Bull. Korean Math. Soc. **43** (2006), No. 2, pp. 425–441

ON WEAKLY-BERWALD SPACES OF SPECIAL (α,β)-METRICS

IL-YONG LEE AND MYUNG-HAN LEE

ABSTRACT. We have two concepts of Douglas spaces and Landsberg spaces as generalizations of Berwald spaces. S. Bácsó gave the definition of a weakly-Berwald space [2] as another generalization of Berwald spaces. In the present paper, we find the conditions that the Finsler space with an (α,β)-metric be a weakly-Berwald space and the Finsler spaces with some special (α,β)-metrics be weakly-Berwald spaces, respectively.

1. Introduction

Let M^n be an n-dimensional differential manifold and let $F^n = (M^n, L)$ be an n-dimensional Finsler space where L is a fundamental function. Let $g_{ij} = \dot{\partial}_i\dot{\partial}_j L^2/2$ be the fundamental tensor, where the symbol $\dot{\partial}_i$ means $\partial/\partial y^i$ and we define G_i as

$$G_i = \{y^r(\partial_r\dot{\partial}_i L^2) - \partial_i L^2\}/4,$$

and $G^i = g^{ij}G_j$ where the symbol ∂_i means $\partial/\partial x^i$ and (g^{ij}) is the inverse matrix of (g_{ij}). The coefficients $(G_j{}^i{}_k, G^i{}_j)$ of the Berwald connection $B\Gamma$ are defined as $G^i{}_j = \dot{\partial}_j G^i$ and $G_j{}^i{}_k = \dot{\partial}_k G^i{}_j$.

A Berwald space is a Finsler space which satisfies the condition $G_i{}^h{}_{jk} = 0$, that is to say, whose coefficients $G_i{}^h{}_j$ of the Berwald connection are functions of the position (x^i) alone. Therefore the equations $y_r G_i{}^r{}_{jk} = 0$ hold, so $2G^i = G_r{}^i{}_s y^r y^s$ are homogeneous polynomials in (y^i) of degree two, so $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of

Received April 29, 2005.
2000 Mathematics Subject Classification: 53B40.
Key words and phrases: Berwald space, cubic metric space, Douglas space, Finsler space with $L = \alpha + \beta^2/\alpha$, infinite series (α,β)-metric space, weakly-Berwald space.
This Research was supported by Kyungsung University Research Grants in 2005.

degree three. Then we can consider the notions of Landsberg spaces and Douglas spaces as two generalizations of Berwald spaces. The notion of weakly-Berwald spaces is the third generalization of Berwald spaces. Thus if a Finsler space satisfies the condition $G_{ij} = 0$, we call it a weakly-Berwald space.

Let (M^n, L) be a Finsler space with an (α,β)-metric $L(\alpha,\beta)$, where $\alpha = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}$ and $\beta = b_i(x)y^i$. In [6], the functions G^i of a Finsler space with an (α,β)-metric are given by $2G^i = \gamma_0{}^i{}_0 + 2B^i$, where $\gamma_j{}^i{}_k$ stand for the Christoffel symbols in the space (M,α), then we have $G^i{}_j = \gamma_0{}^i{}_j + B^i{}_j$ and $G_j{}^i{}_k = \gamma_j{}^i{}_k + B_j{}^i{}_k$, where $\dot{\partial}_j B^i = B^i{}_j$ and $\dot{\partial}_k B^i{}_j = B_j{}^i{}_k$.

Thus a Finsler space with an (α,β)-metric is a weakly-Berwald space, if and only if $B^m{}_m = \partial B^m/\partial y^m$ is a one form.

Recently S. Bácsó and R. Yoshikawa [3] investigated the conditions that Randers and Kropina spaces be weakly-Berwlad spaces. R. Yoshikawa and K. Okubo [12] studied the conditions that generalized Kropina spaces and Matsumoto spaces be weakly-Berwald spaces and Berwald spaces, too.

In the present paper, first we study the condition that the Finsler space F^n with an (α,β)-metric be a weakly-Berwald space. Next we find the conditions that Finsler spaces with an infinite series (α,β)-metric $L = \beta^2/(\beta-\alpha)$, a special metric $L = \alpha + \beta^2/\alpha$ and a special cubic metric $L^3 = c_1\alpha^2\beta + c_2\beta^3$ be weakly-Berwald spaces, respectively.

2. Weakly-Berwald space with respect to (α,β)-metric

In the present section, we deal with the condition that a Finsler space with an (α,β)-metric be a weakly-Berwald space.

Let M be an n-dimensional differential manifold and let $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with an (α,β)-metric $L(\alpha,\beta)$, where $\alpha = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}$ and $\beta = b_i(x)y^i$. In this section, the symbol $(_/)$ stands for h-covariant derivation with respect to the Riemannian connection in the space (M,α) and $\gamma_j{}^i{}_k$ stand for the Christoffel symbols in the space (M,α). Let us list the symbols for the later use:

$$b^i = a^{ir}b_r, \quad b^2 = a^{rs}b_r b_s,$$

$$2r_{ij} = b_{i/j} + b_{j/i}, \quad 2s_{ij} = b_{i/j} - b_{j/i},$$

$$r^i{}_j = a^{ir}r_{rj}, \quad s^i{}_j = a^{ir}s_{rj}, \quad r_i = b_r r^r{}_i, \quad s_i = b_r s^r{}_i.$$

Now we consider the functions $G^i(x,y)$ of F^n with an (α,β)-metric. According to [7], they are written in the form

$$\begin{aligned}
2G^m &= \gamma_0{}^m{}_0 + 2B^m, \\
B^m &= (E^*/\alpha)y^m + (\alpha L_\beta/L_\alpha)s^m{}_0 \\
&\quad - (\alpha L_{\alpha\alpha}/L_\alpha)C^*\{(y^m/\alpha) - (\alpha/\beta)b^m\},
\end{aligned} \tag{2.1}$$

where we put

$$\begin{aligned}
E^* &= (\beta L_\beta/L)C^*, \\
C^* &= \{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)\}/\{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})\}, \\
\gamma^2 &= b^2\alpha^2 - \beta^2, \\
L_\alpha &= \partial L/\partial\alpha, \quad L_\beta = \partial L/\partial\beta, \quad L_{\alpha\alpha} = \partial^2 L/\partial\alpha\partial\alpha, \\
L_{\alpha\beta} &= \partial^2 L/\partial\alpha\partial\beta, \quad L_{\alpha\alpha\alpha} = \partial^3 L/\partial\alpha\partial\alpha\partial\alpha.
\end{aligned} \tag{2.2}$$

Since $\gamma_0{}^i{}_0 = \gamma_j{}^i{}_k(x)y^j y^k$ are homogeneous polynomials in (y^i) of degree two, a Finsler space F^n with an (α,β)-metric is a weakly-Bewald space, if and only if $B^m{}_m = \partial B^m/\partial y^m$ is a homogeneous polynomial in (y^i) of degree one. On the other hand, it is well-known [6] that a Finsler space with an (α,β)-metric is a Berwald space, if and only if B^m are homogeneous polynomials in (y^i) of degree two.

Then differentiating the latter of (2.1) by y^n and contracting m and n in the obtained equation, we have
(2.3)

$$\begin{aligned}
B^m{}_m = &\left\{\dot\partial_m\left(\frac{\beta L_\beta}{\alpha L}\right)y^m + \frac{n\beta L_\beta}{\alpha L} - \dot\partial_m\left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right)\right\}C^* \\
&- \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left\{\dot\partial_m\left(\frac{1}{\alpha}\right)y^m + \frac{1}{\alpha}\delta^m_m - \dot\partial_m\left(\frac{\alpha}{\beta}\right)b^m\right\}C^* \\
&+ \left(\frac{\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}}{\alpha L L_\alpha}\right)(\dot\partial_m C^*)y^m + \left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)(\dot\partial_m C^*)b^m \\
&+ \dot\partial_m\left(\frac{\alpha L_\beta}{L_\alpha}\right)s^m{}_0.
\end{aligned}$$

Since $L = L(\alpha,\beta)$ is a positively homogeneous function of α and β of degree one, we have

$$\begin{aligned}
L_\alpha\alpha + L_\beta\beta = L, &\quad L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = 0, \\
L_{\beta\alpha}\alpha + L_{\beta\beta}\beta = 0, &\quad L_{\alpha\alpha\alpha}\alpha + L_{\alpha\alpha\beta}\beta = -L_{\alpha\alpha}.
\end{aligned}$$

Using the above and the homogeneity of (y^i), we obtain

$$\dot{\partial}_m\left(\frac{\beta L_\beta}{\alpha L}\right)y^m = -\frac{\beta L_\beta}{\alpha L}, \tag{2.4}$$

$$\begin{aligned}&\dot{\partial}_m\left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right)\\ &= \frac{\gamma^2}{(\beta L_\alpha)^2}\left\{L_\alpha L_{\alpha\alpha} + \alpha L_\alpha L_{\alpha\alpha\alpha} - \alpha(L_{\alpha\alpha})^2\right\},\end{aligned} \tag{2.5}$$

$$\left\{\left(\dot{\partial}_m\frac{1}{\alpha}\right)y^m + \frac{1}{\alpha}\delta^m_m - \left(\dot{\partial}_m\frac{\alpha}{\beta}\right)b^m\right\} = \frac{1}{\alpha\beta^2}\{\gamma^2 + (n-1)\beta^2\}, \tag{2.6}$$

$$(\dot{\partial}_m C^*)y^m = 2C^*, \tag{2.7}$$

$$\begin{aligned}&(\dot{\partial}_m C^*)b^m\\ &= \frac{1}{2\alpha\beta\Omega^2}\left[\Omega\{\beta(\gamma^2+2\beta^2)W + 2\alpha^2\beta^2 L_\alpha r_0 - \alpha\beta\gamma^2 L_{\alpha\alpha} r_{00}\right.\\ &\quad - 2\alpha(\beta^3 L_\beta + \alpha^2\gamma^2 L_{\alpha\alpha})s_0\} - \alpha^2\beta W\{2b^2\beta^2 L_\alpha - \gamma^4 L_{\alpha\alpha\alpha}\\ &\quad \left. - b^2\alpha\gamma^2 L_{\alpha\alpha}\}\right],\end{aligned} \tag{2.8}$$

$$\dot{\partial}_m\left(\frac{\alpha L_\beta}{L_\alpha}\right)s^m{}_0 = \frac{\alpha^2 L L_{\alpha\alpha} s_0}{(\beta L_\alpha)^2}, \tag{2.9}$$

where

$$\begin{aligned}&W = (r_{00}L_\alpha - 2\alpha s_0 L_\beta),\\ &\Omega = (\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \quad \text{provided that } \Omega \neq 0.\\ &Y_i = a_{ir}y^r, \quad s_{00} = 0, \quad b^r s_r = 0, \quad a^{ij}s_{ij} = 0.\end{aligned} \tag{2.10}$$

Substituting (2.4), (2.5), (2.6), (2.7), (2.8) and (2.9) into (2.3), we have

$$\begin{aligned}B^m{}_m = &\frac{1}{2\alpha L(\beta L_\alpha)^2\Omega^2}\{2\Omega^2 AC^* + 2\alpha L\Omega^2 Bs_0\\ &+ \alpha^2 L L_\alpha L_{\alpha\alpha}(Cr_{00} + Ds_0 + Er_0)\},\end{aligned} \tag{2.11}$$

where
(2.12)

$$\begin{aligned} A &= (n+1)\beta^2 L_\alpha(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}) \\ &\quad + \alpha\gamma^2 L\{\alpha(L_{\alpha\alpha})^2 - 2L_\alpha L_{\alpha\alpha} - \alpha L_\alpha L_{\alpha\alpha\alpha}\}, \\ B &= \alpha^2 L L_{\alpha\alpha}, \\ C &= \beta\gamma^2\{-\beta^2(L_\alpha)^2 + 2b^2\alpha^3 L_\alpha L_{\alpha\alpha} - \alpha^2\gamma^2(L_{\alpha\alpha})^2 + \alpha^2\gamma^2 L_\alpha L_{\alpha\alpha\alpha}\}, \\ D &= 2\alpha\{\beta^3(\gamma^2-\beta^2)L_\alpha L_\beta - \alpha^2\beta^2\gamma^2 L_\alpha L_{\alpha\alpha} \\ &\quad - 2\alpha\beta\gamma^2(\gamma^2+2\beta^2)L_\beta L_{\alpha\alpha} - \alpha^3\gamma^4(L_{\alpha\alpha})^2 - \alpha^2\beta\gamma^4 L_\beta L_{\alpha\alpha\alpha}\}, \\ E &= 2\alpha^2\beta^2 L_\alpha \Omega. \end{aligned}$$

Summarizing up the above, we obtain

THEOREM 2.1. *The necessary and sufficient condition for a Finsler space F^n with an (α,β)-metric to be a weakly-Berwald space is that $G^m{}_m = \gamma_0{}^m{}_m + B^m{}_m$ and $B^m{}_m$ is a homogeneous polynomial in (y^m) of degree one, where $B^m{}_m$ is given by (2.11) and (2.12), provided that $\Omega \neq 0$.*

REMARK. The results (2.11) and (2.12) of Theorem 2.1 are rather different from the result (1.2) of Theorem 1 given by R. Yoshigawa and K. Okubo [12].

Here we state the following Lemma and Remark for the later frequent use:

LEMMA 2.2. [4] *If $\alpha^2 \equiv 0 \pmod{\beta}$, that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

REMARK. Throughout the present paper, we say "homogeneous polynomial(s) in (y^i) of degree r" as $hp(r)$ for brevity. Thus $\gamma_0{}^i{}_0$ is $hp(2)$ and, if the Finsler space is a weakly-Berwald space, then $B^m{}_m$ is $hp(1)$.

3. Infinite series (α,β)-metric space

In the present section, we consider the condition that the Finsler space with an infinite series (α,β)-metric be a weakly-Berwlad space. The notion of this space is recently introduced by [10]: Let us consider the r-th series (α,β)-metric

$$L(\alpha,\beta)=\beta\sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^k,$$

where we put $\alpha<\beta$.
If $r=\infty$, then the above is expressed as the form

$$L(\alpha,\beta)=\beta^2/(\beta-\alpha), \tag{3.1}$$

which is called an *infinite series (α,β)-metric.*

For the Finsler space F^n with (3.1), we have

$$\begin{aligned} &L_\alpha=\beta^2/(\beta-\alpha)^2, \quad L_\beta=\beta(\beta-2\alpha)/(\beta-\alpha)^2,\\ &L_{\alpha\alpha}=2\beta^2/(\beta-\alpha)^3, \quad L_{\alpha\alpha\alpha}=6\beta^2/(\beta-\alpha)^4. \end{aligned} \tag{3.2}$$

Owing to [10], we have

$$\begin{aligned} &2G^m=\gamma_0{}^m{}_0+2B^m,\\ &B^m=\frac{\alpha^3P}{\beta Q}\left\{b^m+\frac{\beta(\beta-4\alpha)}{2\alpha^3}y^m\right\}+\frac{\alpha}{\beta}(\beta-2\alpha)s^m{}_0, \end{aligned} \tag{3.3}$$

where $P=\beta r_{00}-2\alpha(\beta-2\alpha)s_0$ and $Q=2\alpha^3b^2-3\alpha\beta^2+\beta^3$, provided that $Q\neq 0$.

Substituting (3.2) into (2.12), (2.2) and (2.10), we have

$$\begin{aligned} A&=\frac{\beta^6}{(\beta-\alpha)^7}\{(n+1)(\beta-\alpha)(\beta-4\alpha)\beta^2+2\alpha(\alpha-2\beta)\gamma^2\},\\ B&=\frac{2\alpha^2\beta^4}{(\beta-\alpha)^4},\\ C&=\frac{\gamma^2\beta^5}{(\beta-\alpha)^6}\{-\beta^2(\beta-\alpha)^2+4b^2(\beta-\alpha)\alpha^3+2\alpha^2\gamma^2\},\\ D&=\frac{2\alpha\beta^4}{(\beta-\alpha)^6}\{(\beta-2\alpha)\beta^2(\beta-\alpha)^2(\gamma^2-\beta^2)-2(\beta-\alpha)\alpha^2\beta^2\gamma^2\\ &\quad-4\alpha(\beta-\alpha)(\beta-2\alpha)\gamma^2(\gamma^2+2\beta^2)-2(3\beta-4\alpha)\alpha^2\gamma^4\},\\ E&=\frac{2\alpha^2\beta^6}{(\beta-\alpha)^5}Q, \quad \Omega=\frac{\beta^2}{(\beta-\alpha)^3}Q,\\ W&=\frac{\beta}{(\beta-\alpha)^2}P, \quad C^*=\frac{\alpha(\beta-\alpha)P}{2Q}. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (2.11), we get
(3.5)

$$\begin{aligned}
&\{8b^4\alpha^7\beta^2 - 8b^4\alpha^6\beta^3 - 24b^2\alpha^5\beta^4 + 32b^2\alpha^4\beta^5 \\
&- 2(4b^2-9)\alpha^3\beta^6 - 30\alpha^2\beta^7 + 14\alpha\beta^8 - 2\beta^9\}B^m{}_m \\
&+ [4(n-1)b^2\alpha^5\beta^3 - 2(5n-4)b^2\alpha^4\beta^4 + 2\{(n-2)b^2 - 6n\}\alpha^3\beta^5 \\
&+ (19n+1)\alpha^2\beta^6 - 2(4n+1)\alpha\beta^7 + (n+1)\beta^8]r_{00} \\
&+ [8(4n-3)b^2\alpha^7\beta^2 - 8(7n-6)b^2\alpha^6\beta^3 \\
&- 4\{3(4n-1) - (7n-8)b^2\}\alpha^5\beta^4 + 4\{(25n-3) - (n-2)b^2\}\alpha^4\beta^5 \\
&- 2(35n+3)\alpha^3\beta^6 + 4(5n+2)\alpha^2\beta^7 - 2(n+1)\alpha\beta^8]s_0 \\
&+ \{-8b^2\alpha^7\beta^2 + 8b^2\alpha^6\beta^3 + 12\alpha^5\beta^4 - 16\alpha^4\beta^5 + 4\alpha^3\beta^6\}r_0 \\
&= 0.
\end{aligned}$$

Suppose that F^n be a weakly-Berwald space, that is, $B^m{}_m$ is $hp(1)$. Since α is irrational in (y^i), the equation (3.5) is divided into two equations as follows:

$$F_1 B^m{}_m + \beta G_1 r_{00} + \alpha^2 H_1 s_0 + \alpha^4 I_1 r_0 = 0, \tag{3.6}$$

$$F_2 B^m{}_m + \beta G_2 r_{00} + H_2 s_0 + \alpha^2 I_2 r_0 = 0, \tag{3.7}$$

where

$$\begin{aligned}
F_1 &= -8b^4\alpha^6 + 32b^2\alpha^4\beta^2 - 30\alpha^2\beta^4 - 2\beta^6, \\
F_2 &= 8b^4\alpha^6 - 24b^2\alpha^4\beta^2 - 2(4b^2-9)\alpha^2\beta^4 + 14\beta^6, \\
G_1 &= -2(5n-4)b^2\alpha^4 + (19n+1)\alpha^2\beta^2 + (n+1)\beta^4, \\
G_2 &= 4(n-1)b^2\alpha^4 + 2\{(n-2)b^2 - 6n\}\alpha^2\beta^2 - 2(4n+1)\beta^4, \\
H_1 &= -8(7n-6)b^2\alpha^4 + 4\{(25n-3) - (n-2)b^2\}\alpha^2\beta^2 + 4(5n+2)\beta^4, \\
H_2 &= 8(4n-3)b^2\alpha^6 - 4\{3(4n-1) - (7n-8)b^2\}\alpha^4\beta^2 \\
&\quad - 2(35n+3)\alpha^2\beta^4 - 2(n+1)\beta^6, \\
I_1 &= 8b^2\alpha^2 - 16\beta^2, \\
I_2 &= -8b^2\alpha^4 + 12\alpha^2\beta^2 + 4\beta^4.
\end{aligned}$$

Eliminating $B^m{}_m$ from these equations, we obtain

$$\beta R r_{00} + S s_0 + \alpha^2\beta^2 T r_0 = 0, \tag{3.8}$$

where

$$R = F_2G_1 - F_1G_2, \quad S = \alpha^2 F_2H_1 - F_1H_2,$$
$$T = 32b^4\alpha^8 + 24b^2(2b^2-7)\alpha^6\beta^2 + 24(4b^2+3)\alpha^4\beta^4 - 80\alpha^2\beta^6 + 8\beta^8.$$

Since only the term $-192(n-1)b^6\alpha^{12}s_0$ of Ss_0 in (3.8) seemingly does not contain β, we must have $hp(12)$ V_{12} such that

$$\alpha^{12}s_0 = \beta V_{12}. \tag{3.9}$$

First we are concerned with $\alpha^2 \not\equiv 0 \pmod{\beta}$ and $b^2 \neq 0$. (3.9) shows the existence of a function $k(x)$ satisfying $V_{12} = k\alpha^{12}$, and hence $s_0 = k\beta$. Then (3.8) is reduced to

$$Rr_{00} + kS + \alpha^2\beta T r_0 = 0.$$

Only the term $-16b^6\alpha^{10}\{(3n-2)r_{00} + 12k(n-1)\alpha^2\}$ of the above does not contain β. Thus there must exist $hp(1)$ U_1 satisfying $(3n-2)r_{00} + 12k(n-1)\alpha^2 = \beta U_1$. It is a contradiction, which leads to $k = 0$. Hence we obtain $s_0 = 0$; $s_j = 0$. Substituting $s_0 = 0$ into (3.8), we have

$$Rr_{00} + \alpha^2\beta T r_0 = 0. \tag{3.10}$$

Then only the term $2(5-n)\beta^{10}r_{00}$ of (3.10) seemingly does not contain α^2, and hence we must have $hp(10)$ V_{10} such that $\beta^{10}r_{00} = \alpha^2 V_{10}$. From $\alpha^2 \not\equiv 0 \pmod{\beta}$ there exists a function $f(x)$ such that

$$r_{00} = \alpha^2 f(x); \quad r_{ij} = a_{ij}f(x). \tag{3.11}$$

Transvecting (3.11) by b^iy^j, we have

$$r_0 = \beta f(x); \quad r_j = b_j f(x). \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10), we have

$$f(x)(R + \beta^2 T) = 0. \tag{3.13}$$

Let us assume $f(x) \neq 0$. Then (3.13) implies

$$-16(3n-2)b^6\alpha^{10} = \beta V_9,$$

where V_9 is $hp(9)$. Analogously to the above, this implies $V_9 = 0$, provided that $b^2 \neq 0$. Hence $f(x) = 0$ must hold and we obtain

$$r_{00} = 0; \quad r_{ij} = 0 \quad \text{and} \quad r_0 = 0; \quad r_j = 0.$$

Conversely, substituting $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$ into (3.5), we have $B^m{}_m = 0$. That is, the Finsler space with (3.1) is a weakly-Berwald space.

On the other hand, we suppose that the Finsler space with (3.1) be a Berwald space. Then we have $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$, because the space is a weakly-Berwald space from the above discussion. Substituting the above into (3.3), we have $B^m = 0$, that is, the Finsler space with (3.1) is a Berwald space. Hence $s_{ij} = 0$ hold good.

Next we deal with $\alpha^2 \equiv 0 \pmod{\beta}$, that is, Lemma 2.2 shows that $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$. From these conditions (3.8) is rewritten in the form

$$3R' r_{00} + 2S' s_0 + 4\delta T' r_0 = 0, \tag{3.14}$$

where

$$R' = -(7\delta - \beta)(6\delta + \beta),$$
$$S' = 216\delta^3 - 263\delta^2\beta + 50\delta\beta^2 - 3\beta^3,$$
$$T' = (\delta - \beta)(9\delta - \beta).$$

Since only the term $-3\beta^2(r_{00} + 2\beta s_0)$ of $3R'r_{00} + 2S's_0$ in (3.14) seemingly does not contain δ, we must have $hp(1)$ V_1 such that $r_{00} + 2\beta s_0 = \delta V_1$. Thus the above shows the existence of a function $g(x)$ satisfying $s_0 = \delta g(x)$; $s_i = d_i g(x)$. Transvecting this equation by b^i and paying attention to $d_i b^i = 2$, we have $g(x) = 0$. Hence we obtain $s_0 = 0$. Substituting $s_0 = 0$ into (3.14), we have

$$3R' r_{00} + 4\delta T' r_0 = 0. \tag{3.15}$$

Only the term $3\beta^2 r_{00}$ of $3R' r_{00}$ in (3.15) seemingly does not contain δ, and hence we must have $hp(3)$ W_3 such that $3\beta^2 r_{00} = \delta W_3$. Further there exists $hp(1)$ W_1 satisfying $r_{00} = \delta W_1$. Substituting this result into (3.15), we have $3(7\delta - \beta)(6\delta + \beta)W_1 = 4(\delta - \beta)(9\delta - \beta)r_0$. Hence there exists a function $\rho(x)$ such that $W_1 = \rho(x)(\delta - \beta)$, and thus substitution of $W_1 = \rho(x)(\delta - \beta)$ into $3(7\delta - \beta)(6\delta + \beta)W_1 = 4(\delta - \beta)(9\delta - \beta)r_0$ leads to $3\rho(x)(7\delta - \beta)(6\delta + \beta) = 4(9\delta - \beta)r_0$. Similarly to the above, this implies $\rho(x) = 0$.

Consequently, we obtain

$$r_{00} = 0; \quad r_{ij} = 0 \quad \text{and} \quad r_0 = 0; \quad r_j = 0.$$

Conversely, from $r_{00} = 0$, $r_0 = 0$ and $s_0 = 0$ we have $B^m{}_m = 0$. Thus the space with (3.1) is a weakly-Berwald space.

Summarizing up all the above, we have

THEOREM 3.1. *A Finsler space with an infinite series (α, β)-metric (3.1) is a weakly-Berwald space, if and only if $r_{ij} = 0$ and $s_j = 0$ are satisfied.*

4. Finsler space with $L = \alpha + \beta^2/\alpha$

The present section is devoted to a Finsler space with the metric

$$L(\alpha, \beta) = \alpha + \frac{\beta^2}{\alpha}. \tag{4.1}$$

This metric is proposed and is thought of as desirable in the viewpoint of geometry and of applications. We quote the proposition as follows:

PROPOSITION. [8] *Let F^n be a Finsler space with the (α, β)-metric (4.1), and suppose that $\alpha^2 = a_{ij}y^iy^j$ be positive-definite. The fundamental tensor of F^n is positive-definite*

(1) *if all the powers ≥ 4 of b_i are neglected,*
(2) *if $n = 2$ and $(1 + 2b^2)\alpha^2 - 3\beta^2$ are positive.*

Now we consider the condition that F^n with (4.1) be a weakly-Berwald space. For F^n with (4.1), we have

$$\begin{aligned} &L_\alpha = (\alpha^2 - \beta^2)/\alpha^2, \quad L_\beta = 2\beta/\alpha, \\ &L_{\alpha\alpha} = 2\beta^2/\alpha^3, \quad L_{\alpha\alpha\alpha} = -6\beta^2/\alpha^4. \end{aligned} \tag{4.2}$$

Substituting (4.2) into (2.1), we have

$$\begin{aligned} &2G^m = \gamma_0{}^m{}_0 + 2B^m, \\ &B^m = \frac{\alpha^2 P}{(\alpha^2 - \beta^2)Q}\left(b^m - \frac{2\beta^3}{\alpha^2(\alpha^2 + \beta^2)}y^m\right) + \frac{2\alpha^2\beta}{\alpha^2 - \beta^2}s^m{}_0, \end{aligned} \tag{4.3}$$

where $P = (\alpha^2 - \beta^2)r_{00} - 4\alpha^2\beta s_0$ and $Q = (1 + 2b^2)\alpha^2 - 3\beta^2$, provided that $Q > 0$. Substituting (4.2) into (2.12), (2.2) and (2.10), and substituting the obtained results into (2.11), we obtain

$$FB^m{}_m + \beta G r_{00} - 2\alpha^2 H s_0 - 2\alpha^2 I r_0 = 0, \tag{4.4}$$

where

$$\begin{aligned}
F &= (1+2b^2)^2\alpha^{10} - (1+2b^2)(7+2b^2)\alpha^8\beta^2 + 2(7+4b^2-2b^4)\alpha^6\beta^4 \\
&\quad - 2(1-8b^2-2b^4)\alpha^4\beta^6 - 3(5+4b^2)\alpha^2\beta^8 + 9\beta^{10}, \\
G &= 5(1-b^2)\alpha^8 - \{(11-2n) - (17+4n)b^2\}\alpha^6\beta^2 \\
&\quad - 2\{(6+5n) + (3-4n)b^2\}\alpha^4\beta^4 \\
&\quad + 2\{(12+7n) - (3-2n)b^2\}\alpha^2\beta^6 - 6(1+2n)\beta^8, \\
H &= (1+2b^2)\alpha^8 - 2(2+b^2)(1+4b^2)\alpha^6\beta^2 \\
&\quad + 2\{(9+2n) + (3+2n)b^2\}\alpha^4\beta^4 \\
&\quad - 2\{2(3+4n) - (3-4n)b^2\}\alpha^2\beta^6 - 12(1-n)\beta^8, \\
I &= (1+2b^2)\alpha^8 - 2(2+b^2)\alpha^6\beta^2 + 2(1-b^2)\alpha^4\beta^4 \\
&\quad + 2(2+b^2)\alpha^2\beta^6 - 3\beta^8.
\end{aligned}$$

Suppose that F^n be a weakly-Berwald space, that is, $B^m{}_m$ is $hp(1)$. Only the term $3\beta^9\{3\beta B^m{}_m - 2(1+n)r_{00}\}$ of (4.4) seemingly does not contain α^2, and hence we must have $hp(9)$ V_9 satisfying $3\beta^9\{3\beta B^m{}_m - 2(1+n)r_{00}\} = \alpha^2 V_9$. For the sake of brevity we suppose $\alpha^2 \not\equiv 0 \pmod{\beta}$. Then the above is reduced to

$$3\beta B^m{}_m - 2(1+n)r_{00} = k\alpha^2 \tag{4.5}$$

with a function $k(x)$. Thus (4.4) is reduced to

$$kF + 2(1+n)F' r_{00} + 3\beta\{\beta G' r_{00} - 2Hs_0 - 2Ir_0\} = 0, \tag{4.4'}$$

where

$$F' = (F - 9\beta^{10})/\alpha^2,$$
$$G' = \{G + 6(1+n)\beta^8\}/\alpha^2.$$

The terms of (4.4′) which seemingly does not contain β are

$$(1+2b^2)^2\alpha^8\{k\alpha^2 + 2(1+n)r_{00}\}.$$

Consequently, we must have $hp(1)$ V i.e., $V = v_i y^i$ such that the above is equal to $(1+2b^2)^2\alpha^8\beta V$. Thus we have

$$(4.6) \qquad k\alpha^2 + 2(1+n)r_{00} = \beta V.$$

Since (4.6) is a contradiction, we have $k = 0$, and hence we get, under the assumption that $n > 2$,

$$(4.7) \qquad r_{00} = \frac{1}{2(1+n)}\beta V; \quad r_{ij} = \frac{1}{4(1+n)}(b_i v_j + b_j v_i).$$

Transvecting (4.7) by $b^i y^j$, we have

$$(4.8) \qquad r_0 = \frac{1}{4(1+n)}(b^2 V + v_b \beta); \quad r_j = \frac{1}{4(1+n)}(b^2 v_j + v_b b_j),$$

where $v_b = v_i b^i$. Substituting $k = 0$, (4.7) and (4.8) into (4.4′), we have

$$(4.9) \quad \{2(1+n)VF' - 12(1+n)Hs_0 - 3b^2 VI\} = 3\beta\{v_b I - \beta VG'\}.$$

The terms of (4.9) which seemingly does not contain β are

$$(1+2b^2)\alpha^8\Big[\{(12n-1)b^2 + 6n\}V - 12n(2b^2-1)s_0\Big].$$

Thus we must have $hp(8)$ V_8 such that

$$(1+2b^2)\alpha^8\Big[\{2(1+n) + (1+4n)b^2\}V - 12(1+n)s_0\Big] = \beta V_8.$$

Hence there must exist a function $h(x)$ such that

$$(4.10) \qquad \begin{aligned} s_0 &= \frac{1}{12(1+n)}\Big[\{2(1+n) + (1+4n)b^2\}V - h\beta\Big]; \\ s_j &= \frac{1}{12(1+n)}\Big[\{2(1+n) + (1+4n)b^2\}V_j - hb_j\Big]. \end{aligned}$$

Consequently, we obtain, under assumption that $n > 2$,

$$\begin{aligned} r_{00} &= \frac{1}{2(1+n)}\beta V, \quad r_0 = \frac{1}{4(1+n)}(b^2 V + v_b\beta), \\ s_0 &= \frac{1}{12(1+n)}\Big[\{2(1+n) + (1+4n)b^2\}V - h\beta\Big]. \end{aligned}$$

Conversely, substituting $k = 0$ and (4.7) into (4.5), we have $3B^m{}_m = V$, that is, $B^m{}_m$ is $hp(1)$.

Next we deal with $\alpha^2 \equiv 0 \pmod{\beta}$, that is, $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$, $d_i b^i = 2$. Since the dimension is equal to two and (b_i, d_i) are independent pair, we can put $v_i = f(x)b_i + g(x)d_i$ under two functions $f(x)$ and $g(x)$, and then $v_b = 2g$. Transvection of (4.10) by b^i leads to $g = 0$. Hence we obtain $v_i = f(x)b_i$ and $v_b = 0$. Substituting the above into (4.7), we have

$$(4.11) \qquad 2(1+n)r_{00} = f(x)\beta^2; \quad 2(1+n)r_{ij} = f(x)b_i b_j.$$

Further from (4.8), we get $r_j = 0$. Furthermore from (4.10), we obtain

$$(4.12) \qquad \begin{aligned} 12(1+n)s_0 &= \{2(1+n)f(x) - h(x)\}\beta; \\ 12(1+n)s_j &= \{2(1+n)f(x) - h(x)\}b_j. \end{aligned}$$

Conversely, substituting $\alpha^2 = \beta\delta$ and (4.11) into (4.5), we have $3B^m{}_m = f\beta + k\delta$, that is, $B^m{}_m$ is $hp(1)$.

Summarizing up the above, we have

THEOREM 4.1. *A Finsler space with $L = \alpha + \beta^2/\alpha$ is a weakly-Berwald space, if and only if*

(1) $\alpha^2 \not\equiv 0 \pmod{\beta}$: (4.7) *and* (4.10) *are satisfied under* $n > 2$ *and* $v_b = v_i b^i$.

(2) $\alpha^2 \equiv 0 \pmod{\beta}$: $n = 2$, $b^2 = 0$ *and* (4.11), (4.12) *are satisfied, where* $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$ *and* $f(x)$, $h(x)$ *are functions of* (x^i).

5. Cubic Finsler space with an (α, β)-metric

In the present section, we find the condition that the cubic Finsler space be a weakly-Berwald space.

Let the so-called *cubic metric* on a differentiable manifold with the local coordinates x^i be defined by

$$L(x, y) = (a_{ijk}(x)y^i y^j y^k)^{\frac{1}{3}} \quad (y^i = \dot{x}^i),$$

where a_{ijk} are components of a symmetric tensor of $(0,3)$-type, depending on the position x, and a Finsler space with a cubic metric is called the *cubic Finsler space*. It is regarded as a direct generalization of Riemannian metric in a sense. We quote from the proposition as follows:

PROPOSITION. [9] *Let F^n be a Finsler space with a cubic metric $L(x,y)$*

(1) *In case of $n > 2$, if L is an (α,β)-metric where α is non-degenerate, then L^3 can be written in the form $L^3 = c_1\alpha^2\beta + c_2\beta^3$ with two constants c_1 and c_2.*
(2) *In case of $n = 2$, L is always written in a generalized $(-1/3)$-Kropina type $L = \alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}$, where α may be degenerate.*

Now the cubic metric $L(\alpha,\beta)$ of F^n is given by

$$L^3(\alpha,\beta) = c_1\alpha^2\beta + c_2\beta^3, \tag{5.1}$$

where c_1 and c_2 are constants. For this case we have

(5.2)
$$3L^2L_\alpha = 2c_1\alpha\beta, \quad 3L^2L_\beta = c_1\alpha^2 + 3c_2\beta^2,$$
$$9L^5L_{\alpha\alpha} = 2c_1\beta^2(3c_2\beta^2 - c_1\alpha^2), \quad 27L^8L_{\alpha\alpha\alpha} = 8c_1^2\alpha\beta^3(c_1\alpha^2 - 9c_2\beta^2).$$

By means of [11], we have

$$G^m = \gamma_0{}^m{}_0 + 2B^m,$$
$$B^m = \frac{P}{Q}\left\{y^m + \frac{(3c_2\beta^2 - c_1\alpha^2)}{2c_1\beta}b^m\right\} + \frac{(c_1\alpha^2 + 3c_2\beta^2)}{2c_1\beta}s^m{}_0, \tag{5.3}$$

where

$$P = c_1\beta r_{00} - (c_1\alpha^2 + 3c_2\beta^2)s_0,$$
$$Q = 3a\beta^2 - c_1\gamma^2, \quad a = c_2b^2 + c_1.$$

Substituting (5.2) into (2.12), (2.2) and (2.10), and substituting the obtained results into (2.11), we obtain

(5.4)
$$\begin{aligned} B^m{}_m = \frac{1}{48c_1\alpha^2\beta^2Q^2L^3}\big[&c_1\beta^2\{8QA' + 81\gamma^2(3c_2\beta^2 - c_1\alpha^2)C'\}r_{00} \\ &- 8\{\beta(c_1\alpha^2 + 3c_2\beta^2)QA' - 3\alpha^2(3c_2\beta^2 - c_1\alpha^2)Q^2L^3 \\ &- 3\beta(3c_2\beta^2 - c_1\alpha^2)D'\}s_0 + 48c_1\alpha^2\beta^2(3c_2\beta^2 - c_1\alpha^2)QL^3r_0\big], \end{aligned}$$

where

$$\begin{aligned} A' &= 9c_2^2\beta^6 + c_2\{-9c_2b^2 + 2(n-8)c_1\}\beta^4\alpha^2 \\ &\quad + c_1\{18c_2b^2 + (2n-1)c_1\}\beta^2\alpha^4 + 3c_1^2\beta^2\alpha^6, \\ C' &= 3c_2(3a + c_1)\beta^4 - 6c_1(3a - c_1)\beta^2\alpha^2 - 3c_1^2b^2\alpha^4, \\ D' &= 3c_2^2(3a + c_1)\beta^8 - c_2(18b^4c_2^2 + 45b^2c_1c_2 + 16c_1^2)\beta^6\alpha^2 \\ &\quad + 2c_1\{9b^4c_2^2 - b^2c_2(5c_1 + 3c_2) - 6c_1^2\}\beta^4\alpha^4 \\ &\quad + b^2c_1^2(14b^2c_2 + 9c_1)\beta^2\alpha^6 - 2b^4c_1^3\alpha^8. \end{aligned}$$

Before discussing our problem, we have to check the assumption $c_1 \neq 0$, $Q \neq 0$ and $L^3 \neq 0$ because $c_1 Q^2 L^3$ appears in the denominator of (5.4). If $L^3 = 0$, then $c_1 = 0$ and $c_2 = 0$. Thus $c_1 \neq 0$ or $c_2 \neq 0$. Also, if $Q = 0$, then $c_2 b^2 + c_1 = 0$ and $c_1 = 0$, that is, $c_2 b^2 = 0$. Thus $c_1 \neq 0$, $c_2 \neq 0$ and $b^2 \neq 0$. Consequently, $c_1 \neq 0$, $c_2 \neq 0$ and $b^2 \neq 0$ are proper assumptions in the present section.

The above (5.4) can be rewritten in the form

$$(5.5) \qquad 48c_1\alpha^2\beta^2 S B^m{}_m - c_1\beta T r_{00} + 8\alpha^2 U s_0 - 48c_1\alpha^2\beta^2 V r_0 = 0,$$

where

$$\begin{aligned}
S = {} & c_1(3a + c_1)^2\beta^6 + c_1(3a + c_1)(a + 3c_1)\beta^4\alpha^2 \\
& + c_1 b^2\{c_2(1 - 6c_1)b^2 - 8c_1^2\}\beta^2\alpha^4 + c_1^2 b^4\alpha^6, \\
T = {} & -657c_2^2(3a + c_1)\beta^8 + c_2[1971c_2^3 b^4 + c_1c_2b^2\{8(6n - 75) + 8019c_2\} \\
& + c_1^2\{64(n - 8) + 3688c_2\}]\beta^6\alpha^2 \\
& + c_1[-4599c_2^2 b^4 + (8n - 3937)c_1c_2b^2 + 4c_1^2(16n - 251)]\beta^4\alpha^4 \\
& + c_1 b^2\{657c_2b^2 + c_1(833 - 16n)\}\beta^2\alpha^6 + 219\alpha^8, \\
U = {} & 3[9c_2^4 b^4 + 3c_1c_2b^2\{(2n + 7)c_2^2 + 3c_1^2\} + 4c_1^2\{2(n - 9)c_2^2 + 3c_1^2\}]\beta^8 \\
& - 9c_1c_2^2[8c_2^2 b^4 - b^2\{(n + 4)c_1 + 6c_2\}]\beta^6\alpha^2 \\
& + c_1^2[21c_2^2 b^4 + c_2b^2\{(133 - 2n)c_1 - 18c_2\} + 8(n + 1)c_1^2]\beta^4\alpha^4 \\
& + 2c_1^3 b^2\{9c_2b^2 - (n - 8)c_1\}\beta^2\alpha^6 - 6b^4c_1^4\alpha^8, \\
V = {} & 3c_2^2(3c_2b^2 + 4c_1)\beta^6 + c_1c_2(3c_2b^2 + 8c_1)\beta^4\alpha^2 \\
& - c_1^2(5c_2b^2 + 4c_1)\beta^2\alpha^4 + c_1^3 b^2\alpha^6.
\end{aligned}$$

Since $B^m{}_m$ is supposed to be $hp(1)$, the term in (5.5) which seemingly does not contain α^2 is $657c_1c_2^2(3a + c_1)\beta^9 r_{00}$ only, and hence we must have $hp(9)$ V_9 satisfying

$$(5.6) \qquad 657c_1c_2^2(3a + c_1)\beta^9 r_{00} = \alpha^2 V_9.$$

Since $b^2 \neq 0$, we are concerned with the general case $\alpha^2 \not\equiv 0 \pmod{\beta}$. (5.6) shows the existence of a function $k(x)$ satisfying $V_9 = k\beta^9$, and hence we have

$$(5.7) \qquad r_{00} = \alpha^2 f(x); \quad r_{ij} = a_{ij} f(x),$$

where $f(x) = k(x)/657c_1c_2^2(3a + c_1)$.

Transvecting (5.7) by b^iy^j, we obtain

$$r_0 = \beta f(x); \quad r_j = b_j f(x). \tag{5.8}$$

Substituting (5.7) and (5.8) into (5.5), we have

$$\beta\{48c_1\beta SB^m{}_m - c_1 f(x)(T + 48\beta^2 V)\} + 8Us_0 = 0. \tag{5.9}$$

Since only the term $-48b^4c_1^4\alpha^8 s_0$ of (5.9) seemingly does not contain β, we must have $hp(8)$ V_8 such that

$$\alpha^8 s_0 = \beta V_8.$$

The above shows the existence of a function $g(x)$ satisfying $V_8 = g(x)\alpha^8$, and hence

$$s_0 = \beta g(x). \tag{5.10}$$

Consequently, we obtain $r_{00} = \alpha^2 f(x)$, $r_0 = \beta f(x)$ and $s_0 = \beta g(x)$.

Conversely, substituting (5.7), (5.8) and (5.10) into (5.5), we have

$$\beta\{48c_1 SB^m{}_m - 48c_1 f(x)\beta V\} = c_1 f(x)T - 8g(x)U. \tag{5.11}$$

Only the term $(219c_1 f(x) - 48b^4c_1^4 g(x))\alpha^8$ of (5.11) seemingly does not contain β, we must have $hp(7)$ V_7 such that $\alpha^8 = \beta V_7$. From Lemma 2.2 it is a contradiction. Thus we have $SB^m{}_m = f(x)\beta V$, that is, we must have a function $h(x)$ such that $B^m{}_m = h(x)\beta$, which is $hp(1)$.

Therefore we have

THEOREM 5.1. *Let F^n $(n > 2)$ be a cubic Finsler space with $L^3 = c_1\alpha^2\beta + c_2\beta^3$ and suppose $c_1 \neq 0$, $c_2 \neq 0$ and $b^2 \neq 0$. F^n is a weakly-Berwald space, if and only if there exist functions $f(x)$ and $g(x)$ such that* (5.7) *and* (5.10) *are satisfied.*

References

[1] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type. A generalization of the notion of Berwald space*, Publ. Math. Debrecen **51** (1997), no. 3-4, 385–406.

[2] S. Bácsó and B. Szilágyi, *On a weakly Berwald Finsler space of Kropina type*, Math. Pannon. **13** (2002), no. 1, 91–95.

[3] S. Bácsó and R. Yoshikawa, *Weakly-Berwald spaces*, Publ. Math. Debrecen **61** (2002), no. 2, 219–231.
[4] M. Hashiguchi, S. Hōjō, and M. Matsumoto, *Landsberg spaces of dimension two with (α, β)-metric*, Tensor (N. S.) **57** (1996), no. 2, 145–153.
[5] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Otsu, Saikawa (1986).
[6] ———, *The Berwald connection of a Finsler space with an (α, β)-metric*, Tensor (N. S.) **50** (1991), no. 1, 18–21.
[7] ———, *Theory of Finsler spaces with (α, β)-metric*, Rep. Math. Phys. **31** (1992), 43–83.
[8] ———, *Finsler spaces with (α, β)-metric of Douglas type*, Tensor (N. S.) **60** (1998), no. 2, 123–134.
[9] M. Matsumoto and S. Numata, *On Finsler space with a cubic metric*, Tensor (N. S.) **33** (1979), no. 2, 153–162.
[10] I. Y. Lee and H. S. Park, *Finsler spaces with infinite series (α, β)-metric*, J. Korean Math. Soc. **41** (2004), no. 3, 567–589.
[11] I. Y. Lee and D. G. Jun, *On two-dimensional Landsberg space of a cubic Finsler space*, East Asian Math. J. **19** (2003), no. 2, 305–316.
[12] R. Yoshikawa and K. Okubo, *The conditions for some (α, β)-metric spaces to be weakly-Berwald spaces*, Proceedings of the 38-th Symposium on Finsler geometry, Nov. 12–15, (2003), 54–57.

IL-YONG LEE AND MYUNG-HAN LEE, DEPARTMENT OF MATHEMATICS, KYUNGSUNG UNIVERSITY, BUSAN 608-736, KOREA
E-mail: iylee@star.ks.ac.kr