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THE QUASIHYPERBOLIC METRIC AND ANALOGUES OF THE HARDY-LITTLEWOOD PROPERTY FOR $\alpha = 0$ IN UNIFORMLY JOHN DOMAINS

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ABSTRACT. We characterize the class of uniformly John domains in terms of the quasihyperbolic metric and from the result we get some analogues of the Hardy-Littlewood property for $\alpha = 0$ in uniformly John domains.

1. Introduction

Suppose that D is a subdomain of euclidean n-space $\mathbb{R}^n$, $n \geq 2$. Let $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. Let $\mathbb{B}(x,r) = \{w : |w-x| < r\}$ for $x \in \mathbb{R}^n$ and $r > 0$. Let $\ell(\gamma)$ denote the euclidean length of a curve γ, and $\operatorname{dist}(A,B)$ denote the euclidian distance from A to B for two sets $A, B \subset \overline{\mathbb{R}}^n$. Let $\operatorname{dia}(\gamma)$ denote a diameter of γ.

A domain $D \subset \overline{\mathbb{R}}^2$ is a *conformal disk* if it is conformally equivalent to $\mathbb{B}(0,1)$; i.e., D is a conformal disk if and only if ∂D is a non-degenerate continuum.

A domain D in $\mathbb{R}^n$ is said to be *b-uniform* if there exists a constant $b \geq 1$ such that each pair of points x_1 and x_2 in D can be joined by a rectifiable arc γ in D with

$$\ell(\gamma) \leq b|x_1 - x_2|$$

and with

$$\min_{j=1,2} \ell(\gamma(x_j, x)) \leq b \operatorname{dist}(x, \partial D) \tag{1.1}$$

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for each $x \in \gamma$, where $\gamma(x_j, x)$ is the part of γ between x_j and x.

We define two internal metrics $\rho_D(x,y)$ and $\lambda_D(x,y)$ by

$$\rho_D(x,y) = \inf \operatorname{dia}(\gamma), \qquad \lambda_D(x,y) = \inf \ell(\gamma)$$

for x, $y \in D$. Here infimums are taken over all open arcs γ which join x and y in D. Obviously $|x-y| \le \rho_D(x,y) \le \lambda_D(x,y)$.

We say that D is a *b-uniformly John domain* if there exists a constant $b \ge 1$ such that each pair of points x_1, $x_2 \in D$ can be joined by an arc $\gamma \subset D$ which satisfies (1.1) and

$$\ell(\gamma) \le b\rho_D(x_1, x_2). \tag{1.2}$$

A domain D is said to be a *b-John domain* if there is a constant $b \ge 1$ such that each pair of points $x_1, x_2 \in D$ can be joined by an arc γ in D which satisfies (1.1) [16]. We call a simply connected John domain in $\mathbb{R}^2$ a *John disk.*

A uniformly John domain is a domain intermediate between a uniform domain and a John domain. By definition

$$\text{uniform} \subsetneqq \text{uniformly John} \subsetneqq \text{John}.$$

Balogh and Volberg [1], [2] introduced a uniformly John domain in connection with conformal dynamics.

Given a set A in $\mathbb{R}^n$, we let $Lip_\alpha(A)$, $0 < \alpha \le 1$, denote the *Lipschitz class* of mapping $f : A \to \mathbb{R}^p$ satisfying for some constant $m < \infty$ such that

$$|f(x_1) - f(x_2)| \le m|x_1 - x_2|^\alpha \tag{1.3}$$

for all x_1 and x_2 in A. If D is a domain in $\mathbb{R}^n$, then $f : D \to \mathbb{R}^p$ is said to belong to the *local Lipschitz class*, $locLip_\alpha(D)$, if there is a constant $m < \infty$ such that (1.3) holds whenever x_1, x_2 lie in any open ball which is contained in D.

In $Lip_\alpha(D)$ and $locLip_\alpha(D)$ we shall use seminorms $||f||_\alpha$ and $||f||_\alpha^{loc}$, respectively, which mean the infimum of the numbers m for which (1.3) holds in the corresponding set.

A domain $D \subset \mathbb{R}^n$ is called a *Lip_α-extension domain* if there exists a constant a depending on D, α and p such that $f \in locLip_\alpha(D)$ implies $f \in Lip_\alpha(D)$ with

$$||f||_\alpha \le a||f||_\alpha^{loc}.$$

Suppose that f is analytic in $D \subset \mathbb{R}^2$. If f is in $Lip_\alpha(D)$, then it is not difficult to show that

$$|f'(z)| \leq m \operatorname{dist}(z, \partial D)^{\alpha-1}$$

in D. Conversely, we have the following well known result of Hardy and Littlewood.

THEOREM 1.1. [8] *If D is an open disk and f is analytic in D with*

$$|f'(z)| \leq m \operatorname{dist}(z, \partial D)^{\alpha-1} \tag{1.4}$$

for all $z \in D$ and for every $\alpha \in (0,1]$, then $f \in Lip_\alpha(D)$ with

$$||f||_\alpha \leq \frac{cm}{\alpha},$$

where c is an absolute constant.

The above theorem leads to the following notion, introduced in [4].

DEFINITION 1.2. A proper subdomain D in $\mathbb{R}^2$ is said to have the *Hardy-Littlewood property of order α*, $\alpha \in (0,1]$, if there exists a constant $c = c(D)$ such that whenever f is analytic in D with (1.4) for all $z \in D$ and for some $\alpha \in (0,1]$, then $f \in Lip_\alpha(D)$ with

$$||f||_\alpha \leq \frac{cm}{\alpha}.$$

Theorem 1.1 tells that each open disk has the Hardy-Littlewood property of order α for all $\alpha \in (0,1]$. In [4, Corollary 2.2] it is proved that uniform domains have the same property. Also it is showed that there exist domains having the Hardy-Littlewood property of order α without being uniform [15].

We define the *quasihyperbolic metric* k_D in a domain $D \subset \mathbb{R}^n$ by

$$k_D(x_1, x_2) = \inf_\gamma \int_\gamma \frac{ds}{\operatorname{dist}(x, \partial D)},$$

where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in D.

Furthermore, we define the distance function δ_D on a domain $D \subset \overline{\mathbb{R}}^2$ by

$$\delta_D(z_1, z_2) = \sup |f(z_1) - f(z_2)|,$$

where the supremum is taken over all analytic functions f on D satisfying

$$|f'(z)| \leq \operatorname{dist}(z, \partial D)^{-1}$$

for all $z \in D$.

Now let us recall a relation of the distance functions k_D and δ_D on a domain D.

LEMMA 1.3. [12, Theorem 1][14, Lemma 4.1] *In a conformal disk* $D \subset \mathbb{R}^2$,

$$\delta_D(z_1, z_2) \le k_D(z_1, z_2) \le c_0 \delta_D(z_1, z_2)$$

for all $z_1, z_2 \in D$, *where* c_0 *is an absolute constant.*

In Section 2 we give Theorem 2.1 which characterizes uniformly John domains in terms of the inner diameter metric and the quasihyperbolic metric. In Section 3 we give two applications of Theorem 2.1 which are analogues of the Hardy-Littlewood Property for $\alpha = 0$ in uniformly John domains in $\mathbb{R}^n, n \ge 2$.

Results in this paper, [9], [10] and [11] show that a uniformly John domain is a domain intermediate between a uniform domain and a John domain.

2. Quasihyperbolic metric in uniformly John domains

In [6], Gehring and Osgood essentially showed (up to an additive constant) that a domain $D \subset \mathbb{R}^n$ is uniform if and only if it satisfies

$$k_D(x_1, x_2) \le c j_D(x_1, x_2)$$

for all $x_1, x_2 \in D$ and some constant c, where

$$j_D(x_1, x_2) = \frac{1}{2} \log \left(\frac{|x_1 - x_2|}{\operatorname{dist}(x_1, \partial D)} + 1 \right) \left(\frac{|x_1 - x_2|}{\operatorname{dist}(x_2, \partial D)} + 1 \right).$$

We define a similar metric j_D^* by

$$j_D^*(x_1, x_2) = \frac{1}{2} \log \left(\frac{\rho_D(x_1, x_2)}{\operatorname{dist}(x_1, \partial D)} + 1 \right) \left(\frac{\rho_D(x_1, x_2)}{\operatorname{dist}(x_2, \partial D)} + 1 \right).$$

We find that k_D and j_D^* are related in uniformly John domains.

THEOREM 2.1. *Suppose that* D *is a proper subdomain in* $\mathbb{R}^n$. *Then* D *is a* b*-uniformly John domain if and only if there exists a constant* c *such that*

$$k_D(x_1, x_2) \le c j_D^*(x_1, x_2) \tag{2.1}$$

for all $x_1, x_2 \in D$, *where* b *and* c *depend only on each other.*

To prove Theorem 2.1 we need two lemmas.

LEMMA 2.2. [11, *Lemma* 4.3] *For any* $c \geq 1$ *and* $x \geq 0$,

$$\log(cx+1) \leq c\log(x+1).$$

LEMMA 2.3. [7, *Lemma 2.1*]

$$\left|\log \frac{\operatorname{dist}(x_1, \partial D)}{\operatorname{dist}(x_2, \partial D)}\right| \leq k_D(x_1, x_2).$$

The proof of Theorem 2.1 is similar to that of Theorem 1 and Theorem 2 in [6].

Proof of necessity of Theorem 2.1. Suppose that D is a b-uniformly John domain. Then by definition there exists a constant $b \geq 1$ such that each pair of points x_1, $x_2 \in D$ can be joined by an arc $\gamma \subset D$ which satisfies (1.1) and (1.2). Choose $x_0 \in \gamma$ so that $\ell(\gamma(x_0, x_1)) = \ell(\gamma(x_0, x_2))$. Then by the triangle inequality it is sufficient to show that

$$k_D(x_j, x_0) \leq c\log\left(\frac{\rho_D(x_1, x_2)}{\operatorname{dist}(x_j, \partial D)} + 1\right) \tag{2.2}$$

for $j = 1, 2$, where $c = 2b(2b+1)$. By symmetry we may assume that $j = 1$.

Suppose first that

$$\ell(\gamma(x_1, x_0)) \leq \frac{b}{b+1}\operatorname{dist}(x_1, \partial D). \tag{2.3}$$

Then $x_0 \in \overline{\mathbb{B}}\left(x_1, \frac{b}{b+1}\operatorname{dist}(x_1, \partial D)\right)$. If $x \in [x_1, x_0]$, then

$$\operatorname{dist}(x, \partial D) \geq \operatorname{dist}(x_1, \partial D) - |x_1 - x| \geq \frac{1}{b+1}\operatorname{dist}(x_1, \partial D)$$

and hence by (1.1)

$$\begin{aligned} |x_1 - x| + \operatorname{dist}(x_1, \partial D) &\leq \ell(\gamma(x_1, x)) + (b+1)\operatorname{dist}(x, \partial D) \\ &< b\operatorname{dist}(x, \partial D) + (b+1)\operatorname{dist}(x, \partial D) \\ &= (2b+1)\operatorname{dist}(x, \partial D). \end{aligned} \tag{2.4}$$

Thus by (1.2), (2.4) and Lemma 2.2

$$\begin{aligned}
k_D(x_1, x_0) &\leq \int_{[x_1,x_0]} \frac{ds}{\operatorname{dist}(x, \partial D)} \leq \int_0^{|x_1-x_0|} \frac{2b+1}{s+\operatorname{dist}(x_1, \partial D)} ds \\
&\leq (2b+1) \log \left(\frac{\ell(\gamma)}{\operatorname{dist}(x_1, \partial D)} + 1 \right) \\
&\leq (2b+1) b \log \left(\frac{\rho_D(x_1, x_2)}{\operatorname{dist}(x_1, \partial D)} + 1 \right).
\end{aligned}$$

This implies (2.2).

Next suppose that (2.3) does not hold and choose $y_1 \in \gamma(x_1, x_0)$ so that

$$\ell(\gamma(x_1, y_1)) = \frac{b}{b+1} \operatorname{dist}(x_1, \partial D).$$

If $x \in \gamma(y_1, x_0)$, then by (1.1)

$$\operatorname{dist}(x, \partial D) \geq \frac{1}{b} \ell(\gamma(x_1, x))$$

and hence again by (1.2) and Lemma 2.2

$$\begin{aligned}
k_D(y_1, x_0) &\leq \int_{\gamma(y_1,x_0)} \frac{ds}{\operatorname{dist}(x, \partial D)} \\
&\leq b \int_{\gamma(y_1,x_0)} \frac{ds}{\ell(\gamma(x_1, y_1)) + \ell(\gamma(y_1, x))} \\
&= b \int_0^{\ell(\gamma(y_1,x_0))} \frac{ds}{\frac{b}{b+1} \operatorname{dist}(x_1, \partial D) + s} \\
&\leq b \log \left(\frac{b+1}{b} \frac{\ell(\gamma(x_1, x_0))}{\operatorname{dist}(x_1, \partial D)} + 1 \right) \\
&\leq (b+1) \log \left(\frac{\ell(\gamma)}{\operatorname{dist}(x_1, \partial D)} + 1 \right) \\
&\leq (b+1) b \log \left(\frac{\rho_D(x_1, x_2)}{\operatorname{dist}(x_1, \partial D)} + 1 \right).
\end{aligned}$$

We also have

$$k_D(x_1, y_1) \leq (2b+1) b \log \left(\frac{\rho_D(x_1, x_2)}{\operatorname{dist}(x_1, \partial D)} + 1 \right)$$

by what was proved above. Then (2.2) follows from the triangle inequality. □

Proof of sufficiency of Theorem 2.1. Suppose that (2.1) holds. Fix $x_1, x_2 \in D$ and let γ be the quasihyperbolic geodesic joining x_1, x_2 in D. We may assume that $\operatorname{dist}(x_1, \partial D) \geq \operatorname{dist}(x_2, \partial D)$. We want to show that (1.1) and (1.2). Set

$$r = \min\{\sup_{x \in \gamma} \operatorname{dist}(x, \partial D), 2\rho_D(x_1, x_2)\}.$$

We shall consider the cases where

$$r < \operatorname{dist}(x_1, \partial D)$$

and where

$$r \geq \operatorname{dist}(x_1, \partial D) \tag{2.5}$$

separately.

Suppose first that $r < \operatorname{dist}(x_1, \partial D)$. Then $r = 2\rho_D(x_1, x_2)$ and

$$|x_1 - x_2| < \frac{1}{2} \operatorname{dist}(x_1, \partial D) \leq \operatorname{dist}(x, \partial D)$$

for all x on the segment β joining x_1 and x_2. Thus

$$x_2 \in \overline{\mathbb{B}}\left(x_1, \frac{1}{2} \operatorname{dist}(x_1, \partial D)\right) \subset D$$

and hence $\rho_D(x_1, x_2) = |x_1 - x_2|$ and $\beta \subset D$, and therefore

$$k_D(x_1, x_2) \leq \int_\beta \frac{ds}{\operatorname{dist}(x, \partial D)} \leq \frac{2|x_1 - x_2|}{\operatorname{dist}(x_1, \partial D)} \leq 1.$$

Since $k_D(x, x_1) \leq k_D(x_1, x_2)$ for $x \in \gamma$, Lemma 2.3 yields the estimate

$$e^{-1} \operatorname{dist}(x_1, \partial D) \leq \operatorname{dist}(x, \partial D) \leq e \operatorname{dist}(x_1, \partial D)$$

for each $x \in \gamma$. These inequalities imply that

$$\begin{aligned} \ell(\gamma) &\leq \int_\gamma e \frac{\operatorname{dist}(x_1, \partial D)}{\operatorname{dist}(x, \partial D)} ds = e \operatorname{dist}(x_1, \partial D) k_D(x_1, x_2) \\ &\leq e \operatorname{dist}(x_1, \partial D) \frac{2|x_1 - x_2|}{\operatorname{dist}(x_1, \partial D)} \leq 2e\rho_D(x_1, x_2) \end{aligned}$$

and that for each $x \in \gamma$

$$\begin{aligned}\ell(\gamma(x_1, x)) &\le \ell(\gamma) \le e \operatorname{dist}(x_1, \partial D) k_D(x_1, x_2) \\ &\le e \operatorname{dist}(x_1, \partial D) \le e^2 \operatorname{dist}(x, \partial D)\end{aligned}$$

and hence (1.1) and (1.2) are obtained.

Suppose next that (2.5) holds. By compactness there exists a point $x_0 \in \gamma$ with

$$r \le \sup_{x \in \gamma} \operatorname{dist}(x, \partial D) = \operatorname{dist}(x_0, \partial D).$$

Next for $j = 1, 2$ let m_j denote the largest integer for which

$$2^{m_j} \operatorname{dist}(x_j, \partial D) \le r,$$

and let y_j be the first point of $\gamma(x_j, x_0)$ with

$$\operatorname{dist}(y_j, \partial D) = 2^{m_j} \operatorname{dist}(x_j, \partial D)$$

as we traverse γ from x_j towards x_0. Obviously

$$\operatorname{dist}(y_j, \partial D) \le r < 2 \operatorname{dist}(y_j, \partial D). \tag{2.6}$$

We first show that for $j = 1, 2$

$$\begin{cases} \ell(\gamma(x_j, y_j)) \le b' \operatorname{dist}(y_j, \partial D), \\ \ell(\gamma(x_j, x)) \le b' e^{b'} \operatorname{dist}(x, \partial D) \text{ for } x \in \gamma(x_j, y_j). \end{cases} \tag{2.7}$$

Clearly we need only consider the case where $j = 1$ and $m_1 \ge 1$. For this choose points $z_1, \dots, z_{m_1+1} \in \gamma(x_1, y_1)$ so that $z_1 = x_1$ and so that z_j is the first point of $\gamma(x_1, y_1)$ for which

$$\operatorname{dist}(z_j, \partial D) = 2^{j-1} \operatorname{dist}(x_1, \partial D) \tag{2.8}$$

as we traverse γ from x_1 towards y_1. Then $z_{m_1+1} = y_1$. Fix j and set

$$t = \frac{\ell(\gamma(z_j, z_{j+1}))}{\operatorname{dist}(z_j, \partial D)}.$$

If $x \in \gamma(z_j, z_{j+1})$, then

$$\operatorname{dist}(x, \partial D) \le \operatorname{dist}(z_{j+1}, \partial D) = 2 \operatorname{dist}(z_j, \partial D),$$

and hence

$$t \le 2 \int_{\gamma_j} \frac{ds}{\operatorname{dist}(z, \partial D)} = 2k_D(z_j, z_{j+1}),$$

where $\gamma_j = \gamma(z_j, z_{j+1})$, since γ is a quasihyperbolic geodesic. Now

$$j_D^*(z_j, z_{j+1}) \le 2 \log \left(\frac{\rho_D(z_j, z_{j+1})}{\operatorname{dist}(z_j, \partial D)} + 1 \right) \le 2 \log(t+1),$$

whence (2.1) implies that

$$\begin{aligned} \frac{t}{4} &\le k_D(z_j, z_{j+1}) \le c j_D^*(z_j, z_{j+1}) \\ &\le 2c \log(t+1) \le 2c(t+1)^{\frac{1}{2}}, \end{aligned}$$

since $\log x \le x^{\frac{1}{2}}$ for $x > 0$.

If $t \ge 1$, we see from above inequalities that

$$t \le 2k_D(z_j, z_{j+1}) \le 4c(t+1)^{\frac{1}{2}} \le 4c(2t)^{\frac{1}{2}}$$

and hence

$$t \le 32c^2 = b'. \tag{2.9}$$

Thus

$$k_D(z_j, z_{j+1}) \le 2c(2b')^{\frac{1}{2}} < b'. \tag{2.10}$$

If $t < 1$, then $t < b'$ and again we have (2.10). Next if $x \in \gamma(z_j, z_{j+1})$, then from Lemma 2.3

$$0 < \log \frac{\operatorname{dist}(z_{j+1}, \partial D)}{\operatorname{dist}(x, \partial D)} \le k_D(x, z_{j+1}) \le k_D(z_j, z_{j+1}),$$

and with (2.9) and (2.10) we conclude that

$$\begin{cases} \ell(\gamma(z_j, z_{j+1})) \le b' \operatorname{dist}(z_j, \partial D), \\ \operatorname{dist}(z_{j+1}, \partial D) \le e^{b'} \operatorname{dist}(x, \partial D) \text{ for } x \in \gamma(z_j, z_{j+1}), \end{cases} \tag{2.11}$$

for $j = 1, \dots, m_1$.

Hence

$$\ell(\gamma(x_1,y_1)) = \sum_{j=1}^{m_1} \ell(\gamma(z_j,z_{j+1})) \le b' \sum_{j=1}^{m_1} \operatorname{dist}(z_j,\partial D)$$
$$= b'(2^{m_1}-1)\operatorname{dist}(x_1,\partial D) < b' \operatorname{dist}(y_1,\partial D)$$

by (2.8) and (2.11). This proves the first inequality in (2.7). Next if $x \in \gamma(x_1,y_1)$, then $x \in \gamma(z_j,z_{j+1})$ for some j and

$$\ell(\gamma(x_1,x)) = \sum_{i=1}^{j} \ell(\gamma(z_i,z_{i+1})) \le b' \sum_{i=1}^{j} \operatorname{dist}(z_i,\partial D)$$
$$< b' \operatorname{dist}(z_{j+1},\partial D) \le b'e^{b'} \operatorname{dist}(x,\partial D)$$

again by (2.8) and (2.11). This completes the proof of (2.7).

We show next that if $\operatorname{dist}(y_1,\partial D) \le \operatorname{dist}(y_2,\partial D)$, then

$$(2.12) \qquad \begin{cases} \ell(\gamma(y_1,y_2)) \le b'e^{b'} \operatorname{dist}(y_1,\partial D), \\ \operatorname{dist}(y_2,\partial D) \le e^{b'} \operatorname{dist}(x,\partial D) \text{ for } x \in \gamma(y_1,y_2). \end{cases}$$

Obviously we may assume that $y_1 \neq y_2$ since otherwise there is nothing to prove.

Suppose first that

$$r = \sup_{x\in\gamma} \operatorname{dist}(x,\partial D)$$

and set

$$t = \frac{\ell(\gamma(y_1,y_2))}{\operatorname{dist}(y_1,\partial D)}.$$

If $x \in \gamma(y_1,y_2)$, then

$$\operatorname{dist}(x,\partial D) \le r \le 2\operatorname{dist}(y_1,\partial D),$$

by (2.6) and we can repeat the proof of (2.11), with z_j replaced by y_1 and z_{j+1} by y_2, to obtain (2.12).

Suppose next that

$$r = 2\rho_D(x_1,x_2).$$

Then the triangle inequality, (2.6) and (2.7) imply that

$$\rho_D(y_1,y_2) \le \ell(\gamma(x_1,y_1)) + \ell(\gamma(x_2,y_2)) + \rho_D(x_1,x_2)$$
$$\le b' \operatorname{dist}(y_1,\partial D) + b' \operatorname{dist}(y_2,\partial D) + \frac{r}{2}$$
$$\le 4b' \operatorname{dist}(y_1,\partial D).$$

Therefore

$$j_D^*(y_1, y_2) \le \log\left(\frac{\rho_D(y_1, y_2)}{\operatorname{dist}(y_1, \partial D)} + 1\right)^2 = 2\log(4b' + 1) \le 2\log 5b'$$

and

$$k_D(y_1, y_2) \le c j_D^*(y_1, y_2) \le c\log(5b') \le 2c(5b')^{\frac{1}{2}} < b'$$

by (2.1). If $x \in \gamma(y_1, y_2)$, then by Lemma 2.3

$$e^{-b'} \operatorname{dist}(y_2, \partial D) \le \operatorname{dist}(x, \partial D) \le e^{b'} \operatorname{dist}(y_1, \partial D)$$

and from this

$$\ell(\gamma(y_1, y_2)) \le e^{b'} \operatorname{dist}(y_1, \partial D) k_D(y_1, y_2) \le b' e^{b'} \operatorname{dist}(y_1, \partial D)$$

and again we obtain (2.12).

We now complete the proof of Theorem 2.1 as follows. By relabelling we may assume that $\operatorname{dist}(y_1, \partial D) \le \operatorname{dist}(y_2, \partial D)$. Then

$$\begin{aligned} \ell(\gamma) &= \ell(\gamma(x_1, y_1)) + \ell(\gamma(x_2, y_2)) + \ell(\gamma(y_1, y_2)) \\ &\le 4b' e^{b'} \operatorname{dist}(y_2, \partial D) \\ &\le 4b' e^{2b'} r \le 8b' e^{2b'} \rho_D(x_1, x_2) \end{aligned}$$

by (2.6), (2.7) and (2.12). This establishes (1.2). Next if $x \in \gamma$, then either $x \in \gamma(x_j, y_j)$ and

$$\min_{j=1,2} \ell(\gamma(x_j, x)) \le \ell(\gamma(x_j, x)) \le b' e^{b'} \operatorname{dist}(x, \partial D)$$

by (2.7), or $x \in \gamma(y_1, y_2)$ and

$$\min_{j=1,2} \ell(\gamma(x_j, x)) \le \frac{1}{2}\ell(\gamma) \le b' e^{b'} \operatorname{dist}(y_2, \partial D) \le b' e^{2b'} \operatorname{dist}(x, \partial D)$$

by (2.12). In each case we obtain (1.1) and the proof is complete. □

REMARK 2.4. Theorem 4.1 and Remark 4.14 in [11] show that a proper subdomain $D \subset \mathbb{R}^2$ is b-John disk if and only if for some constant $c > 0$

$$k_D(x_1, x_2) \le c j'_D(x_1, x_2) \tag{2.13}$$

for all $x_1, x_2 \in D$, where b and c depend only on each other. Here $j'_D(x_1, x_2)$ is a metric obtained by replacing $\rho_D(x_1, x_2)$ in $j_D^*(x_1, x_2)$ with $\lambda_D(x_1, x_2)$. But Theorem 3.6 in [13] shows that for $n > 2$, $D \subset \mathbb{R}^n$ is a b-John domain if (2.13) holds.

3. Analogues of the Hardy-Littlewood property for $\alpha = 0$ in uniformly John domains

In this section we give two applications of Theorem 2.1 which are analogues of the Hardy-Littlewood Property for $\alpha = 0$ in uniformly John domains in $\mathbb{R}^n, n \geq 2$.

In [9] we have an analogue of the Hardy-Littlewood Property of order $\alpha \in (0, 1]$ for uniformly John domains in $\mathbb{R}^2$ as follows.

LEMMA 3.1. [9] *If a proper subdomain D in $\mathbb{R}^2$ is a b-uniformly John domain and if f is analytic and satisfies*

$$|f'(z)| \leq m \operatorname{dist}(z, \partial D)^{\alpha-1}$$

for all z in D and for some $\alpha \in (0, 1]$, then

$$|f(z_1) - f(z_2)| \leq \frac{cm}{\alpha} \rho_D(z_1, z_2)^{\alpha}$$

for all z_1 and z_2 in D, where $c = c(b)$.

Now we examine the case $\alpha = 0$.

THEOREM 3.2. *A conformal disk $D \subset \mathbb{R}^2$ is a b-uniformly John domain if and only if every analytic function f in D satisfying*

$$|f'(z)| \leq \operatorname{dist}(z, \partial D)^{-1} \tag{3.1}$$

for all z in D satisfies

$$|f(z_1) - f(z_2)| \leq c \log\left(1 + \frac{\rho_D(z_1, z_2)}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right) \tag{3.2}$$

for all z_1 and z_2 in D. Here b and c depend only on each other.

Proof. First suppose that D is a b-uniformly John domain. Then by Theorem 2.1,

$$k_D(z_1, z_2) \leq a \log\left(1 + \frac{\rho_D(z_1, z_2)}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right)$$

for all z_1 and z_2 in D, where a depends only on b. If f is analytic and satisfies (3.1) in D, then

$$|f(z_1) - f(z_2)| \leq k_D(z_1, z_2) \leq a \log\left(1 + \frac{\rho_D(z_1, z_2)}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right)$$

as desired.

Now suppose that every f analytic and satisfying (3.1) in D also satisfies (3.2). By Lemma 1.3,

$$k_D(z_1, z_2) \le c_0 \delta_D(z_1, z_2) \le c_0 a \log\left(1 + \frac{\rho_D(z_1, z_2)}{\min_{j=1,2} \operatorname{dist}(z_j, \partial D)}\right)$$

for all z_1 and z_2 in D. Thus by Theorem 2.1, D is a b-uniformly John domain. □

REMARK 3.3. For a b-uniform domain and a b-John disk, we need to replace $\rho_D(z_1, z_2)$ in (3.2) by $|z_1 - z_2|$ and $\lambda_D(z_1, z_2)$, respectively [14].

By Theorem 1.1 and elementary calculus we know that for functions analytic in a domain $D \subset \mathbb{R}^2$ and for $0 < \alpha \le 1$, $f \in locLip_\alpha(D)$ is equivalent to the bound on the derivative

$$|f'(z)| \le m \operatorname{dist}(z, \partial D)^{\alpha - 1}$$

in D.

For higher dimensions Gehring and Martio show the following.

LEMMA 3.4. [5, 2.13 *Theorem*] *Suppose that D is a domain in $\mathbb{R}^n$ and that $0 < \alpha \le 1$. Then $f : D \to \mathbb{R}^p$ belongs to $locLip_\alpha(D)$ if and only if there are constants $m < \infty$ and $0 < c < 1$ such that*

$$|f(x_1) - f(x_2)| \le m|x_1 - x_2|^\alpha$$

whenever $|x_1 - x_2| \le c \operatorname{dist}(x_1, \partial D)$.

Then Gehring and Martio extend the Hardy-Littlewood property to higher dimensions by using the concept of $locLip_\alpha(D)$ and show that uniform domains in $\mathbb{R}^n$, $n \ge 2$ are *Lip_α-extension domain* for all $0 < \alpha \le 1$ [5].

Now we examine the case of uniformly John domains in $\mathbb{R}^n$, $n \ge 2$ with $\alpha = 0$ and obtain a higher dimensional version of Theorem 3.2.

THEOREM 3.5. *A domain $D \subset \mathbb{R}^n$ is a b-uniformly John domain if and only if every function f in D satisfying*

$$|f(x_1) - f(x_2)| \le m \log\left(1 + \frac{|x_1 - x_2|}{\min_{j=1,2} \operatorname{dist}(x_j, \partial D)}\right) \tag{3.3}$$

for all x_1 and x_2 in D with $|x_1 - x_2| \le \operatorname{dist}(x_1, \partial D)$ satisfies

$$|f(x_1) - f(x_2)| \le mc \log\left(1 + \frac{\rho_D(x_1, x_2)}{\min_{j=1,2} \operatorname{dist}(x_j, \partial D)}\right) \tag{3.4}$$

for all x_1 and x_2 in D. Here b, c and m depend only on each other.

Proof. Suppose that $D \subset \mathbb{R}^n$ is a b-uniformly John domain. Then by Theorem 2.1,

$$k_D(x_1, x_2) \le a \log\left(1 + \frac{\rho_D(x_1, x_2)}{\min_{j=1,2} \operatorname{dist}(x_j, \partial D)}\right)$$

for all x_1 and x_2 in D. Let f satisfy (3.3). Fix x_1, $x_2 \in D$ and let $\gamma \subset D$ be the quasihyperbolic geodesic with endpoints x_1 and x_2. Let $\gamma(s)$ be the parameterization of γ with respect to arc length measured from x_1, $\ell = \ell(\gamma)$. Let $y_1 = x_1$. We choose positive numbers r_i and ℓ_i, and points $y_i \in \gamma$ as follows:

$$r_1 = \frac{1}{2} \operatorname{dist}(y_1, \partial D), \ell_1 = \max\{s : \gamma(s) \in \overline{\mathbb{B}}^n(y_1, r_1)\}, y_2 = \gamma(\ell_1);$$

$$r_2 = \frac{1}{2} \operatorname{dist}(y_2, \partial D), \ell_2 = \max\{s : \gamma(s) \in \overline{\mathbb{B}}^n(y_2, r_2)\}, y_3 = \gamma(\ell_2);$$

and so on. After a finite number of steps, N, say, $\ell_N = \ell$ and the process stops. Let $y_{N+1} = x_2$. So by [3, Lemma 2.6], we have

$$\begin{aligned}
|f(x_1) - f(x_2)| &\le \sum_{i=1}^{N} m \log\left(1 + \frac{|y_i - y_{i+1}|}{\operatorname{dist}(y_{i+1}, \partial D)}\right) \\
&\le m \sum_{i=1}^{N} k_D(\gamma(y_i, y_{i+1})) \\
&\le ma \log\left(1 + \frac{\rho_D(x_1, x_2)}{\min_{j=1,2} \operatorname{dist}(x_j, \partial D)}\right)
\end{aligned}$$

as desired.

Now suppose (3.3) implies (3.4) in D. Fix $x_0 \in D$. Let

$$f(x) = k_D(x, x_0).$$

If x_1, $x_2 \in B \subset D$, B an open ball, then

$$|f(x_1) - f(x_2)| \le k_D(x_1, x_2).$$

Let $\gamma \subset B$ be the segment of the circle through x_1, x_2 perpendicular to ∂B with endpoints x_1, x_2. Then

$$\ell(\gamma) \le \pi |x_1 - x_2|$$

and

$$\min_{j=1,2} \ell(\gamma(x_j, x)) \le \pi \text{ dist}(x, \partial B) \le \pi \text{ dist}(x, \partial D)$$

for all $x \in \gamma$. Following the same argument used in the proof of [11, Theorem 4.1] we get

$$k_D(x_1, x_2) \le \int_\gamma \frac{ds}{\text{dist}(x, \partial D)} \le m \log\left(1 + \frac{|x_1 - x_2|}{\min_{j=1,2} \text{dist}(x_j, \partial D)}\right),$$

where m is independent of x_0 and B, i.e., (3.3) holds. So

$$k_D(x, x_0) \le cm \log\left(1 + \frac{\rho_D(x, x_0)}{\min\{\text{dist}(x, \partial D), \text{dist}(x_0, \partial D)\}}\right)$$

for all $x \in D$, where cm is independent of x_0. Thus

$$k_D(x_1, x_2) \le cm \log\left(1 + \frac{\rho_D(x_1, x_2)}{\min_{j=1,2} \text{dist}(x_j, \partial D)}\right)$$

for all x_1 and x_2 in D, and hence D is a uniformly John domain by Theorem 2.1. □

REMARK 3.6. For a uniform domain $D \subset \mathbb{R}^n$, we need to replace $\rho_D(x_1, x_2)$ in (3.4) by $|x_1 - x_2|$ [14]. Also a domain $D \subset \mathbb{R}^n$ is a b-John domain if (3.3) implies (3.4) obtained by replacing $\rho_D(x_1, x_2)$ with $\lambda_D(x_1, x_2)$ [13].

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