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GREEN'S EQUIVALENCES OF BIRGET-RHODES EXPANSIONS OF FINITE GROUPS

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ABSTRACT. In this paper we establish a counting method for the Green classes of the Birget-rhodes expansion of finite groups. As an application of the results, we derive explicit enumeration formulas for the Green classes for finite groups of order pq and a finite cyclic group of order p^m, where p and q are arbitrary given distinct prime numbers.

1. Introduction

An expansion defined by Birget and Rhodes [2] can be though of a systematic way of writing semigroups S as homomorphic images of other semigroups $\overline{S}$; some important properties of S are preserved in $\overline{S}$. Among various almost finite expansions of semigroups investigated by Birget and Rhodes, we are mainly interested in the following particular expansion of a semigroup S, which is called the *Birget-Rhodes expansion* of S: For any finite sequence $(s_1, s_2, \ldots, s_n)$ of elements $s_1, s_2, \ldots, s_n$ in S, put

$$P(s_1, s_2, \ldots, s_n) := \{1, s_1, s_1s_2, \ldots, s_1s_2\cdots s_n\},$$

where 1 is the identity of S^1. Define

$$\tilde{S}^{\mathscr{R}} := \{(P(s_1, s_2, \ldots, s_n), s_1s_2\cdots s_n) : s_1, s_2, \ldots, s_n \in S, n \geq 1\}$$

with the multiplication

$$\begin{aligned}&(P(s_1, s_2, \ldots, s_n), s_1s_2\cdots s_n)(P(t_1, t_2, \ldots, t_m), t_1t_2\cdots t_m)\\ =\ &(P(s_1, s_2, \ldots, s_n) \cup (s_1s_2\cdots s_n)\\ &\cdot P(t_1, t_2, \ldots, t_m), s_1s_2\cdots s_nt_1t_2\cdots t_m)\end{aligned}$$

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where $s{\cdot}U = \{su : u \in U\}$ for $s \in S$ and $U \subset S$. Then $\tilde{S}^{\mathscr{R}}$ is a semigroup. And it turns out [9] that when $S = G$ is a group,

$$\tilde{G}^{\mathscr{R}} = \{(A, g) \in P_1(G) \times G : g \in G\},$$

where $P_1(G)$ denotes the set of all finite subsets of G containing the identity 1_G of G.

In [9] Szendrei showed that the Birget-Rhodes expansion $\tilde{(\cdot)}^{\mathscr{R}}$ as a natural functor from the category of groups into the category of F-inverse semigroups is the left adjoint of the functor assigning the greatest group homomorphic image to every F-inverse semigroup. In [3] the authors derived a new approach to the Burnside problem using the residually finiteness of the Birget-Rhodes expansion $\tilde{(\cdot)}^{\mathscr{R}}$.

In [5] Exel constructed, in a canonical way, an inverse monoid $\mathcal{S}(G)$ associated with a group G defined via generators and relations. He established the one-to-one correspondence between actions of $\mathcal{S}(G)$ on a set X (an action of an inverse semigroup S on the set X is a unital homomorphism from S to the symmetric inverse monoid $I(X)$) and partial actions of G on X, with its applications on graded C^*-algebras. In [7] Kellendonk and Lawson observed that the inverse monoid $\mathcal{S}(G)$ constructed by Exel is exactly the same as the Birget-Rhodes expansion $\tilde{G}^{\mathcal{R}}$ of the group G. In [4], the authors prove that if a group G acts faithfully on a Hausdorff space X and acts freely at a non-isolated point, then the Birget-Rhodes expansion $\tilde{G}^{\mathcal{R}}$ of the group G is isomorphic to an inverse monoid of Möbius type which mainly aries in conformal geometry.

In this paper, we restrict our attention to the "finite" Birget-Rhodes expansion $\tilde{(\cdot)}^{\mathscr{R}}$ functor from the category of finite groups into the category of finite F-inverse semigroups in which natural counting problems depending on the group structures arise. Beside its importance in studying finite semigroups, the problem counting Green $\mathscr{L}, \mathscr{R}$ $\mathscr{D}$, and $\mathscr{H}$-classes of $\tilde{G}^{\mathscr{R}}$ of a finite group G looks very natural in the theory of finite inverse semigroups (cf. [1]). Although it is shown by a direct approach (Theorem 2.5) that both the Green $\mathscr{L}$ and $\mathscr{R}$-classes of the Birget-Rhodes expansion $\tilde{G}^{\mathscr{R}}$ consist of $2^{|G|-1}$ classes that looks independent on the group structures of G, but the number of the Green $\mathscr{D}$ or $\mathscr{H}$-classes are heavily depend on the group structures of G.

Our main objective of this paper is to count the Green classes of the Birget-Rhodes expansion of a finite group.

As an application of the results, when G is a finite group of order pq or a finite cyclic group of order p^m, we obtain an explicit formula on the number of the Green classes for G, where p and q are distinct primes is given in section 3.

2. Green's relations on $\tilde{G}^{\mathscr{R}}$

In the following, we always *assume* that G is a finite group of order n. For a subset A of G, we denote $|A|$ by the number of elements of A. By $A \leq G$ we shall mean that A is a subgroup of G.

Green's relations on an inverse monoid S are defined as follow: for $s, t \in S$,

$$s \mathscr{R} t \Longleftrightarrow ss^{-1} = tt^{-1}; s \mathscr{L} t \Longleftrightarrow s^{-1}s = t^{-1}t;$$
$$s \mathscr{J} t \Longleftrightarrow SsS = StS; \mathscr{H} = \mathscr{R} \cap \mathscr{L}.$$

The relations $\mathscr{R}$ and $\mathscr{L}$ commute under composition. The Green $\mathscr{D}$-relation is then defined by

$$\mathscr{D} = \mathscr{R} \circ \mathscr{L} = \mathscr{L} \circ \mathscr{R}.$$

These relations are equivalence relations on S, and they play an important role in the investigation of the structure of semigroups ([6, 8]). Observe that

$$\mathscr{H} = \mathscr{R} \cap \mathscr{L} \subset \mathscr{R} \cup \mathscr{L} \subset \mathscr{D} \subset \mathscr{J}.$$

For $a \in S$, $\mathscr{L}_a, \mathscr{R}_a, \mathscr{J}_a, \mathscr{H}_a$, and $\mathscr{D}_a$ denote the $\mathscr{L}, \mathscr{R}, \mathscr{J}, \mathscr{H}$, and $\mathscr{D}$-classes of a in S, respectively.

The set $\tilde{G}^{\mathscr{R}}$ defined by

$$\tilde{G}^{\mathscr{R}} = \{(A, g) \in \mathcal{P}_1(G) \times G \,:\, g \in A\}$$

is an inverse monoid, called the *Birget-Rhodes expansion* ([2, 3, 9]) of the group G, under the multiplication $(A, g)(B, h) = (A \cup g \cdot B, gh)$, where $g \cdot B = \{gb : b \in B\}$. It is known [9] that the Birget-Rhodes expansion $\tilde{G}^{\mathscr{R}}$ of G is an F-inverse monoid whose maximum group image is isomorphic to the given group G.

For a subset A of G, the *stablizer* of A is defined by

$$\mathrm{Stab}(A) := \{g \in G : gA = A\}.$$

Then $\mathrm{Stab}(A)$ is a subgroup of G.

LEMMA 2.1. *Let $(A, g), (B, h) \in \tilde{G}^{\mathscr{R}}$. Then we have*

(1) *$(A, g) \mathscr{R} (B, h)$ if and only if $A = B$, and hence*

$$\mathscr{R}_{(A,g)} = \{(A, a) \,:\, a \in A\}.$$

(2) $(A,g)\mathscr{L}(B,h)$ *if and only if* $g^{-1}A = h^{-1}B$, *and hence*

$$\mathscr{L}_{(A,g)} = \{(a^{-1}A, a^{-1}g) \,:\, a \in A\}.$$

(3) $(A,g)\mathscr{D}(B,h)$ *if and only if* $A = kB$ *for some* $k \in G$, *and hence*

$$\mathscr{D}_{(A,g)} = \{(k^{-1}A, h) : k \in A \text{ and } h \in k^{-1}A\}.$$

(4) $\mathscr{D} = \mathscr{J}$.

(5) $(A,g)\mathscr{H}(B,h)$ *if and only if* $A = B = hg^{-1}A$, *and hence*

$$\mathscr{H}_{(A,g)} = \{(A, sg) \,:\, s \in \mathrm{Stab}(A)\}.$$

In particular, the maximal subgroup $H(A,1)$ *is isomorphic to* $\mathrm{Stab}(A)$.

Proof. (1) is straightforward.

(2) The first statement is clear and the second comes from

$$\mathscr{L}_{(A,g)} = \{(B,h) \in \tilde{G}^{\mathscr{R}} : B = hg^{-1}A\} = \{(hg^{-1}A, h) : h \in A^{-1}g\}.$$

(3) Suppose that $(A,g)\mathscr{D}(B,h)$. Then there exists an element (C,f) in $\tilde{G}^{\mathscr{R}}$ such that $(A,g)\mathscr{L}(C,f)$ and $(C,f)\mathscr{R}(B,h)$. By (1) and (2), we have $g^{-1}A = f^{-1}C$ and $C = B$ and hence $A = gf^{-1}B$. Conversely, suppose that $A = kB$ for some $k \in G$. Then $(B, k^{-1}g) \in \tilde{G}^{\mathscr{R}}$, $(A,g)\mathscr{L}(B,k^{-1}g)$, and also $(B,k^{-1}g)\mathscr{R}(B,h)$. This implies that $(A,g)\mathscr{D}(B,h)$. Moreover,

$$\mathscr{D}_{(A,g)} = \{(k^{-1}A, h) \in \tilde{G}^{\mathscr{R}} : k \in G\} = \{(k^{-1}A, h) : k \in A \text{ and } h \in k^{-1}A\}.$$

(4) Let $(A,g)\mathscr{D}(B,h)$ and $(A,g) \leq (B,h)$. Then by (3), we have $A = kB$ for some $k \in G$, $B \subset A$, and $g = h$. This implies that $(A,g) = (B,h)$. By Corollary 19 of 3.2 in [8], we have $\mathscr{D} = \mathscr{J}$.

(5) The first statement comes (1) and (2), and the second comes from the fact that $H(A,1)$ is equal to the Green $\mathscr{H}$-class of $(A,1)$ in $\tilde{G}^{\mathscr{R}}$. Moreover,

$$\begin{aligned}\mathscr{H}_{(A,g)} &= \{(A,h) \in \tilde{G}^{\mathscr{R}} : hg^{-1}A = A\} = \{(A,h) \in \tilde{G}^{\mathscr{R}} : h \in \mathrm{Stab}(A)g\} \\ &= \{(A, sg) : s \in \mathrm{Stab}(A)\}.\end{aligned}$$

This completes the proof. □

Notice that $\mathrm{Stab}(A)$ acts freely on the set A by left multiplication

$$\mathrm{Stab}(A) \times A \to A, \quad g \cdot a = ga.$$

By the Burnside Lemma, the number of orbits is

$$\Big|A/\mathrm{Stab}(A)\Big| = \frac{|A|}{|\mathrm{Stab}(A)|}. \tag{2.1}$$

COROLLARY 2.2. *Let* $(A,g) \in \tilde{G}^{\mathscr{R}}$. *Then*

(1) $|\mathscr{L}_{(A,g)}| = |\mathscr{R}_{(A,g)}| = |A|$,

(2) $|\mathscr{D}_{(A,g)}| = |\mathscr{J}_{(A,g)}| = \frac{|A|^2}{|\mathrm{Stab}(A)|} \leq |A|^2$,

(3) $|\mathscr{H}_{(A,g)}| = |\mathrm{Stab}(A)|$.

Proof. (1) It comes from Lemma 2.1 (1) and (2).

(2) We observe that for $k, k' \in A$, $k^{-1}A = (k')^{-1}A$ if and only if $k'k^{-1} \in \mathrm{Stab}(A)$ if and only if $k' \in \mathrm{Stab}(A)k$. Now, by Lemma 2.1 (3) and (2.1), we have

$$|\mathscr{D}_{(A,g)}| = \Big|A/\mathrm{Stab}(A)\Big| \cdot |A| = \frac{|A|}{|\mathrm{Stab}(A)|} \cdot |A| \leq |A|^2.$$

(3) By Lemma 2.1(5), we have $|\mathscr{H}_{(A,g)}| = |\mathrm{Stab}(A)g| = |\mathrm{Stab}(A)|$. □

COROLLARY 2.3. *Let* $(A,g) \in \tilde{G}^{\mathscr{R}}$. *Then* A *is a subgroup of* G *if and only if* $|\mathscr{D}_{(A,g)}| = |A|$. *In this case, we have that* $\mathscr{D}_{(A,g)} = \{(A,g) : g \in A\}$.

Proof. Suppose that A is a subgroup of G. Then $\mathrm{Stab}(A) = A$. This implies that $|A/\mathrm{Stab}(A)| = 1$, and hence by Corollary 2.2 (2), $|\mathscr{D}_{(A,g)}| = |A/\mathrm{Stab}(A)| \cdot |A| = |A|$.

Conversely, suppose that $|\mathscr{D}_{(A,g)}| = |A|$. Let $x, y \in A$. Since $|A/\mathrm{Stab}(A)| = 1$, there exists $g \in \mathrm{Stab}(A)$ such that $gx = y$. This implies that $xy^{-1} = g^{-1} \in \mathrm{Stab}(A)$ and thus $xy^{-1}A = A$. Because $1_G \in A$, $xy^{-1} \in A$ and hence A is a subgroup of G. □

REMARK 2.4. Each $\mathscr{D}$-class in a semigroup is a union of $\mathscr{L}$-classes and also a union of $\mathscr{R}$-classes. If the intersection of an $\mathscr{L}$-class and an $\mathscr{R}$-class is none empty set, then it is an $\mathscr{H}$-class. We may visualize a $\mathscr{D}$-class in a finite semigroup as "eggbox" diagram in Figure 1.

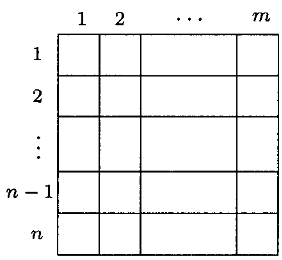


Figure 1. Eggbox

In this diagram each row represents an $\mathscr{R}$-class, each column represents an $\mathscr{L}$-class, and each cell an $\mathscr{H}$-class. In the case of our semigroup $\tilde{G}^{\mathscr{R}}$, each $\mathscr{D}$-class $\mathscr{D}_{(A,g)}$ forms the square eggbox, $n = m = \frac{|A|}{|\mathrm{Stab}(A)|}$ from Corollary 2.2.

THEOREM 2.5. *We have*

(1) $|\tilde{G}^{\mathscr{R}}/\mathscr{L}| = |\tilde{G}^{\mathscr{R}}/\mathscr{R}| = 2^{|G|-1}$.

(2) $|\tilde{G}^{\mathscr{R}}/\mathscr{H}| = \sum_{A \in P_1(G)} \frac{|A|}{|\mathrm{Stab}(A)|}$.

Proof. (1) It is immediate to see that for each $k = 1, 2, \ldots, n = |G|$,

$$|\{(A,g) \in \tilde{G}^{\mathscr{R}} : |A| = k\}| = \binom{n-1}{k-1} \cdot k.$$

By Corollary 2.2 (1), we have

$$|\{(A,g) \in \tilde{G}^{\mathscr{R}} : |A| = k\}/\mathscr{L}| = \frac{1}{k}\left[\binom{n-1}{k-1} \cdot k\right] = \binom{n-1}{k-1}$$

and hence

$$|\tilde{G}^{\mathscr{R}}/\mathscr{L}| = \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}.$$

(2) This results follows from Lemma 2.1 (5) and Corollary 2.2 (3). □

LEMMA 2.6. *Let* $(A,g), (B,h) \in \tilde{G}^{\mathscr{R}}$ *with* $(A,g)\, \mathscr{D}\, (B,h)$. *Then*

(1) $|A| = |B|$,

(2) $|\mathrm{Stab}(A)| = |\mathrm{Stab}(B)|$.

In particular, if G *is abelian, then* $\mathrm{Stab}(A) = \mathrm{Stab}(B)$.

Proof. (1) By Lemma 2.1, $A = kB$ for some $k \in G$, and hence $|A| = |B|$.

(2) By Corollary 2.2,

$$|A|^2/|\mathrm{Stab}(A)| = |\mathscr{D}_{(A,g)}| = |\mathscr{D}_{(B,h)}| = |B|^2/|\mathrm{Stab}(B)|.$$

Thus $|\mathrm{Stab}(A)| = |\mathrm{Stab}(B)|$ from (1). Assume that G is abelian. Since $(A,g)\, \mathscr{D}\, (B,h)$, by Lemma 2.1, $A = kB$ for some $k \in G$. Let $a \in \mathrm{Stab}(A)$. Then $aA = A$, and hence $B = k^{-1}A = k^{-1}aA = ak^{-1}A = aB$. Thus $a \in \mathrm{Stab}(B)$. Conversely, if $b \in \mathrm{Stab}(B)$, then $bA = bkB = kbB = kB = A$, and hence $b \in \mathrm{Stab}(A)$. □

For $1 \leq k, m \leq |G|$ and a subgroup S of G, we set

$$\mathcal{A}_k = \{(A,g) \in \tilde{G}^{\mathscr{R}} :\ |A| = k\},\ d_k(S) = \Big|\{(A,g) \in \mathcal{A}_k : \mathrm{Stab}(A) = S\}\Big|,$$

and

$$d_k(m) = \sum_{S \leq G, |S|=m} d_k(S).$$

THEOREM 2.7. *We have*

$$\left|\tilde{G}^{\mathscr{R}}/\mathscr{D}\right| = \sum_{k=1}^{|G|} |\mathcal{A}_k/\mathscr{D}|$$

and for each k,

$$|\mathcal{A}_k/\mathscr{D}| = \sum_{m=1}^{k} \frac{m}{k^2} \cdot d_k(m) = \frac{1}{k^2} \sum_{S \leq G} |S| \cdot d_k(S).$$

Proof. It follows by Lemma 2.6 (1) that $\left|\tilde{G}^{\mathscr{R}}/\mathscr{D}\right| = \sum_{k=1}^{|G|} |\mathcal{A}_k/\mathscr{D}|$. Note that $\mathcal{A}_k$ is the disjoint union of the sets $\{(A, g) \in \mathcal{A}_k : |\text{Stab}(A)| = m\}$,

$$\mathcal{A}_k = \bigcup_{m=1}^{k} \{(A, g) \in \mathcal{A}_k : |\text{Stab}(A)| = m\}.$$

By Lemma 2.6 (2), we have $|\mathcal{A}_k/\mathscr{D}| = \sum_{m=1}^{k} |\{(A, g) \in \mathcal{A}_k : |\text{Stab}(A)| = m\}/\mathscr{D}|$. Since

$$\{(A, g) \in \mathcal{A}_k : |\text{Stab}(A)| = m\} = \bigcup_{S \leq G, |S|=m} \{(A, g) \in \mathcal{A}_k : \text{Stab}(A) = S\},$$

it follows from Corollary 2.2 that

$$\begin{aligned} &\left|\{(A, g) \in \mathcal{A}_k : |\text{Stab}(A)| = m\}/\mathscr{D}\right| \\ &= \sum_{S \leq G, |S|=m} \frac{|S|}{k^2} \cdot \left|\{(A, g) \in \mathcal{A}_k : \text{Stab}(A) = S\}\right|. \end{aligned}$$

Therefore,

$$|\mathcal{A}_k/\mathscr{D}| = \sum_{m=1}^{k} \sum_{S \leq G, |S|=m} \frac{|S|}{k^2} \cdot d_k(S) = \sum_{m=1}^{k} \frac{m}{k^2} \cdot d_k(m).$$

Now, if $|S| > k$ then $\{(A, g) \in \mathcal{A}_k : \text{Stab}(A) = S\} = \emptyset$ and hence

$$\sum_{m=1}^{k} \sum_{S \leq G, |S|=m} \frac{|S|}{k^2} \cdot d_k(S) = \frac{1}{k^2} \sum_{S \leq G} |S| \cdot d_k(S).$$

This completes the proof. □

COROLLARY 2.8. *We have*

$$\left|\tilde{G}^{\mathscr{R}}/\mathscr{H}\right| = \sum_{k=1}^{|G|}\sum_{S\le G}\frac{d_k(S)}{|S|}.$$

Proof. By Corollary 2.2 (2), $|\mathscr{D}_{(A,g)}| = |A|^2/|\mathrm{Stab}(A)|$. For each $(B,h) \in \mathscr{D}_{(A,g)}$, by Corollary 2.2 (3) and Lemma 2.6, $|A| = |B|$ and $|\mathscr{H}_{(B,h)}| = |\mathrm{Stab}(B)| = |\mathrm{Stab}(A)|$. This implies that the number of $\mathscr{H}$-classes in $\mathscr{D}_{(A,g)}$ is $|A|^2/|\mathrm{Stab}(A)|^2$ (See Figure 1). By Theorem 2.7, we have

$$\left|\tilde{G}^{\mathscr{R}}/\mathscr{H}\right| = \sum_{k=1}^{|G|}\left[\sum_{S\le G}\frac{1}{k^2}|S|d_k(S)\cdot\frac{k^2}{|S|^2}\right] = \sum_{k=1}^{|G|}\sum_{S\le G}\frac{d_k(S)}{|S|}.$$

□

From Theorem 2.7 and Corollary 2.8, we can see that the counting problem on the number of Green $\mathscr{H}$ and $\mathscr{D}$-classes of the Birget-Rhodes expansion of a finite group G is eventually that problem on $d_k(S)$ for each k $(1 \le k \le |G|)$ and for each subgroup S. Let S be a fixed subgroup of G and let

$$\widetilde{d_k}(S) = |\{(A,g) \in \mathcal{A}_k : S \le \mathrm{Stab}(A)\}|.$$

It is clear that

$$\widetilde{d_k}(S) = \sum_{S\le K} d_k(K). \tag{2.2}$$

To calculate $d_k(S)$ in terms of $\widetilde{d_k}(K)$, one can invert the equation (2.2) by introducing the Möbius function for G. This assigns an integer $\mu(K)$ to each super-subgroup K of S by recursive formula

$$\sum_{S\le H}\mu(H) = \begin{cases} 1 & \text{if } S = H \\ 0 & \text{if } S < H.\end{cases}$$

Then we have

$$d_k(S) = \sum_{S\le K}\mu(K)\,\widetilde{d_k}(K). \tag{2.3}$$

LEMMA 2.9. *Let $A \subsetneq G$. Then A is a union of right cosets of the subgroup* Stab(A) *in G.*

Proof. Straightforward. □

LEMMA 2.10. *Let K be a subgroup of G and let $1 \leq k \leq |G|$. Then*

$$\widetilde{d_k}(K) = \begin{cases} \begin{pmatrix} \frac{|G|}{|K|} - 1 \\ \frac{k}{|K|} - 1 \end{pmatrix} \cdot k & \text{if } |K| \text{ is a divisor of } k, \\ 0 & \text{otherwise} \end{cases}$$

where $\binom{0}{0}$ is defined to be 1.

Proof. Let $l = \frac{k}{|K|}$. By definition, $\widetilde{d_k}(K)$ is non-empty implies that $|K|$ divides k. Thus it suffices to show that

$$\begin{aligned} &\{(A, g) \in \mathcal{A}_k : K \leq \mathrm{Stab}(A)\} \\ &= \{(B, h) \in \mathcal{A}_k : B = K \cup Kg_1 \cup \cdots \cup Kg_{l-1},\ g_i \in G\}. \end{aligned}$$

Let $(A, g) \in \mathcal{A}_k$ such that $K \leq \mathrm{Stab}(A)$. Then Stab(A) is a union of right cosets of the group K in Stab(A). By Lemma 2.9, the set A is a union of right cosets of the group Stab(A) in G. Conversely, if $B = K \cup Kg_1 \cup \cdots \cup Kg_{l-1}$ with $|B| = k$, then obviously the group K is contained in Stab(B). This completes the proof. □

LEMMA 2.11. *Let $1 \leq k, m \leq |G|$. Then*

(1) *If $m \nmid k$, then $d_k(m) = 0$.*

(2) *If $\gcd(|G|, k) = 1$, then*

$$d_k(m) = \begin{cases} \begin{pmatrix} |G| - 1 \\ k - 1 \end{pmatrix} \cdot k, & \text{if } m = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and hence

$$|\mathcal{A}_k/\mathscr{D}| = \frac{1}{|G|}\begin{pmatrix} |G| \\ k \end{pmatrix} \text{ and } |\mathcal{A}_k/\mathscr{H}| = |\mathcal{A}_k|.$$

(3) $d_k(|G|) = \begin{cases} |G|, & \text{if } k = |G| \\ 0, & \text{otherwise.} \end{cases}$

(4) $|\mathcal{A}_k/\mathscr{D}| = |\mathcal{A}_{n-k}/\mathscr{D}|$.

Proof. (1) Suppose that $d_k(m) \neq 0$. Then by definition there exists a subgroup S of order m and a subset A with $|A| = k$ such that Stab$(A) = S$. By (2.1), $m|k$.

(2) Suppose that $d_k(m) \neq 0$. Then m divides $|G|$ since it is an order of a subgroup. Since $\gcd(|G|, k) = 1$, m must be equal to 1. Thus

$$d_k(1) = \sum_{S \leq G, |S|=1} d_k(S) = d_k(\{1_G\}) = \begin{pmatrix} |G| - 1 \\ k - 1 \end{pmatrix} \cdot k.$$

It then follows by Theorem 2.7 and Corollary 2.8 that

$$|\mathcal{A}_k/\mathscr{D}| = \frac{1}{k^2}\sum_{m=1}^{k} m\cdot d_k(m) = \frac{1}{k}\binom{|G|-1}{k-1} = \frac{1}{|G|}\binom{|G|}{k}$$
$$|\mathcal{A}_k/\mathscr{H}| = \binom{|G|-1}{k-1}\cdot k = |\mathcal{A}_k|.$$

(3) It follows from the fact that $d_k(|G|) = d_k(G) = \widetilde{d}_k(G)$ and from Lemma 2.10.

(4) By Theorem 2.7 and Lemma 2.10,

$$|\mathcal{A}_k/\mathscr{D}| = \sum_{S\le G}|S|\left[\sum_{S\le K}\mu(K)\cdot\frac{1}{k}\binom{\frac{n}{|K|}-1}{\frac{k}{|K|}-1}\right]\ (|K|\,\big|\,k).$$

Since

$$\frac{1}{k}\binom{\frac{n}{|K|}-1}{\frac{k}{|K|}-1} = \frac{1}{n-k}\binom{\frac{n}{|K|}-1}{\frac{n-k}{|K|}-1},$$

we have

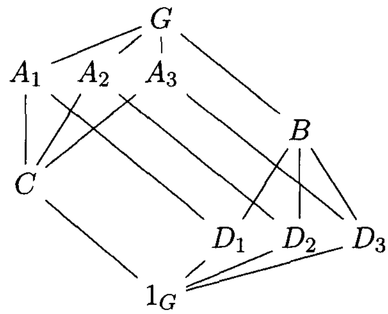
$$|\mathcal{A}_k/\mathscr{D}| = \sum_{S\le G}|S|\left[\sum_{S\le K}\mu(K)\cdot\frac{1}{n-k}\binom{\frac{n}{|K|}-1}{\frac{n-k}{|K|}-1}\right] = |\mathcal{A}_{n-k}/\mathscr{D}|.$$

This completes the proof. □

REMARK 2.12. If G is a trivial group, then $|\tilde{G}^{\mathscr{R}}/\mathscr{D}| = 1$. By Lemma 2.11 (4), we have that if $|G| = 2$, then $|\tilde{G}^{\mathscr{R}}/\mathscr{D}| = 2$ and if $|G|\ge 3$, then

$$|\tilde{G}^{\mathscr{R}}/\mathscr{D}| = \sum_{k=1}^{|G|}|\mathcal{A}_k/\mathscr{D}|$$
$$= \begin{cases} 3 + |\mathcal{A}_{\frac{|G|}{2}}/\mathscr{D}| + 2\displaystyle\sum_{k=2}^{\lfloor\frac{|G|-1}{2}\rfloor}|\mathcal{A}_k/\mathscr{D}|, & \text{if } |G| \text{ is even,}\\ 3 + 2\displaystyle\sum_{k=2}^{\frac{|G|-1}{2}}|\mathcal{A}_k/\mathscr{D}|, & \text{if } |G| \text{ is odd.}\end{cases}$$

EXAMPLE 2.13. Let $G = \mathbb{Z}_6 \oplus \mathbb{Z}_2$. Then G has the following lattice diagram of subgroups:



where

$$A_1 = \langle(1,0)\rangle,\ A_2 = \langle(1,1)\rangle,$$
$$A_3 = \{(0,0),(2,0),(2,1),(4,0),(4,1),(0,1)\},$$
$$B = \{(0,0),(0,1),(3,0),(3,1)\},\ C = \langle(2,0)\rangle,$$
$$D_1 = \langle(3,0)\rangle,\ D_2 = \langle(3,1),\ D_3 = \langle(0,1)\rangle.$$

Applying Lemma 2.11, we have that

$$|\mathcal{A}_{12}/\mathscr{D}| = 1, |\mathcal{A}_{12}/\mathscr{H}| = 1, |\mathcal{A}_{11}/\mathscr{D}| = 1, |\mathcal{A}_{11}/\mathscr{H}| = 121,$$
$$|\mathcal{A}_7/\mathscr{D}| = 66, |\mathcal{A}_7/\mathscr{H}| = 3234, |\mathcal{A}_5/\mathscr{H}| = 1650, |\mathcal{A}_1/\mathscr{H}| = 1.$$

Next, we will compute the case $|\mathcal{A}_6/\mathscr{D}|$. The remaining cases are similar.

Let $(A, g) \in \mathcal{A}_6$. Then $|\mathrm{Stab}(A)| = 6$ or 3 or 2 or 1. Using (2.3) and Lemma 2.10,

$$d_6(A_i) = \sum_{A_i \leq K} \mu(K)\widetilde{d_6}(K) = \binom{1}{0} \cdot 6 = 6 \text{ for each } i = 1,2,3,$$

$$d_6(C) = \sum_{C \leq K} \mu(K)\widetilde{d_6}(K) = 1 \cdot \binom{3}{1} \cdot 6 - 3 \cdot 6 = 0,$$

$$\begin{aligned} d_6(D_i) &= \sum_{D_i \leq K} \mu(K)\widetilde{d_6}(K) \\ &= 1 \cdot \binom{5}{2} \cdot 6 - 1 \cdot 6 = 54 \text{ for each } i = 1,2,3, \end{aligned}$$

$$\begin{aligned} d_6(\{1_G\}) &= \sum_{1_G \leq K} \mu(K)\widetilde{d_6}(K) = 1 \cdot \binom{11}{5} \cdot 6 - 1 \cdot \binom{3}{1} \cdot 6 \\ &\quad + 3 \cdot (-1) \cdot \binom{5}{2} \cdot 6 + 3 \cdot 1 \cdot 6 = 2592, \end{aligned}$$

Now, we evaluate $|\mathcal{A}_6/\mathscr{D}|$ and $|\mathcal{A}_6/\mathscr{H}|$ as follows.

$$\begin{aligned}|\mathcal{A}_6/\mathscr{D}| &= \frac{1}{6^2}\sum_{S\le G}|S|\cdot d_6(S)\\ &= \frac{|\{1_G\}|}{6^2}\cdot d_6(\{1_G\}) + \sum_{i=1}^{3}\frac{|A_i|}{6^2}\cdot d_6(A_i) + \sum_{i=1}^{3}\frac{|D_i|}{6^2}\cdot d_6(D_i)\\ &= 72+3+9 = 84\end{aligned}$$

$$\begin{aligned}|\mathcal{A}_6/\mathscr{H}| &= \sum_{S\le G}\frac{d_6(S)}{|S|} = \frac{d_6(\{1_G\})}{1} + \sum_{i=1}^{3}\frac{d_6(A_i)}{6} + \sum_{i=1}^{3}\frac{d_6(D_i)}{2}\\ &= 2592+3+81 = 2676.\end{aligned}$$

We have the following table on $|\mathcal{A}_k/\mathscr{D}|$ and $|\mathcal{A}_k/\mathscr{H}|$:

k	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert\mathcal{A}_k/\mathscr{D}\rvert$	1	7	19	45	66	84	66	45	19	7	1	1
$\lvert\mathcal{A}_k/\mathscr{H}\rvert$	1	19	163	633	1650	2676	3234	2532	1467	475	121	1

TABLE 1. $|\mathcal{A}_k/\mathscr{D}|$ and $|\mathcal{A}_k/\mathscr{H}|$

We conclude from Theorem 2.7, Corollary 2.8, and Table 1 that

$$|\tilde{G}^{\mathscr{R}}/\mathscr{D}| = 361 \text{ and } |\tilde{G}^{\mathscr{R}}/\mathscr{H}| = 12972.$$

3. Enumeration formulas

In this section we will find an explicit formula on the number of the Green $\mathscr{H}$ or $\mathscr{D}$-classes for finite groups of order pq or for finite cyclic groups of order p^m where p and q are distinct prime numbers.

Let $k_p(G)$ denote the number of subgroups of G having order p.

THEOREM 3.1. *If G is a group of order pq ($p \neq q, p, q$: prime), then we have*

$$(1)\ |\tilde{G}^{\mathscr{R}}/\mathscr{D}| = 1 + \frac{1}{pq}\Big[2^{pq} - 2 + k_p(G)(p-1)(2^q-2) + k_q(G)(q-1)(2^p-2)\Big].$$

(2) $|\tilde{G}^{\mathscr{R}}/\mathscr{H}| = 1 + (pq+1)2^{pq-2} - pq + k_p(G)(1-p)\Big[(q+1)2^{q-2} - q\Big] + k_q(G)(1-q)\Big[(p+1)2^{p-2} - p\Big].$

Proof. Let $1 \leq k \leq pq = |G|$. We consider the following cases:
Case (i) $k = pq$. By Lemma 2.11,

$$|\mathcal{A}_k/\mathscr{D}| = 1 \text{ and } |\mathcal{A}_k/\mathscr{H}| = 1.$$

Case (ii) $\gcd(pq, k) = 1$. By Lemma 2.11, we find that

$$|\mathcal{A}_k/\mathscr{D}| = \frac{1}{pq}\binom{pq}{k} \text{ and } |\mathcal{A}_k/\mathscr{H}| = \binom{pq-1}{k-1} \cdot k = \frac{1}{pq} \cdot k^2 \binom{pq}{k}.$$

Case (iii) $k = lp,\ 1 \leq l \leq q-1$: In this case,

$$d_k(p) = \sum_{S \leq G, |S|=p} d_k(S) = k_p(G) \cdot \binom{q-1}{l-1} \cdot lp,$$
$$d_k(1) = \sum_{S \leq G, |S|=1} d_k(S) = \left[\binom{pq-1}{lp-1} - k_p(G)\binom{q-1}{l-1}\right] \cdot lp.$$

Therefore

$$\begin{aligned}
|\mathcal{A}_k/\mathscr{D}| &= \frac{1}{k^2}\{p \cdot d_k(p)(G) + 1 \cdot d_k(1)\} \\
&= \frac{1}{l^2p^2}\Bigg\{p \cdot k_p(G) \cdot \left[\binom{q-1}{l-1} \cdot lp\right] \\
&\quad + \left[\binom{pq-1}{lp-1} - k_p(G)\binom{q-1}{l-1}\right] \cdot lp\Bigg\} \\
&= \frac{k_p(G)}{q}\binom{q}{l} + \frac{1}{pq}\binom{pq}{lp} - \frac{k_p(G)}{pq}\binom{q}{l}
\end{aligned}$$

and

$$\begin{aligned}|\mathcal{A}_k/\mathscr{H}| &= d_k(\{1_G\}) + \sum_{S\le G, |S|=p} \frac{d_k(S)}{|S|}\\ &= \left[\binom{pq-1}{lp-1}\cdot lp - k_p(G)\cdot\binom{q-1}{l-1}\cdot lp\right] + k_p(G)\binom{q-1}{l-1}\cdot l\\ &= \frac{1}{pq}\cdot(lp)^2\binom{pq}{lp} - \frac{k_p(G)\cdot p}{q}\cdot l^2\binom{q}{l} + \frac{k_p(G)}{q}\cdot l^2\binom{q}{l}\\ &= \frac{k_p(G)}{q}\left[(1-p)\cdot l^2\binom{q}{l}\right] + \frac{1}{pq}\cdot(lp)^2\binom{pq}{lp}.\end{aligned}$$

Case (iv) $k = lq,\ 1 \le l \le p-1$: In this case, we have that

$$|\mathcal{A}_k/\mathscr{D}| = \frac{k_q(G)}{p}\binom{p}{l} + \frac{1}{pq}\binom{pq}{lq} - \frac{k_q(G)}{pq}\binom{p}{l}$$

and

$$|\mathcal{A}_k/\mathscr{H}| = \frac{k_q(G)}{p}\left[(1-q)\cdot l^2\binom{p}{l}\right] + \frac{1}{pq}\cdot(lq)^2\binom{pq}{lq}.$$

Therefore by Theorem 2.7, we have

$$\begin{aligned}|\tilde{G}^{\mathscr{R}}/\mathscr{D}| &= \sum_{k=1}^{pq}|\mathcal{A}_k/\mathscr{D}|\\ &= |\mathcal{A}_{pq}/\mathscr{D}| + \sum_{\substack{1\le k<pq\\ \gcd(k,pq)=1}}|\mathcal{A}_k/\mathscr{D}| + \sum_{l=1}^{q-1}|\mathcal{A}_{lp}/\mathscr{D}| + \sum_{l=1}^{p-1}|\mathcal{A}_{lq}/\mathscr{D}|\\ &= 1 + \sum_{l=1}^{q-1}\left[\frac{k_p(G)}{q}\binom{q}{l} - \frac{k_p(G)}{pq}\binom{q}{l}\right]\\ &\quad + \sum_{l=1}^{p-1}\left[\frac{k_q(G)}{p}\binom{p}{l} - \frac{k_q(G)}{pq}\binom{p}{l}\right]\\ &\quad + \sum_{\substack{1\le k<pq\\ \gcd(k,pq)=1}}\frac{1}{pq}\binom{pq}{k} + \sum_{l=1}^{q-1}\frac{1}{pq}\binom{pq}{lp} + \sum_{l=1}^{p-1}\frac{1}{pq}\binom{pq}{lq}\end{aligned}$$

$$
\begin{aligned}
&= 1 + \frac{k_p(G)}{q}\Big(1 - \frac{1}{p}\Big)\sum_{l=1}^{q-1}\binom{q}{l} \\
&\quad + \frac{k_q(G)}{p}\Big(1 - \frac{1}{q}\Big)\sum_{l=1}^{p-1}\binom{p}{l} + \frac{1}{pq}\sum_{k=1}^{pq-1}\binom{pq}{k} \\
&= 1 + \frac{1}{pq}\big[2^{pq} - 2 + k_p(G)(p-1)(2^q - 2) \\
&\quad + k_q(G)(q-1)(2^p - 2)\big]
\end{aligned}
$$

and also by Corollary 2.8, we have

$$
\begin{aligned}
&|\tilde{G}^{\mathscr{R}}/\mathscr{H}| \\
&= \sum_{k=1}^{pq}|\mathcal{A}_k/\mathscr{H}| \\
&= \big|\mathcal{A}_{pq}/\mathscr{H}\big| + \sum_{\substack{1\leq k<pq \\ \gcd(k,pq)=1}} |\mathcal{A}_k/\mathscr{H}| \\
&\quad + \sum_{l=1}^{q-1}|\mathcal{A}_{lp}/\mathscr{H}| + \sum_{l=1}^{p-1}|\mathcal{A}_{lq}/\mathscr{H}| \\
&= 1 + \frac{1}{pq}\Bigg[\sum_{\substack{1\leq k<pq \\ \gcd(k,pq)=1}} k^2\binom{pq}{k} + \sum_{l=1}^{q-1}(lp)^2\binom{pq}{lp} + \sum_{l=1}^{p-1}(lq)^2\binom{pq}{lq}\Bigg] \\
&\quad + (1-p)\frac{k_p(G)}{q}\sum_{l=1}^{q-1}l^2\binom{q}{l} + (1-q)\frac{k_q(G)}{p}\sum_{l=1}^{p-1}l^2\binom{p}{l} \\
&= 1 + \frac{1}{pq}\sum_{k=1}^{pq-1}k^2\binom{pq}{k} + (1-p)\frac{k_p(G)}{q}\sum_{l=1}^{q-1}l^2\binom{q}{l} \\
&\quad + (1-q)\frac{k_q(G)}{p}\sum_{l=1}^{p-1}l^2\binom{p}{l}.
\end{aligned}
$$

Since

$$
\sum_{i=1}^{n-1}k^2\binom{n}{k} = n(n+1)2^{n-2} - n^2 = n\big[(n+1)2^{n-2} - n\big], \tag{3.1}
$$

we can obtain the result

$$|\tilde{G}^{\mathscr{R}}/\mathscr{H}| = 1 + (pq+1)2^{pq-2} - pq + k_p(G)(1-p)\Big[(q+1)2^{q-2} - q\Big] + k_q(G)(1-q)\Big[(p+1)2^{p-2} - p\Big].$$

This completes the proof. □

REMARK 3.2. Let p and q be distinct prime numbers. Then we have the following two cases:

(i) $p < q,\ p \nmid (q-1)$. In this case, there are only one non-isomorphic group of order pq, the cyclic group $\mathbb{Z}_{pq}$ which is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_q$.

(ii) $p < q,\ p \mid (q-1)$. In this case, there are only two non-isomorphic groups of order pq. one of them is the cyclic group $\mathbb{Z}_{pq}$ and the other is a non-abelian group $\mathcal{K}_{pq}$ generated by two elements a and b such that

$$|\langle a\rangle| = p, \quad |\langle b\rangle| = q, \quad ab = b^s a,$$

where $s \neq 1$ and $s^p \equiv 1 \pmod q$. In $\mathcal{K}_{pq}$, there are q subgroups of order p and only one subgroup of order q.

By (i) and (ii) of the above remark, we have the following results.

COROLLARY 3.3. *Let p and q are distinct prime numbers. Then*

(1) $$|\widetilde{\mathbb{Z}_{pq}}^{\mathscr{R}}/\mathscr{D}| = 1 + \frac{1}{pq}\big[2^{pq} - 2 + (p-1)(2^q-2) + (q-1)(2^p-2)\big],$$

$$|\widetilde{\mathbb{Z}_{pq}}^{\mathscr{R}}/\mathscr{H}| = 1 + (pq+1)2^{pq-2} - pq + (1-p)\Big[(q+1)2^{q-2} - q\Big] + (1-q)\Big[(p+1)2^{p-2} - p\Big].$$

(2) $$|\tilde{\mathcal{K}}_{pq}^{\mathscr{R}}/\mathscr{D}| = 1 + \frac{1}{pq}\big[2^{pq} - 2 + q(p-1)(2^q-2) + (q-1)(2^p-2)\big],$$

$$|\tilde{\mathcal{K}}_{pq}^{\mathscr{R}}/\mathscr{H}| = 1 + (pq+1)2^{pq-2} - pq + q(1-p)\Big[(q+1)2^{q-2} - q\Big] + (1-q)\Big[(p+1)2^{p-2} - p\Big].$$

(3) $$|\widetilde{D_p}^{\mathscr{R}}/\mathscr{D}| = 1 + \frac{1}{2p}\big[2^{2p} - 2 + p(2^p-2) + 2(p-1)\big],$$

$$|\tilde{D}_p^{\mathscr{R}}/\mathscr{H}| = 1 + (2p+1)2^{pq-2} - 2p - p\Big[(p+1)2^{p-2} - p\Big] + (1-p),$$

where D_p $(p \geq 3)$ denotes the Diheral group of order $2p$.

Proof. (1) It follows from that $k_p(\mathbb{Z}_{pq}) = 1 = k_q(\mathbb{Z}_{pq})$.

(2) It follows from the fact $k_p(\mathcal{K}_{pq}) = q$ and $k_q(\mathcal{K}_{pq}) = 1$.

(3) It follows from the fact $|D_p| = 2p$, $k_2(D_p) = p$ and $k_p(D_p) = 1$. □

To obtain an explicit formula on the number of the Green $\mathscr{H}$ or $\mathscr{D}$-classes of the Birget-Rhodes expansion of a finite cyclic group of order p^m (p : prime), we begin with the following basic lemma of natural numbers.

LEMMA 3.4. *Let n ($\neq 1$) be a natural number. Then we have*

$$\sum_{k=1}^{m} \sum_{\substack{1 \le l < n^{m-k+1} \\ n \nmid l}} \binom{n^m}{ln^{k-1}} = 2^{n^m} - 2.$$

Proof. It is immediate that for each $i = 0, 1, \dots, m-1$,

$$|\{ln^i : 1 \le l < n^{m-i},\ n \nmid l\}| = (n-1)n^{m-i-1}$$

and therefore, $\sum_{i=0}^{m-1} |\{ln^i : 1 \le l < n^{m-i},\ n \nmid l\}| = n^m - 1$. This implies that

$$\begin{aligned} &\sum_{k=1}^{m} \sum_{\substack{1 \le l < n^{m-k+1} \\ n \nmid l}} \binom{n^m}{ln^{k-1}} \\ &= \sum_{\substack{1 \le l < n^m \\ n \nmid l}} \binom{n^m}{l} + \sum_{\substack{1 \le l < n^{m-1} \\ n \nmid l}} \binom{n^m}{ln} + \cdots + \sum_{\substack{1 \le l < n \\ n \nmid l}} \binom{n^m}{ln^{m-1}} \\ &= \sum_{s=1}^{n^m - 1} \binom{n^m}{s} = 2^{n^m} - 2. \end{aligned}$$

This completes the proof. □

THEOREM 3.5. *Let G be a cyclic group of order p^m (p, prime). Then we have*

(1) $$|\tilde{G}^{\mathscr{R}}/\mathscr{D}| = 1 + \frac{1}{p^m}\left[(2^{p^m} - 2) + (p-1)\sum_{k=1}^{m-1} p^{k-1}(2^{p^{m-k}} - 2)\right].$$

(2) $$\begin{aligned} |\tilde{G}^{\mathscr{R}}/\mathscr{H}| = 1 &+ \left[(p^m + 1)2^{p^m - 2} - p^m\right] \\ &+ (1-p)\sum_{k=1}^{m-1}\left[(p^{m-k} + 1)2^{p^{m-k} - 2} - p^{m-k}\right]. \end{aligned}$$

Proof. Let $1 \le k \le p^m = |G|$.

Case (i) $k = p^m$. By Lemma 2.11,

$$|\mathcal{A}_k/\mathscr{D}| = |\mathcal{A}_k/\mathscr{H}| = 1.$$

Case (ii) $k \neq p^m$. We divide the following two subcases;

(a) k is not a multiple of p : In this case, $\gcd(p^m, k) = 1$. By Lemma 2.11, we have that

$$|\mathcal{A}_k/\mathscr{D}| = \frac{1}{p^m}\binom{p^m}{k} \text{ and } |\mathcal{A}_k/\mathscr{H}| = \binom{p^m-1}{k-1}\cdot k = \frac{k^2}{p^m}\binom{p^m}{k}.$$

(b) k is a multiple of p : In this case, the number k can be expressed by the form

$$k = lp^i,\ 1 \le l < p^{m-i},\ p \nmid l.$$

If $(A, g) \in \mathcal{A}_k$, then $\mathrm{Stab}(A)$ is a subgroup of G with its order p^s for s $(0 \le s \le i)$. Let P_t be the subgroup of G with its order p^t. Then by the equation (2.3), we have

$$\begin{aligned}
d_k(P_i) &= \binom{p^{m-i}-1}{l-1}\cdot lp^i, \\
d_k(P_s) &= \sum_{P_s \le K} \mu(K)\widetilde{d}_k(K) = \mu(P_s)\widetilde{d}_k(P_s) + \mu(P_{s+1})\widetilde{d}_k(P_{s+1}) \\
&= \left[\binom{p^{m-s}-1}{lp^{i-s}-1} - \binom{p^{m-s-1}-1}{lp^{i-s-1}-1}\right]\cdot lp^i \qquad \text{for } s\ (0 \le s < i).
\end{aligned}$$

This implies that

$$\begin{aligned}
|\mathcal{A}_k/\mathscr{D}| &= \frac{1}{l^2p^{2i}}\left[\sum_{s=0}^{i} p^s \cdot d_k(P_s)\right] \\
&= \frac{1}{lp^i}\left[\binom{p^m-1}{lp^i-1} + \sum_{j=1}^{i}(p^j - p^{j-1})\binom{p^{m-j}-1}{lp^{i-j}-1}\right] \\
&= \frac{1}{lp^i}\binom{p^m-1}{lp^i-1} + \sum_{j=1}^{i}\left(\frac{1}{lp^{i-j}} - \frac{1}{p}\cdot\frac{1}{lp^{i-j}}\right)\binom{p^{m-j}-1}{lp^{i-j}-1} \\
&= \frac{1}{p^m}\binom{p^m}{lp^i} + \sum_{j=1}^{i}\left(\frac{1}{p^{m-j}} - \frac{1}{p^{m-j+1}}\right)\binom{p^{m-j}}{lp^{i-j}} \\
&= \frac{1}{p^m}\left[\binom{p^m}{lp^i} + (p-1)\sum_{j=1}^{i} p^{j-1}\binom{p^{m-j}}{lp^{i-j}}\right]
\end{aligned}$$

and

$$
\begin{aligned}
&|\mathcal{A}_k/\mathscr{H}| \\
&= \sum_{s=0}^{i} \frac{d_k(P_s)}{p^s} \\
&= lp^i \cdot \binom{p^m-1}{lp^i-1} + \sum_{j=1}^{i} \left[lp^{i-j} \cdot \binom{p^{m-j}-1}{lp^{i-j}-1} - lp^{i-j+1} \cdot \binom{p^{m-j}-1}{lp^{i-j}-1} \right] \\
&= \frac{(lp^i)^2}{p^m} \binom{p^m}{lp^{i-j}} + \sum_{j=1}^{i} \left[\frac{(lp^{i-j})^2}{p^{m-j}} \binom{p^{m-j}}{lp^{i-j}} - \frac{p(lp^{i-j})^2}{p^{m-j}} \binom{p^{m-j}}{lp^{i-j}} \right] \\
&= \frac{1}{p^m} \left[(lp^i)^2 \binom{p^m}{lp^i} + (1-p) \sum_{j=1}^{i} p^j (lp^{i-j})^2 \binom{p^{m-j}}{lp^{i-j}} \right].
\end{aligned}
$$

Thus we have that

$$
\begin{aligned}
&\sum_{\substack{k \\ p|k}} |\mathcal{A}_k/\mathscr{D}| \\
&= \sum_{\substack{1 \le l < p^{m-1} \\ p \nmid l}} |\mathcal{A}_{lp}/\mathscr{D}| + \sum_{\substack{1 \le l < p^{m-2} \\ p \nmid l}} |\mathcal{A}_{lp^2}/\mathscr{D}| + \cdots + \sum_{\substack{1 \le l < p \\ p \nmid l}} |\mathcal{A}_{lp^{m-1}}/\mathscr{D}| \\
&= \frac{1}{p^m} \left[\sum_{k=1}^{m-1} \sum_{\substack{1 \le l < p^{m-k} \\ p \nmid l}} \binom{p^m}{lp^k} + (p-1) \sum_{k=1}^{m-1} p^{k-1} (2^{p^{m-k}} - 2) \right],
\end{aligned}
$$

where the last equality follows from Lemma 3.4, and

$$
\begin{aligned}
\sum_{\substack{k \\ p|k}} |\mathcal{A}_k/\mathscr{H}| = &\sum_{\substack{1 \le l < p^{m-1} \\ p \nmid l}} |\mathcal{A}_{lp}/\mathscr{H}| + \sum_{\substack{1 \le l < p^{m-2} \\ p \nmid l}} |\mathcal{A}_{lp^2}/\mathscr{H}| \\
&+ \cdots + \sum_{\substack{1 \le l < p \\ p \nmid l}} |\mathcal{A}_{lp^{m-1}}/\mathscr{H}|
\end{aligned}
$$

$$= \frac{1}{p^m}\sum_{k=1}^{m-1}\sum_{\substack{1\le l<p^{m-k}\\ p\nmid l}}(lp^k)^2\binom{p^m}{lp^k}$$
$$+ (1-p)\sum_{k=1}^{m-1}\Big[(p^{m-k}+1)2^{p^{m-k}-2} - p^{m-k}\Big],$$

where the last equality follows from (3.1).

By Theorem 2.7 and by Corollary 2.8, we conclude that

$$\begin{aligned}
|\tilde{G}^{\mathscr{R}}/\mathscr{D}| &= \sum_{k=1}^{p^m}|\mathcal{A}_k/\mathscr{D}| \\
&= |\mathcal{A}_{p^m}/\mathscr{D}| + \sum_{\substack{k\\ p\nmid k}}|\mathcal{A}_k/\mathscr{D}| + \sum_{\substack{k\\ p|k}}|\mathcal{A}_k/\mathscr{D}| \\
&= 1 + \frac{1}{p^m}\Bigg[\Bigg\{\sum_{\substack{k\\ p\nmid k}}\binom{p^m}{k} + \sum_{k=1}^{m-1}\sum_{\substack{1\le l<p^{m-k}\\ p\nmid l}}\binom{p^m}{lp^k}\Bigg\} \\
&\qquad + (p-1)\sum_{k=1}^{m-1}p^{k-1}(2^{p^{m-k}}-2)\Bigg] \\
&= 1 + \frac{1}{p^m}\Bigg[\sum_{k=1}^{m}\sum_{\substack{1\le l<p^{m-k+1}\\ p\nmid l}}\binom{p^m}{lp^{k-1}} \\
&\qquad + (p-1)\sum_{k=1}^{m-1}p^{k-1}(2^{p^{m-k}}-2)\Bigg] \\
&= 1 + \frac{1}{p^m}\left[(2^{p^m}-2) + (p-1)\sum_{k=1}^{m-1}p^{k-1}(2^{p^{m-k}}-2)\right],
\end{aligned}$$

and

$$|\tilde{G}^{\mathscr{R}}/\mathscr{H}| = \sum_{k=1}^{p^m}|\mathcal{A}_k/\mathscr{H}|$$

$$= |\mathcal{A}_{p^m}/\mathscr{H}| + \sum_{\substack{k \\ p\nmid k}} |\mathcal{A}_k/\mathscr{H}| + \sum_{\substack{k \\ p|k}} |\mathcal{A}_k/\mathscr{D}|$$

$$= 1 + \frac{1}{p^m}\left[\sum_{\substack{k \\ p\nmid k}} k^2 \binom{p^m}{k} + \sum_{k=1}^{m-1} \sum_{\substack{1\le l<p^{m-k} \\ p\nmid l}} (lp^k)^2 \binom{p^m}{lp^k}\right]$$

$$+ (1-p)\sum_{k=1}^{m-1}\left[(p^{m-k}+1)2^{p^{m-k}-2} - p^{m-k}\right]$$

$$= 1 + \frac{1}{p^m}\sum_{k=1}^{m} \sum_{\substack{1\le l<p^{m-k+1} \\ p\nmid l}} (lp^{k-1})^2 \binom{p^m}{lp^{k-1}}$$

$$+ (1-p)\sum_{k=1}^{m-1}\left[(p^{m-k}+1)2^{p^{m-k}-2} - p^{m-k}\right]$$

$$= 1 + \left[(p^m+1)2^{p^m-2} - p^m\right]$$

$$+ (1-p)\sum_{k=1}^{m-1}\left[(p^{m-k}+1)2^{p^{m-k}-2} - p^{m-k}\right].$$

This completes the proof. □

REMARK 3.6. Let G be a group of order p^2 (p, prime). Then G is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$ and therefore

(1) $|\widetilde{\mathbb{Z}_{p^2}}^{\mathscr{R}}/\mathscr{D}| = 1 + \frac{1}{p^2}\left[2^{p^2} - 2 + (p-1)(2^p - 2)\right]$,

$|\widetilde{\mathbb{Z}_{p^2}}^{\mathscr{R}}/\mathscr{H}| = 1 + (p^2+1)2^{p^2-2} - p^2 + (1-p)\left[(p+1)2^{p-2} - p\right]$.

(2) $|\widetilde{\mathbb{Z}_p \times \mathbb{Z}_p}^{\mathscr{R}}/\mathscr{D}| = 1 + \frac{1}{p^2}\left[2^{p^2} - 2 + (p+1)(p-1)(2^p - 2)\right]$,

$|\widetilde{\mathbb{Z}_p \times \mathbb{Z}_p}^{\mathscr{R}}/\mathscr{H}| = 1 + (p^2+1)2^{p^2-2} - p^2 + (p+1)(1-p)\left[(p+1)2^{p-2} - p\right]$.

Finally, we give a Table 2 on the number of the Green $\mathscr{H}$ and $\mathscr{D}$-classes of the Birget-Rhodes expansions of groups with order ≤ 10.

$\|G\| = n$	The type of groups	$\|\tilde{G}^{\mathscr{R}}/\mathscr{D}\|$	$\|\tilde{G}^{\mathscr{R}}/\mathscr{H}\|$
1	trivial group	1	1
2	$\mathbb{Z}_2$	2	2
3	$\mathbb{Z}_3$	3	6
4	$\mathbb{Z}_4$	5	16
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	6	14
5	$\mathbb{Z}_5$	7	44
6	$\mathbb{Z}_6$	13	100
	$S_3 = D_3$	15	90
7	$\mathbb{Z}_7$	17	250
8	$\mathbb{Z}_8$	35	552
	Q_4 = Quatenion	36	550
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	39	511
	D_4	42	494
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	45	478
9	$\mathbb{Z}_9$	59	1262
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	63	1232
10	$\mathbb{Z}_{10}$	107	2760
	D_5	119	2588

TABLE 2. The number of the Green $\mathscr{D}$ and $\mathscr{H}$-classes of $\tilde{G}^{\mathscr{R}}$.

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