

COLLAR LEMMA IN QUATERNIONIC HYPERBOLIC MANIFOLD

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ABSTRACT. In this paper, we show that a short simple closed geodesic in quaternionic hyperbolic 2-manifold has an embedded tubular neighborhood whose width only depends on the length of the geodesic.

1. Introduction

The motivation of this work comes from the open question of S. Markham and J. R. Parker on collar lemma in quaternionic hyperbolic space. In [6], S. Markham and J. R. Parker showed that if a simple closed geodesic in a quaternionic hyperbolic 2-manifold is sufficiently short, it has an embedded tubular neighborhood. It is almost obtained by following word by word the proof of complex hyperbolic space case ([6]). And essential technique of the proof of complex hyperbolic space case comes from Zagier's inequality ([7]). However it is not sufficient with only Zagier's inequality to get the relation of the width of an embedded tubular neighborhood and the length of a simple closed geodesic with it as an embedded tubular neighborhood, and so they left this as an open question in [6]. Thus in this paper, as an application of the quaternionic hyperbolic Jørgensen's inequality ([4]), we show that if a simple closed geodesic in a quaternionic hyperbolic 2-manifold is sufficiently short, then there exists an embedded tubular neighborhood of this geodesic, called a *collar*, whose width only depends on the length of the closed geodesic. To do this, we get hint from Meyerhoff's original idea ([7]) and apply the technique of the proof of complex hyperbolic space case used in [6].

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2. Preliminary

Quaternionic hyperbolic 2-space, $\mathbf{H}_{\mathbb{H}}^2$ is a rank one symmetric space of non-compact type and can be defined as the set of negative (quaternionic) lines in a 3-dimensional (so real 12-dimensional) quaternionic vector space

$$\mathbb{H}^{2,1} = \{(q_1 \ q_2 \ q_3)^t \mid q_i \in \mathbb{H}, i = 1, 2, 3\}$$

with the Epstein's second Hermitian form $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ of signature

(2, 1). And quaternionic hyperbolic 2-space has a useful model, called *Siegel domain model*,

$$\begin{aligned} \mathfrak{S} &= \{(q_1, q_2) \in \mathbb{H}^2 \mid |q_1|^2 + 2\Re(q_2) < 0\} \\ &\equiv \{(q_2, q_1, 1) \in \mathbb{P}\mathbb{H}^{2,1} \mid |q_1|^2 + 2\Re(q_2) < 0\} \end{aligned}$$

which can be considered as a generalization of upper half space model of real hyperbolic space. By using *Siegel domain model*, we can give horospherical coordinate to each point of $\mathbf{H}_{\mathbb{H}}^2$ associated a point in $\mathbb{H} \times \mathfrak{S}\mathbb{H} \times \mathbb{R}^+$ as follows ;

$$\begin{aligned} \mathbb{H} \times \mathfrak{S}\mathbb{H} \times \mathbb{R}^+ &\xrightarrow{\cong} \mathfrak{S} \\ \begin{pmatrix} \zeta \\ v \\ u \end{pmatrix} &\mapsto \begin{pmatrix} -|\zeta|^2 - u + v \\ \sqrt{2}\zeta \\ 1 \end{pmatrix} = \begin{pmatrix} q_2 \\ q_1 \\ 1 \end{pmatrix} \Leftrightarrow \zeta = \frac{q_1}{\sqrt{2}}, \\ &u = -\Re(q_2) - \frac{|q_1|^2}{2}, \\ &v = \Im(q_2). \end{aligned}$$

Similarly, we can identify $\partial\mathfrak{S}$ with $\mathbb{H} \times \mathfrak{S}\mathbb{H} \cup \{\infty\}$ as follows ;

$$\begin{aligned} \mathbb{H} \times \mathfrak{S}\mathbb{H} \cup \{\infty\} &\xrightarrow{\cong} \partial\mathfrak{S} \\ \begin{pmatrix} \zeta \\ v \end{pmatrix} &\mapsto \begin{pmatrix} -|\zeta|^2 + v \\ \sqrt{2}\zeta \\ 1 \end{pmatrix} = \begin{pmatrix} q_2 \\ q_1 \\ 1 \end{pmatrix} \Leftrightarrow \zeta = \frac{q_1}{\sqrt{2}}, v = \Im(q_2) \\ \infty &\mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

This identification gives homeomorphism between $\overline{\mathbf{H}_{\mathbb{H}}^2}$ and $\{\mathbf{q} \in \mathbb{P}\mathbb{H}^{2,1} : \langle \mathbf{q}, \mathbf{q} \rangle < 0\}$, and this homeomorphism defines Bergman metric ρ on $\mathbf{H}_{\mathbb{H}}^2$

as follows ([1]) :

$$\cosh^2\left(\frac{\rho(P\mathbf{z}, P\mathbf{w})}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle},$$

where P is a right projection map by third coordinate, i.e.,

$$P : \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} z_1 z_3^{-1} \\ z_2 z_3^{-1} \\ 1 \end{pmatrix}.$$

The isometry group of $\mathbf{H}_{\mathbb{H}}^2$ with respect to the Bergman metric is the non-compact Lie group

$$\begin{aligned} PSp(2, 1) &= \{[A] : A \in GL(3, \mathbb{H}), \langle \mathbf{q}, \mathbf{q}' \rangle = \langle A\mathbf{q}, A\mathbf{q}' \rangle, \mathbf{q}, \mathbf{q}' \in \mathbb{H}^{2,1}\} \\ &= \{[A] : A \in GL(3, \mathbb{H}), J = A^*JA\} \\ &= \{[A] : A \in Sp(2, 1)\}, \end{aligned}$$

where $[A] : \mathbb{P}\mathbb{H}^2 \rightarrow \mathbb{P}\mathbb{H}^2; x\mathbb{H} \mapsto (Ax)\mathbb{H}$ for $A \in Sp(2, 1), x \in \mathbb{H}^{2,1}$ and A^* is the quaternionic Hermitian transpose of A . Thus $PSp(2, 1) = Sp(2, 1)/\pm I$ since for $A \in PSp(2, 1), |\det A| = 1$. The matrix forms of loxodromic elements fixing 0 and ∞ among elements of $PSp(2, 1)$ are

$$\begin{pmatrix} \lambda\mu & 0 & \hat{0} \\ 0 & \mu\nu & 0 \\ 0 & 0 & \frac{\mu}{\lambda} \end{pmatrix},$$

where $\lambda \in \mathbb{R}^+, \mu, \nu \in Sp(1)$ ([5]). To prove collar lemma, the discreteness condition related to loxodromic elements of $PSp(2, 1)$ is necessary. Recently the following theorem related to this condition was proved ([3]).

THEOREM 2.1. *Let A be a loxodromic element of $PSp(2, 1)$ fixing \mathbf{p} and \mathbf{q} and let B be any element of $PSp(2, 1)$. If*

$$M\left(\left|[B(\mathbf{p}), \mathbf{q}, \mathbf{p}, B(\mathbf{q})]\right|^{\frac{1}{2}} + 1\right) < 1 \text{ or } M\left(\left|[B(\mathbf{p}), \mathbf{p}, \mathbf{q}, B(\mathbf{q})]\right|^{\frac{1}{2}} + 1\right) < 1,$$

where $M = |\lambda\mu - 1| + 2|\nu - 1| + \left|\frac{\mu}{\lambda} - 1\right|$, then the group $\langle A, B \rangle$ is elementary or not discrete.

For a discrete subgroup Γ of $PSp(2, 1)$ and a loxodromic element A of Γ with axis γ which is a geodesic, let $T_r(\gamma)$ be the tube of radius r about γ and $C_r = T_r(\gamma)/\langle A \rangle$ be the collar of width r about $\gamma' = \gamma/\langle A \rangle$ which is a simple closed geodesic in $\mathbf{H}_{\mathbb{H}}^2/\Gamma$. Then by using the above theorem, there exists an embedded collar of the simple closed geodesic in $\mathbf{H}_{\mathbb{H}}^2/\Gamma$ by the following theorem ([6]).

THEOREM 2.2. *Let Γ be a discrete, non-elementary and torsion-free subgroup of $PSp(2, 1)$. Let $A = \begin{pmatrix} \lambda\mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \frac{\mu}{\lambda} \end{pmatrix}$ be a loxodromic element of Γ with axis γ , where $\lambda > 1, \mu, \nu \in Sp(1)$. Put $M = |\lambda\mu - 1| + 2|\nu - 1| + \left| \frac{\mu}{\lambda} - 1 \right|$ and suppose that $M < \frac{1}{2}$. Let $r > 0$ be defined by $\cosh(r) + 1 = \frac{1}{M}$. Then $T_r(\gamma)$ is precisely invariant under Γ .*

COROLLARY 2.3. *Let Γ, A and r be as in Theorem. Then in $\mathbf{H}_{\mathbb{H}}^2/\Gamma$, the simple closed geodesic $\gamma' = \gamma/\langle A \rangle$ has an embedded collar $C_r(\gamma')$ of width r .*

The proof of Theorem 2.2 is essentially obtained by following step by step the proof of complex hyperbolic space case.

3. Main theorem

Now we show that if a simple closed geodesic in a quaternionic hyperbolic 2-manifold is sufficiently short, then there exists an embedded tubular neighborhood of this geodesic, called a *collar*, whose width only depends on the length of the closed geodesic in spite of the complexity of constant M appearing in quaternionic hyperbolic Jørgensen inequality comparing with the one of complex hyperbolic space. We state main theorem.

THEOREM 3.1. *Let Γ be a discrete, non-elementary and torsion-free subgroup of $PSp(2, 1)$. Let $A = \begin{pmatrix} \lambda\mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \frac{\mu}{\lambda} \end{pmatrix}$ be a loxodromic element of Γ with axis γ where $\lambda = e^{\frac{l}{2}} > 1, \mu, \nu \in Sp(1)$. There exists an increasing smooth function k defined for l satisfying*

$$\sqrt{\left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^3 l}{\sqrt{3}}}\right) + 1} \left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^3 l}{\sqrt{3}}} - 1 + \frac{k(l)}{4}\right) + 2 \sin \frac{k(l)}{2} \leq \frac{1}{4}$$

and $\lim_{l \rightarrow 0} \frac{\sqrt[3]{l}}{k(l)} = 0$ such that if $r > 0$ is defined by $\cosh(r) + 1 =$

$$\frac{1}{2 \left(\sqrt{\left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^3 l}{\sqrt{3}}}\right) + 1} \left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^3 l}{\sqrt{3}}} - 1 + \frac{k(l)}{4}\right) + 2 \sin \frac{k(l)}{2} \right)},$$

then $T_r(\gamma)$ is precisely invariant under Γ . And in this case, $r \rightarrow \infty$ as $l \rightarrow 0$.

Proof. Let $\mu = e^{I\alpha} := \cos \alpha + I \sin \alpha, \nu = e^{J\beta} := \cos \beta + J \sin \beta$, where $\cos \alpha = \Re(\mu), I = \frac{\Im(\mu)}{|\Im(\mu)|} = \frac{\Im(\mu)}{|\sin \alpha|}, \cos \beta = \Re(\nu), J = \frac{\Im(\nu)}{|\Im(\nu)|} = \frac{\Im(\nu)}{|\sin \beta|}$.

Then

$$\begin{aligned} M &= |\lambda\mu - 1| + \left| \frac{\mu}{\lambda} - 1 \right| + 2|\nu - 1| \\ &= |e^{\frac{l}{2}}e^{I\alpha} - 1| + |e^{-\frac{l}{2}}e^{I\alpha} - 1| + 2|e^{J\beta} - 1| \\ &= 2\sqrt{(\cosh \frac{l}{2} + 1)(\cosh \frac{l}{2} - \cos \alpha)} + 4\left| \sin \frac{\beta}{2} \right|. \end{aligned}$$

Here we use the following two facts originated from Zagier and Meyerhoff respectively.

PROPOSITION. [Zagier’s inequality] ([7]) For all $0 < l < 2\pi\sqrt{3}$ and $0 \leq \theta < 2\pi$, there exists $n \in \mathbb{N}$ such that

$$\cosh\left(\frac{nl}{2}\right) - \cos(n\theta) \leq \cosh \sqrt{\frac{2\pi l}{\sqrt{3}}} - 1.$$

PROPOSITION. [Pigeonhole Principle] ([7]) For all $0 \leq \theta < 2\pi$ and $N \in \mathbb{N}$, there exists $n \leq N$ and $n' \leq N$ such that

$$\cos(n\theta) \geq \cos\left(\frac{2\pi}{N}\right) \quad \text{and} \quad \sin(n'\theta) \leq \sin\left(\frac{2\pi}{N}\right).$$

First of all, consider a function $k(l)$ satisfying $\lim_{l \rightarrow 0} \frac{\sqrt[3]{l}}{k(l)} = 0$. Then by the pigeonhole principle, there exists an integer m_μ such that $1 \leq m_\mu \leq 2\left(\left\lceil \frac{4\pi}{k(l)} \right\rceil\right)^2$ and $\cos m_\mu \alpha \geq \cos\left(\frac{\pi}{(\lceil \frac{4\pi}{k(l)} \rceil)^2}\right)$, where $\lceil x \rceil$ is called the ceiling of x , denoting the least integer $\geq x$ (e.g., $\left\lceil \frac{4\pi}{k(l)} \right\rceil = n + 1$ if and only if

$$\frac{4\pi}{k(l)} \leq n + 1 < \frac{4\pi}{k(l)} + 1 \Leftrightarrow \frac{2\pi}{n + 1} \leq \frac{k(l)}{2} < \frac{2\pi}{n}.$$

And there exists an integer m_ν such that $1 \leq m_\nu \leq \lceil \frac{4\pi}{k(l)} \rceil$ and

$$\sin\left(\frac{m_\nu m_\mu \beta}{2}\right) \leq \sin\left(\frac{2\pi}{\lceil \frac{4\pi}{k(l)} \rceil}\right) \leq \sin \frac{k(l)}{2}.$$

Thus we have

$$\begin{aligned} m_\nu m_\mu l &\leq 2\left(\lceil \frac{4\pi}{k(l)} \rceil\right)^3 l \\ &< 2\left(\frac{4\pi}{k(l)} + 1\right)^3 l \leq 2\pi\sqrt{3} \end{aligned}$$

and for l satisfying

$$(*) \quad 2\left(\frac{4\pi}{k(l)} + 1\right)^3 l \leq 2\pi\sqrt{3},$$

by Zagier's inequality, there exists $n_0 \in \mathbb{N}$ such that

$$\cosh\left(\frac{n_0(m_\nu m_\mu l)}{2}\right) - \cos n_0(m_\nu m_\mu \alpha) \leq \cosh \sqrt{\frac{2\pi(m_\nu m_\mu l)}{\sqrt{3}}} - 1.$$

Now denote the corresponding M for A by $M_{m_\nu m_\mu}$ for $A^{m_\nu m_\mu}$. Then

$$\begin{aligned} &M_{m_\nu m_\mu} \\ &:= 2\sqrt{(\cosh \frac{m_\nu m_\mu l}{2} + 1)(\cosh \frac{m_\nu m_\mu l}{2} - \cos m_\nu m_\mu \alpha) + 4|\sin \frac{m_\nu m_\mu \beta}{2}|} \\ &\leq 2\sqrt{(\cosh \frac{m_\nu m_\mu l}{2} + 1)(\cosh \frac{m_\nu m_\mu l}{2} - \cos \frac{m_\nu \pi}{(\lceil \frac{4\pi}{k(l)} \rceil)^2}) + 4|\sin \frac{m_\nu m_\mu \beta}{2}|} \\ &\leq 2\sqrt{(\cosh \frac{n_0(m_\nu m_\mu l)}{2} + 1)(\cosh \frac{n_0(m_\nu m_\mu l)}{2} - \cos \frac{m_\nu \pi}{(\lceil \frac{4\pi}{k(l)} \rceil)^2}) + 4 \sin \frac{k(l)}{2}} \\ &\leq 2\sqrt{(\cosh \sqrt{\frac{2\pi(m_\nu m_\mu l)}{\sqrt{3}}} + 1)} \\ &\quad \times \sqrt{(\cosh \sqrt{\frac{2\pi(m_\nu m_\mu l)}{\sqrt{3}}} - (1 + \cos \frac{m_\nu \pi}{(\lceil \frac{4\pi}{k(l)} \rceil)^2} - \cos n_0(m_\nu m_\mu \alpha)) + 4 \sin \frac{k(l)}{2}} \\ &\leq 2\sqrt{(\cosh \sqrt{\frac{2\pi m_\nu m_\mu l}{\sqrt{3}}} + 1)(\cosh \sqrt{\frac{2\pi m_\nu m_\mu l}{\sqrt{3}}} - 1 + \frac{\lceil \frac{4\pi}{k(l)} \rceil \pi}{(\lceil \frac{4\pi}{k(l)} \rceil)^2}) + 4 \sin \frac{k(l)}{2}} \\ &\leq 2\sqrt{(\cosh \sqrt{\frac{2\pi \cdot \lceil \frac{4\pi}{k(l)} \rceil \cdot 2 \cdot (\lceil \frac{4\pi}{k(l)} \rceil)^2 l}{\sqrt{3}}} + 1)} \\ &\quad \times \sqrt{(\cosh \sqrt{\frac{2\pi \cdot \lceil \frac{4\pi}{k(l)} \rceil \cdot 2 \cdot (\lceil \frac{4\pi}{k(l)} \rceil)^2 l}{\sqrt{3}}} - 1 + \frac{\pi}{\lceil \frac{4\pi}{k(l)} \rceil}) + 4 \sin \frac{k(l)}{2}} \end{aligned}$$

$$\begin{aligned}
 &= 2\sqrt{\left(\cosh \sqrt{\frac{4\pi(\lceil \frac{4\pi}{k(l)} \rceil)^{3l}}{\sqrt{3}}} + 1\right) \left(\cosh \sqrt{\frac{4\pi(\lceil \frac{4\pi}{k(l)} \rceil)^{3l}}{\sqrt{3}}} - 1 + \frac{\pi}{\lceil \frac{4\pi}{k(l)} \rceil}\right) + 4 \sin \frac{k(l)}{2}} \\
 &\leq 2\sqrt{\left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^{3l}}{\sqrt{3}}} + 1\right) \left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^{3l}}{\sqrt{3}}} - 1 + \frac{\pi}{k(l)}\right) + 4 \sin \frac{k(l)}{2}} \\
 &= 2\sqrt{\left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^{3l}}{\sqrt{3}}} + 1\right) \left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^{3l}}{\sqrt{3}}} - 1 + \frac{k(l)}{4}\right) + 4 \sin \frac{k(l)}{2}} \\
 &\leq \frac{1}{2}.
 \end{aligned}$$

In particular, we have used $1 \leq m_\nu \leq \left\lceil \frac{4\pi}{k(l)} \right\rceil$ in third line and mean value theorem in seventh line. Now since $\lim_{l \rightarrow 0} \frac{\sqrt[3]{l}}{k(l)} = 0$, $\lim_{l \rightarrow 0} \left(\frac{4\pi}{k(l)} + 1\right)^3 l = 0$ and so for sufficiently small l ,

$$\begin{aligned}
 (**) \quad &\sqrt{\left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^{3l}}{\sqrt{3}}} + 1\right) \left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^{3l}}{\sqrt{3}}} - 1 + \frac{k(l)}{4}\right)} \\
 &+ 2 \sin \frac{k(l)}{2} \leq \frac{1}{4}
 \end{aligned}$$

is satisfied and for such l , (*) is also satisfied. Thus if we define k for l satisfying (**) and apply Theorem 2.2. with A replaced by $A^{m_\nu m_\mu}$, we obtain the result of this theorem. \square

COROLLARY 3.2. *Let γ' be a simple closed geodesic of length l in a quaternionic hyperbolic 2-manifold. There exists an increasing smooth function k defined for l satisfying*

$$\sqrt{\left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^{3l}}{\sqrt{3}}} + 1\right) \left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^{3l}}{\sqrt{3}}} - 1 + \frac{k(l)}{4}\right) + 2 \sin \frac{k(l)}{2}} \leq \frac{1}{4}$$

and $\lim_{l \rightarrow 0} \frac{\sqrt[3]{l}}{k(l)} = 0$ such that if $r > 0$ is defined by

$$\cosh(r)+1 = \frac{1}{2 \left(\sqrt{\left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^{3l}}{\sqrt{3}}} + 1\right) \left(\cosh \sqrt{\frac{4\pi(\frac{4\pi}{k(l)}+1)^{3l}}{\sqrt{3}}} - 1 + \frac{k(l)}{4}\right) + 2 \sin \frac{k(l)}{2}} \right)},$$

then γ' has an embedded collar $C_r(\gamma')$ of width r .

EXAMPLE. Define a function k as $k(l) = \sqrt[3]{l}$. Then clearly $\lim_{l \rightarrow 0} \frac{\sqrt[3]{l}}{k(l)} = 0$ and for $0 < l \leq 1.103320907 \times 10^{-10}$, (**) is satisfied.

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