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A NOTE ON ASCEND AND DESCEND OF FACTORIZATION PROPERTIES

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ABSTRACT. In this paper we extend the study of ascend and descend of factorization properties (for atomic domains, domains satisfying ACCP, bounded factorization domains, half-factorial domains, pre-Schreier and semirigid domains) to the finite factorization domains and idf-domains for domain extension $A \subseteq B$.

Following Cohn [5], we say that R is *atomic* domain if each nonzero nonunit of R is a product of finite number of irreducible elements (atoms) of R. It is well-known that any UFD or Noetherian domain is atomic. We say that an integral domain R satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of R. An integral domain R satisfies ACCP if and only if $R[\{X_\alpha\}]$ satisfies ACCP for any family of indeterminates $\{X_\alpha\}$(cf. [3, Page 5]). But by Roitman [9] the polynomial extension $R[X]$ is not atomic whenever R is atomic domain, in general. It is well-known that any domain satisfying ACCP is an atomic but the converse does not hold (cf. [7]) (see also [9] and [13]).

By [3] an atomic domain R is a *bounded factorization domain* (BFD) if for each nonzero nonunit $x \in R$, there is a positive integer $N(x)$ such that whenever $x = x_1 \cdots x_n$ as a product of irreducible elements of R, then $n \leq N(x)$ (equivalently, we may just assume that each x_i is a nonunit of R (cf. [3, Theorem 2.4]). Noetherian and Krull domains are BFDs ([3, Proposition 2.2]). Also a BFD satisfies ACCP but the converse is not true (cf. [3, Example 2.1]).

In [12] Zaks introduced the notions of half-factorial domains, by the same, an atomic domain R is a *half-factorial domain* (HFD) if for each nonzero nonunit element x of R, if $x = x_1 \cdots x_m = y_1 \cdots y_n$ with each x_i, y_j irreducible in R, then $m = n$. Obviously a UFD is an HFD but converse is not true, for example $\mathbb{Z}[\sqrt{-3}]$, and an HFD is a BFD (cf. [3]).

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In general HFDs do not behave very well under extensions. By [3, Page 11], if $R[X]$ is an HFD, then certainly R is an HFD. However, $R[X]$ need not be an HFD if R is an HFD. For example the domain $R = \mathbb{R}+X\mathbb{C}[X]$ is an HFD, but $R[Y]$ is not an HFD since $(X(1+iY))(X(1-iY)) = X^2(1+Y^2)$ are factorizations into irreducibles of different lengths (cf. [3, Page 11]).

In order to measure how far an atomic domain R is being an HFD, by [1, page 217], the elasticity of R is defined as

$$\rho(R) = \sup\{m/n : \ x_1 \cdots x_m = y_1 \cdots y_n, \text{ each } x_i, y_j \in R \text{ is irreducible}\}.$$

Thus $1 \leq \rho(R) \leq \infty$ and $\rho(R) = 1$ if and only if R is HFD.

By [3] R is an *idf-domain* if each nonzero element of R has atmost a finite number of non-associate irreducible divisors and UFDs are examples of idf-domains. Moreover, Noetherian domain $R = \mathbb{R} + X\mathbb{C}[X]$ is an HFD but not an idf-domain(cf. [3, Example 4.1(a)]).

By [3] R is a *finite factorization domain*(FFD) if each nonzero nonunit of R has a finite number of non-associate divisors and hence, only a finite number of factorizations upto order and associates. An FFD is not an HFD and vise versa. Moreover, an integral domain R is an FFD if and only if R is an atomic idf-domain (cf. [3, Theorem 5.1]).

In general,

FFD $\Leftarrow$ **UFD** $\Rightarrow$ **HFD** $\Rightarrow$ **BFD** $\Rightarrow$ **ACCP** $\Rightarrow$ **Atomic** and

Atomic $\Leftarrow$ **ACCP** $\Leftarrow$ **BFD** $\Leftarrow$ **FFD** $\Rightarrow$ **idf-domin**.

But none of the above implications is reversible.

According to [5] and [11], an element x of an integral domain R is called primal if whenever x divides a product a_1a_2 with $a_1, a_2 \in R$, then x can be written as $x = x_1x_2$ such that x_i divides a_i, $i = 1, 2$ (an element whose divisors are primal elements is called a completely primal). A domain R is called a pre-Schreier if every nonzero element $x \in R$ is primal. An integrally closed pre-Schreier domain is called a Schreier domain. By [5], any GCD-domain (a domain R is called a GCD-domain if every pair of elements of R has a greatest common divisor) is a Schreier domain.

Following Zafrullah [10], we say that an element x of integral domain R is said to be rigid if whenever r, $s \in R$ and r, s divides x, we have s divides r or r divides s. Also R is known to be semirigid if every nonzero element of R can be expressed as a product of a finite number of rigid elements.

By [6, page 326], for a (commutative) ring extension $A \subseteq B$, the conductor of A in B is the largest common ideal $A : B = \{x \in A : xB \subseteq$

$A\}$ of A and B. In [8], the whole study is based on the conductor ideal $A : B$ and on the *Condition*$*$: *Let* $A \subseteq B$ *be a (commutative) ring extension. For each* $x \in B$ *there exist* $x' \in U(B)$ *and* $x'' \in A$ *such that* $x = x'x''$, in which we established a criterion for ascend and descend of factorization properties.

In the following we restate ([8, Proposition 2.6, Proposition 2.7 and Theorem 2.10]) as:

Let $A \subseteq B$ be a domain extension which satisfies the *Condition*$*$ and $M = A : B$ is a maximal ideal in A.

(1)

(a) Then A is *atomic* if and only if B is atomic.

(b) If A is *atomic*, then $\rho(A) = \rho(B)$.

(c) Then A satisfies $ACCP$ if and only if B satisfies $ACCP$.

(d) Then A is a BFD if and only if B is a BFD.

(e) Then A is an HFD if and only if B is an HFD.

(2) If A is a *pre* – *Schreier ring*, then B is a *pre* – *Schreier ring.*

(3) If A is a *semirigid domain*, then B is a *semirigid daomain.*

In this paper we extend the study of ascend and descend of factorization properties to idf-domains and FFDs for a domain extension $A \subseteq B$ which satisfies the *Condition*$*$ whereas $M = A : B$ is a maximal ideal in A. But first we give some examples of ring extensions satisfying the *Condition* $*$.

EXAMPLE 1. (a) If B is a fraction ring of A, then ring extension $A \subseteq B$ satisfies *Condition*$*$. Hence the ring extension $A \subseteq B$ satisfies *Condition*$*$ is the generalization of a localization.

(b) If B is a field, then ring extension $A \subseteq B$ satisfies *Condition* $*$.

(c) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy *Condition*$*$, then so does the ring extension $A \subseteq C$.

(d) If the ring extensions $A \subseteq B$ satisfies *Condition*$*$, then the ring extension $A + XB[X] \subseteq B[X]$ (or $A + XB[[X]] \subseteq B[[X]]$) also satisfies *Condition*$*$.

THEOREM 1. *Let* $A \subseteq B$ *be the domain extension which satisfies the Condition*$*$ *and* $M = A : B$ *is a maximal ideal in* A. *If* A *is an idf* – *domain, then* B *is an idf* – *domain.*

Proof. Suppose A be an idf-domain and let x be a nonzero element of B. Therefore there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$. Since x'' has finite number of irreducible divisors in A, which are also irreducibles in B, by [8, Theorem 2.5(d)]. Hence each nonzero element of B has finite number of irreducible divisors in B. □

THEOREM 2. *Let $A \subseteq B$ be the domain extension which satisfies the Condition∗ and $M = A : B$ is maximal ideal in A. If A is an FFD, then B is an FFD.*

Proof. Follows by [3, Theorem 5.1], [8, Proposition 2.6(a)] and Theorem 1. □

The converse of Theorem 1 and Theorem 2 is not true because if we consider the domain extension $A = \mathbb{R} + X\mathbb{C}[X] \subseteq \mathbb{C}[X] = B$ which satisfies the *Condition∗* and $M = A : B = X\mathbb{C}[X]$ is a maximal ideal in A. As $\mathbb{C}[X]$ being a UFD is an idf-domain but $\mathbb{R} + X\mathbb{C}[X]$ is not an idf-domain (cf. [3, Example 4.1(a)]). Moreover it is observed that $U(B) \cap A = U(A)$ and $U(B)/U(A)$ is infinite. On the other hand for the field extension $F_1 \subseteq F_2$, the domain $F_1 + XF_2[X]$ (or $F_1 + XF_2[[X]]$) is an FFD if and only if F_2^*/F_1^* is finite which is only possible if $F_1 = F_2$ or F_2 is finite (cf. [4, Example 5]).

REMARK 1. Let $A \subseteq B$ be the domain extension which satisfies the *Condition∗* and $M = A : B$ be a maximal ideal in A. Let B be an idf-domain and let x be a nonzero element of A. Therefore x being an element of B has finite number of irreducible divisors in B, say $d_1, d_2, \ldots, d_n$ and hence $x = bd_1d_2 \cdots d_n$, where $b \in B$. But $x = x'x''$, with $x'' \in A$ and $x' \in U(B)$. Therefore

$$x = x'x'' = bd_1d_2 \cdots d_n.$$

This implies

$$\begin{aligned} x'' &= (x')^{-1}bd_1d_2 \cdots d_n \\ &= (x')^{-1}b'b''d_1'd_1''d_2'd_2'' \cdots d_n'd_n'' \\ &= ((x')^{-1}b'd_1'd_2' \cdots d_n')b''d_1''d_2'' \cdots d_n''. \end{aligned}$$

Here $(x')^{-1}, b', d_1', d_2', \ldots, d_n' \in U(B)$ and $b'', d_1'', d_2'', \ldots, d_n'' \in A$. Hence by [8, Theorem 2.5(b)], whenever $d_i \in M$, d_i is irreducible in A if and only if d_i is irreducible in B, so in this case $d_i = 1d_i$. Now if $d_i \in B\backslash M$, then d_i is irreducible in B if and only if d_i'' is irreducible in A, where $d_i = d_i'd_i''$ with $d_i' \in U(B)$ and $d_i'' \in A$ (cf. [8, Theorem 2.5(c)]). Since $b'd_1'd_2' \cdots d_n' = u \in U(B)$, therefore

$$\begin{aligned} x &= x'((x')^{-1}b'd_1'd_2' \cdots d_n')b''d_1''d_2'' \cdots d_n'' \\ &= ub''d_1''d_2'' \cdots d_n''. \end{aligned}$$

Now if $x \in A\backslash M$, then $u \in A$ (by [8, Lemma 2.3(a)]). Since $U(B) \cap A = U(A)$ (by [8, Proposition 2.2(c)]), therefore $u \in U(A)$. Similarly if $x \in M$, then either $b'' \in M$ or $b'' \in A\backslash M$. It is obvious that for

$b'' \in M$, $ub'' \in A$. However it is not always true that for $b'' \in A \backslash M$ and $u \in U(B) \backslash U(A)$, $ub'' \in A$. To make this always happen that $ub'' \in A$, we may assume that $U(B) = U(A)$. If a domain extension $A \subseteq B$ satisfies the *Condition*$*$ such that $U(B) = U(A)$, then $A = B$. But if $F_1 \subseteq F_2$ is proper finite field extension, then the domain extension $A = F_1[X] \subseteq F_1 + XF_2[X] = B$ is such that $U(B) = U(A)$, which does not satisfies the *Condition* $*$. Surprisingly both A and B are FFDs and hence idf-domains.

REMARK 2. In the domain extension $A = \mathbb{Z}_{(2)} + X\mathbb{R}[[X]] \subseteq \mathbb{R}[[X]] = \mathbb{R} + X\mathbb{R}[[X]] = B$, A and B are idf-domains (cf. [3, Page 13]). Obviously this extension satisfies *Condition*$*$ but $A : B$ is not a maximal ideal in A. On the other hand in the domain extension $A = \mathbb{Z}_{(2)} + X\mathbb{R}[[X]] \subseteq \mathbb{Q} + X\mathbb{R}[[X]] = C$, A is an idf-domain but C is not an idf-domain because $\mathbb{R}^*/\mathbb{Q}^*$ is not finite. Here we have also observed that the domain extension $A \subseteq C$ satisfies the *Condition*$*$, indeed; as $\mathbb{Z}_{(2)} \subseteq \mathbb{Q}$ satisfies *Condition**, so if $h(X) = q + X\sum_{i \geq 0} r_i X^i \in \mathbb{Q} + X\mathbb{R}[[X]]$, then $q = q'q''$, where $q' \in \mathbb{Q}^* = U(\mathbb{Q} + X\mathbb{R}[[X]])$, $q'' \in \mathbb{Z}_{(2)}$, hence $h(X) = q'(q'' + X\sum_{i \geq 0} (q')^{-1} r_i X^i)$, where $q'' + X\sum_{i \geq 0} (q')^{-1} r_i X^i \in \mathbb{Z}_{(2)} + X\mathbb{R}[[X]]$. Moreover $A : C$ is not a maximal ideal in A.

REMARK 3. Following [2, Example 5.3], let V be a valuation domain and F be its quotient field such that F is the countable union of an increasing family $\{V_i\}$ of valuation overrings of V. Let K be a proper field extension of F and K^*/F^* is infinite. Then each $R_i = V_i + XK[[X]]$ is an idf-domain. However $R = \cup R_i = F + XK[[X]]$ is not an idf-domain because K^*/F^* is infinite. By this example we have observed several interesting situations regarding the ascend and descend of factorization properties for domain extension.

(i) The domain extension $V_i + XK[[X]] \subseteq K[[X]]$ satisfies the *Condition*$*$ as the extension $V_i \subseteq K$ satisfies the *Condition* $*$. But $XK[[X]]$ is not a maximal ideal in $V_i + XK[[X]]$ and such that $U(V_i + XK[[X]]) \neq U(K[[X]])$. In this case both $V_i + XK[[X]]$ and $K[[X]]$ are idf-domains.

(ii) The domain extension $V_i + XK[[X]] \subseteq F + XK[[X]]$ satisfies the *Condition*$*$ but $XK[[X]]$ is not a maximal ideal in $V_i + XK[[X]]$ and such that $U(V_i + XK[[X]]) \neq U(F + XK[[X]])$. In this case $V_i + XK[[X]]$ is an idf-domain but $F + XK[[X]]$ is not an idf-domain.

(iii) The domain extension $F + XK[[X]] \subseteq K[[X]]$ satisfies the *Condition*$*$ such that $U(F + XK[[X]]) \neq U(K[[X]])$ and $XK[[X]]$ is maximal ideal in $F + XK[[X]]$. But $F + XK[[X]]$ is not an FFD whereas the domain $K[[X]]$ is an FFD.

References

[1] D. D. Anderson and D. F. Anderson, *Elasticity of factorizations in integral domains,* J. Pure Appl. Algebra **80** (1992), no. 3, 217–235.

[2] D. D. Anderson, D. F. Anderson, and M. Zafrullah, *Factorization in integral domains, II,* J. Algebra **152** (1992), no. 1, 78–93.

[3] ———, *Factorization in integral domains,* J. Pure Appl. Algebra **69** (1990), no. 1, 1–19.

[4] D. D. Anderson and B. Mullinns, *Finite Factorization Domains,* Proc. Amer. Math. Soc. **124** (1996), no. 2, 389–396.

[5] P. M. Cohn, *Bezout rings and their subrings,* Proc. Cambridge Philos. Soc. **64** (1968), 251–264.

[6] R. Gilmer, *Multiplicative Ideal Theory,* Marcel Dekker, New York, 1972.

[7] A. Grams, *Atomic rings and the ascending chain condition for principal ideals,* Proc. Cambridge Philos. Soc. **75** (1974), 321–329.

[8] N. Radu, S. O. Ibrahim Al-Salihi, and T. Shah, *Ascend and descend of factorization properties,* Rev. Roumaine Math. Pures Appl. **45** (2000), 4, 659–669.

[9] M. Roitman, *Polynomial extensions of atomic domains,* J. Pure Appl. Algebra **87** (1993), no. 2, 187–199.

[10] M. Zafrullah, *Semirigid GCD domain,* Manuscripta Math. **17** (1975), no. 1, 55–66.

[11] ———, *On a property of pre-Schreier domains,* Comm. Algebra **15** (1987), no. 9, 1895–1920.

[12] A. Zaks, *Half factorial domain,* Bull. Amer. Math. Soc. **82** (1976), no. 6, 721–723.

[13] ———, *Atomic rings without a.c.c. on principal ideals,* J. Algebra **74** (1982), no. 1, 223–231.

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