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SPECTRAL CONTINUITY OF ESSENTIALLY p–HYPONORMAL OPERATORS

An-Hyun Kim and Eun-Young Kwon

Abstract. In this paper it is shown that the spectrum σ is continuous at every p-hyponormal operator when restricted to the set of essentially p-hyponormal operators and moreover σ is continuous when restricted to the set of compact perturbations of p-hyponormal operators whose spectral pictures have no holes associated with the index zero.

The spectrum σ can be viewed as a function whose domain consists of operators and whose range consists of compact sets, equipped with the Hausdorff metric, in the complex plane $\mathbf{C}$. It is well-known that σ is upper semicontinuous, but σ is not continuous in general. In [12] it was shown that σ is continuous on the set of normal operators (also see [7, Solution 105]). This argument can be easily extended to the set of hyponormal operators. Also the continuity of σ was considered in [4] and [5]. Recently, in [10], it was shown that σ is continuous when restricted to the set of p-hyponormal operators. However we don't guarantee that σ is continuous on the set of compact perturbations of points of spectral continuity since the spectrum can undergo a substantial change even under rank one perturbations. In fact, σ need not be continuous at rank–one perturbations of unitary operators. To see this, consider

$$T_n = \begin{pmatrix} U & \frac{1}{n}(I-UU^*) \\ 0 & U^* \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix},$$

where U is the unilateral shift on ℓ_2. In this note we examine the continuity of the spectrum for the set including compact perturbations of p-hyponormal operators.

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Throughout this paper let $\mathcal{H}$ be a complex Hilbert space, let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$ and let $\mathcal{K}(\mathcal{H})$ denote the ideal of compact operators on $\mathcal{H}$. If $\mathbf{S}$ is a compact subset of $\mathbf{C}$, we write $\operatorname{iso}\mathbf{S}$, $\operatorname{acc}\mathbf{S}$ and $\eta\,\mathbf{S}$ for the isolated points, the accumulation points, and the polynomially convex hull of $\mathbf{S}$, respectively. If $T \in \mathcal{L}(\mathcal{H})$ write $\sigma(T)$, $\sigma_p(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$ and $\omega(T)$ for the spectrum, the point spectrum, the essential spectrum, the left essential spectrum, the right essential spectrum, and the Weyl spectrum (= the set of $\lambda \in \mathbf{C}$ for which $T-\lambda$ is not *Weyl*, i.e., $T-\lambda$ is not Fredholm of index zero), respectively, of T (cf. [8]). If $T \in \mathcal{L}(\mathcal{H})$, a *hole* in $\sigma_e(T)$ is a bounded component of $\mathbf{C} \setminus \sigma_e(T)$ and a *pseudohole* in $\sigma_e(T)$ is a component of $\sigma_e(T) \setminus \sigma_{le}(T)$ or $\sigma_e(T) \setminus \sigma_{re}(T)$. The *spectral picture* of T, denoted $\mathcal{SP}(T)$, is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with those holes and pseudoholes.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$ $(p > 0)$ (cf. [1], [2], [3]). If $p = 1$ then T is called hyponormal. Let $\mathfrak{A}$ denote a unital C^*-algebra. An element $a \in \mathfrak{A}$ is called *normal* if $a^*a = aa^*$; *hyponormal* if $a^*a \geq aa^*$; and *p–hyponormal* if $(a^*a)^p \geq (aa^*)^p$ for $p > 0$. If $\varrho : \mathfrak{A} \to \mathcal{L}(\mathcal{H}_\varrho)$ is an isometric $*$–homomorphism for a Hilbert space H_ϱ and if a is a p-hyponormal element in $\mathfrak{A}$ then we can easily see that $\varrho(a)$ is a p–hyponormal operator on $\mathcal{H}_\varrho$. Let π denote the canonical map of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, which is a unital C^*-algebra. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *essentially p–hyponormal* if $\pi(T)$ is a p–hyponormal element in $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single valued extension property* (SVEP) if for every open set $G \subset \mathbb{C}$, the only analytic function $f : G \to \mathcal{H}$ satisfying $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in G$ is the zero function on G. For example, every p–hyponormal operator has the SVEP because if T is a p-hyponormal operator then $T-\lambda$ has finite ascent for all $\lambda \in \mathbb{C}$ ([2, Theorem 4]), which implies that T has the SVEP ([11, Proposition 1.8]). Recall also ([1]) that every p-hyponormal operator is reduced by each of its eigenspaces.

Our main theorem is an extension of [10, Theorem].

THEOREM 1. *The following operators T are points of spectral continuity when the function σ is restricted to $\mathfrak{P}_e(\mathcal{H})$:*

(i) *T is reduced by its finite-dimensional eigenspaces;*
(ii) *T has the SVEP;*
(iii) *$T-\mu$ has finite ascent for all $\mu \in \sigma_p(T)$.*

Proof. Write $\mathfrak{P}_e(\mathcal{H})$ for the set of essentially p-hyponormal operators. Suppose that $T, T_n \in \mathfrak{P}_e(\mathcal{H})$, for $n \in \mathbb{Z}_+$, are such that T_n converges to T in norm. If $\varrho : \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_\varrho)$ is an isometric $*$–homomorphism for a Hilbert space H_ϱ then as we noticed in the preceding, $\varrho(\pi(T))$ and $\varrho(\pi(T_n))$ $(n \in \mathbb{Z}_+)$ are p–hyponormal operators on $\mathcal{H}_\varrho$ such that $\varrho(\pi(T_n))$ converges to $\varrho(\pi(T))$ in norm. Since σ is continuous on the set of all p–hyponormal operators ([10, Theorem]), it follows that $\lim \sigma(\varrho(\pi(T_n))) = \sigma(\varrho(\pi(T)))$. Since $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}_\varrho)$ are both unital C^*–algebras it follows from the 'spectral permanence' that

$$\begin{aligned}\lim \sigma_e(T_n) &= \lim \sigma(\pi(T_n)) = \lim \sigma(\varrho(\pi(T_n)))\\ &= \sigma(\varrho(\pi(T))) = \sigma(\pi(T)) = \sigma_e(T).\end{aligned}$$

So the essential spectrum σ_e is continuous when restricted to the set $\mathfrak{P}_e(\mathcal{H})$.

Now suppose that $T_n, T \in \mathfrak{P}_e(\mathcal{H})$ $(n \in \mathbb{Z}_+)$ are such that T_n converges to T in norm. Since in general σ is upper semicontinuous, that is, $\limsup_n \sigma(T_n) \subseteq \sigma(T)$, it suffices to show that $\sigma(T) \subseteq \liminf_n \sigma(T_n)$ whenever T satisfies the given condition. We split the proof into two cases.

Case 1 ($\lambda \in \operatorname{iso} \sigma(T)$). In this case we use an argument of Newburgh [Ne, lemma 3]: if $\lambda \in \operatorname{iso}\sigma(T)$ then for every neighborhood $\mathcal{N}(\lambda)$ of λ there exists an $N \in \mathbb{Z}_+$ such that $n > N$ implies $\sigma(T_n) \cap \mathcal{N}(\lambda) \neq \emptyset$. This shows that $\lambda \in \liminf_n \sigma(T_n)$.

Case 2 ($\lambda \in \operatorname{acc}\sigma(T)$). We assume to the contrary that $\lambda \notin \liminf_n \sigma(T_n)$. Then there exists a neighborhood $\mathcal{N}(\lambda)$ of λ such that does not intersect infinitely many $\sigma(T_n)$. Thus we can choose a subsequence $\{T_{n_k}\}_k$ of $\{T_n\}_n$ such that for some $\epsilon > 0$, $\operatorname{dist}(\lambda, \sigma(T_{n_k})) > \epsilon$ for all $k \in \mathbb{Z}_+$. Since evidently, $\operatorname{dist}(\lambda, \sigma(T_{n_k})) \le \operatorname{dist}(\lambda, \sigma_e(T_{n_k}))$ and σ_e is continuous at T, it follows that $T - \lambda$ is Fredholm. By the continuity of the Fredholm index, we have that $\operatorname{ind}(T-\lambda) = \lim_{k\to\infty} \operatorname{ind}(T_{n_k} - \lambda) = 0$, which says that $T - \lambda$ is Weyl, i.e., $\lambda \notin \omega(T)$.

Firstly, we suppose that T is reduced by its finite-dimensional eigenspaces. Let

$$\mathfrak{M} := \bigvee\left\{\ker(T-\mu) : \ 0 < \dim\ker(T-\mu) < \infty\right\}.$$

By assumption, $\mathfrak{M}$ reduces T. Write

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : \mathfrak{M} \oplus \mathfrak{M}^\perp \to \mathfrak{M} \oplus \mathfrak{M}^\perp.$$

Then T_1 is a normal operator and $\sigma(T_2) = \omega(T_2)$. Since $\sigma_e(T_1) = \omega(T_1)$ and $\lambda \notin \omega(T)$ we can see that $T_1 - \lambda$ is Weyl and $T_2 - \lambda$ is invertible. Since T_1 is normal it follows that $\lambda \in \operatorname{iso}\sigma(T_1)$. Therefore we have that $\lambda \in \operatorname{iso}\sigma(T)$, a contradiction.

Secondly, we suppose that T has the SVEP. Since $\lambda \in \sigma(T) \setminus \omega(T)$, there exists a neighborhood $G(\lambda)$ of λ such that for all $\mu \in G(\lambda)$, $T - \mu$ is Weyl but not invertible. Therefore λ is an interior point of $\sigma_p(T)$, and so T does not have the SVEP because every semi-Fredholm operator whose point spectrum contains a neighborhood of zero does not have the SVEP (see [6, Theorem 9]), a contradiction.

Thirdly, we suppose that $T - \mu$ has finite ascent for all $\mu \in \sigma_p(T)$. Since $T - \lambda$ is Weyl but not invertible, we have $\lambda \in \sigma_p(T)$, and by assumption $T - \lambda$ has finite ascent. Now observe that by the "Index Product Theorem"

$$\dim\ker(T-\lambda)^n - \dim(\operatorname{ran}(T-\lambda)^n)^{\perp} = \operatorname{ind}((T-\lambda)^n) = n \operatorname{ind}(T-\lambda) = 0.$$

So if $\dim\ker(T-\lambda)^n$ is constant then so is $\dim(\operatorname{ran}(T-\lambda)^n)^{\perp}$, which shows that finite ascent forces both finite ascent and descent. Thus it follows ([8, Theorem 9.7.6]) that $T-\lambda$ is Fredholm and $T-\mu$ is invertible for sufficiently small $|\lambda-\mu| \neq 0$, and hence $\lambda \in \operatorname{iso}\sigma(T)$, a contradiction. This proves the theorem. $\square$

COROLLARY 2. *The function σ is continuous at every p-hyponormal operator when restricted to the set of essentially p-hyponormal operators.*

Proof. This follows at once from the second assertion of Theorem 1 together with a remark above Theorem 1. $\square$

COROLLARY 3. *The function σ is continuous when restricted to the set $\mathfrak{P}_0(\mathcal{H}) + \mathcal{K}(\mathcal{H})$, where $\mathfrak{P}_0(\mathcal{H})$ denotes the set of p-hyponormal operators whose spectral pictures have no holes associated with the index zero.*

Proof. Observe first that $\mathfrak{P}_0(\mathcal{H}) + \mathcal{K}(\mathcal{H})$ contains only essentially p-hyponormal operators. Now if $\mathcal{SP}(T)$ has no holes associated with the index zero then so does $\mathcal{SP}(T+K)$ for every compact operator K since $\mathcal{SP}(T+K) = \mathcal{SP}(T)$. Thus if $\lambda \in \operatorname{acc}\sigma(T+K)$ then by the same argument of Theorem 1 we have that $\lambda \notin \omega(T+K)$; therefore $\lambda \in \operatorname{iso}(T+K)$, a contradiction. This completes the proof. $\square$

COROLLARY 4. *The function σ is continuous on the set of essentially p-hyponormal operators T satisfying $\sigma(T) = \omega(T)$.*

Proof. This follows from a careful examination of the proof of Theorem 1. □

Corollary 4 can easily be satisfied by "Toeplitz operators" since by Coburn's theorem, $\sigma(T_\varphi) = \omega(T_\varphi)$ for every Toeplitz operator T_φ. Thus we recapture [9, Theorem 11]:

COROLLARY 5. *The function σ is continuous when restricted to the set $\mathfrak{T} = \{T_\varphi : \varphi \in PQC\}$, where PQC denotes the algebra generated by piecewise continuous functions and quasicontinuous functions on the unit circle.*

Proof. This follows from the remark above Corollary 4 together with the fact that if $\varphi \in PQC$ then T_φ is essentially normal. □

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DEPARTMENT OF MATHEMATICS, CHANGWON NATIONAL UNIVERSITY, CHANGWON 641-773, KOREA

E-mail: ahkim@changwon.ac.kr
key3506@changwon.ac.kr