikov additivity of signature,

$$\operatorname{sign}(X_i') = \operatorname{sign}(X_i' - N(\tilde{F}_i)) + \operatorname{sign}(N(\tilde{F}_i)) = \operatorname{sign}(X_i' - N(\tilde{F}_i)), i = 1, 2.$$

Thus we conclude that $2 - 2b_1(X_i') + 2b_2^+(X_i') = 2 - 2b_1(X_i' - N(\tilde{F}_i)) + 2b_2^+(X_i' - N(\tilde{F}_i))$, $i = 1, 2$.

Since $X_1'\sharp_{\tilde{\phi}}X_2'$ is simply-connected and obtained from $(X_1' - \tilde{F}_1)\coprod(X_2' - \tilde{F}_2)$ identifying $N(\tilde{F}_1) - \tilde{F}_1$ with $N(\tilde{F}_2) - \tilde{F}_2$ by using the map $\tilde{\phi} = f\circ\tilde{\psi}$, $N(\tilde{F}_i) - \tilde{F}_i$ are simply-connected and so $X_i' - N(\tilde{F}_i)$ are simply-connected, $i = 1, 2$.

Since X_i are simply-connected and $b_2^+(X_i') = 0$, we have $b_2^+(X_i' - N(\tilde{F}_i)) = 0$. Thus by the definition of the Seiberg-Witten invariant as above, since $b_2^+(X_i' - N(\tilde{F}_i)) = 0$, there is no non-trivial solution of the Seiberg-Witten equations over the cylindrical end spaces $X_i' - N(\tilde{F}_i)$, $i = 1, 2$.

Thus we conclude that there is no non-trivial solution of the Seiberg-Witten equations on $X_1'\sharp_{\tilde{\phi}}X_2'$ and so the quotient X is not symplectic and has vanishing Seiberg-Witten invariants for all Spin^c-structures of X. □

THEOREM 3.3. *The quotient X is homeomorphic, but not diffeomorphic to $\tilde{S}(p,q)$ with type* $(1, 13)$.

Proof. Since X has type $(1, 13)$, by M. Freedman [13] X is homeomorphic to $\mathbb{CP}^2\sharp 13\overline{\mathbb{CP}}^2$.

Since the algebraic surface $\tilde{S}(p,q)$ is the blow-up of Dolgachev surface $S(p,q)$ at 4 points where p and q are respectively prime numbers greater than 1, it is of type $(1, 13)$ and so it is homeomorphic to $\mathbb{CP}^2\sharp 13\overline{\mathbb{CP}}^2$.

Let C_1 be the unique chamber of $\tilde{S}(p,q) = S(p,q)\sharp 4\overline{\mathbb{CP}}^2$ for which $C_1 \cap \operatorname{Im}(i) \neq \emptyset$, where $i : H^2(S(p,q);\mathbb{R}) \to H^2(S(p,q)\sharp 4\overline{\mathbb{CP}}^2;\mathbb{R})$ is the inclusion.

By Z. Szabó [24], the blow-up formula shows that every basic class of C_1 can be written as $tK + \Sigma_{i=1}^4(-1)^{\delta_i}E_i$ with some $|t| \leqslant 1$, $\delta_i = 0, 1$ where K is the canonical class of $\tilde{S}(p,q)$ and E_i denotes the exceptional class of the i-th copy $\overline{\mathbb{CP}}^2$.

Since, by Theorem 3.2, there is no non-trivial solution of the Seiberg-Witten equations over the quotient $X = X_1\sharp_{\phi_1,\phi_2}X_2/\sigma$, we conclude the quotient X is not diffeomorphic to $\tilde{S}(p,q)$ with type $(1, 13)$. □

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